Basic Linear Algebra
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Lecture – 05
Systems of Linear Equations II

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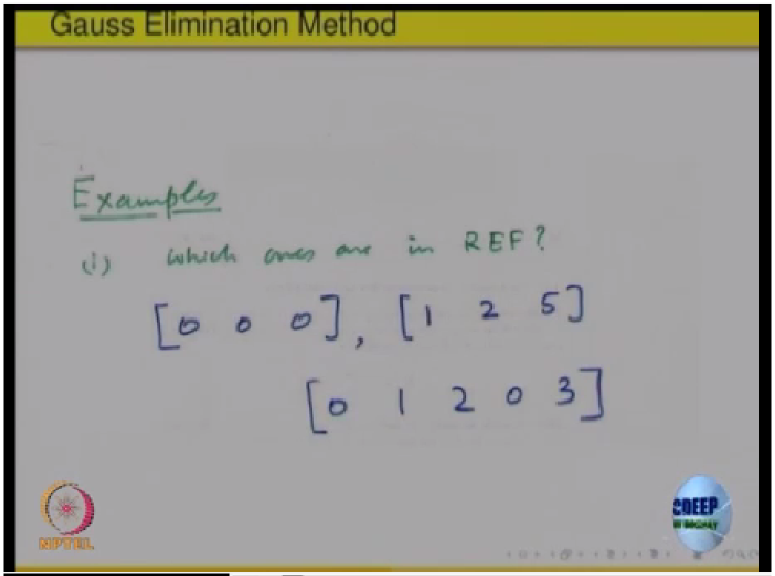

Let us look at the matrix 0 0 0 0. Is it in the row echelon form? We are looking only at the non-0 ones, so we are not really bothered about it, okay. You can say this is not in the row echelon form. It cannot be brought because it is non, there is no non-0 entry at all. Let us look at this 1 2 and 5. Is that in the row echelon form? Yes. It is. There is only 1 row which is non-0, so fine. What about this one? That also is, okay.

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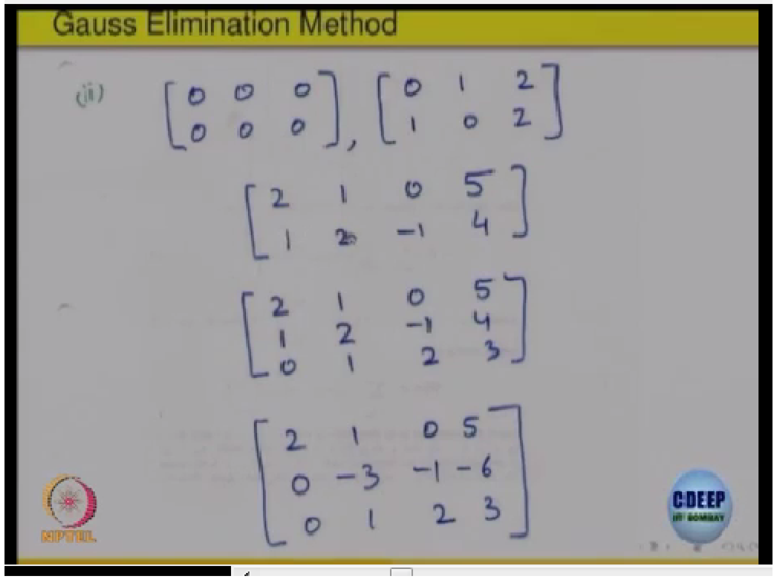

Let us look at some more. For example, let us look at this one. 0 1 2 first row. 1 0 2 then second row, right. Here the first non-0 entry in the first row is coming at 1 but in the next row, it is coming at, first place itself. Whereas in the second column where it is in the first column, so this is not in the row echelon form. If I wanted to bring it in the row echelon form, I will interchange these 2 rows.

I will take it up and bring it down, okay. What about this one? This is not in the row echelon form, right. Because r has to be between 1 and n. There has to be a 0 row at the bottom, right. Sorry, is it in the row echelon form? Let we just see what is. We said there is a r. No this is in the row echelon form. Sorry. It says r between 1 and minimum, right. r could be equal to 2 itself, okay.

Is it okay, that this is in the row echelon form or not. Yes. No it is not in the row echelon form. Why? Why it is not in the row echelon form? Because here the first non-0 entry case coming at the place 2. In the next row, the non-0 entry should not be coming at the same column. It should be on the right side. See the second property, right. So this is not in the row echelon form because the non-0 entry for both the rows are coming at the same column, right.

What about this one? Again it is not, right. Because again the same conflict comes here itself. What about the last one? That is still not because even though for this one it is okay. First non-0 entry, first row is coming at p1, first column. Second is coming at, okay, fine, no problem. But the third one again is not coming at that place, right. It is again at the column 2. So this is also not in the row echelon form, right.

A row is called non-0 row if there is at least 1 non-0 entry in that. That is all. Right. And to be in the row echelon form, what do we want? We want that the non-0 entry in some row it is coming here. In the next row, it should be on the right side of that column, that is all. Other entries could be 0.

But there is at least 1 non-0 entry and where it should, so where should the non-0 entry. 0 0 0 0 non-0, where should it start that is a question? So the non-0 entry in the first one comes at p1. Next one comes at p2 and the next one comes at pr. It is okay? Clear? So out of this, which one is, let us look at in the first one.

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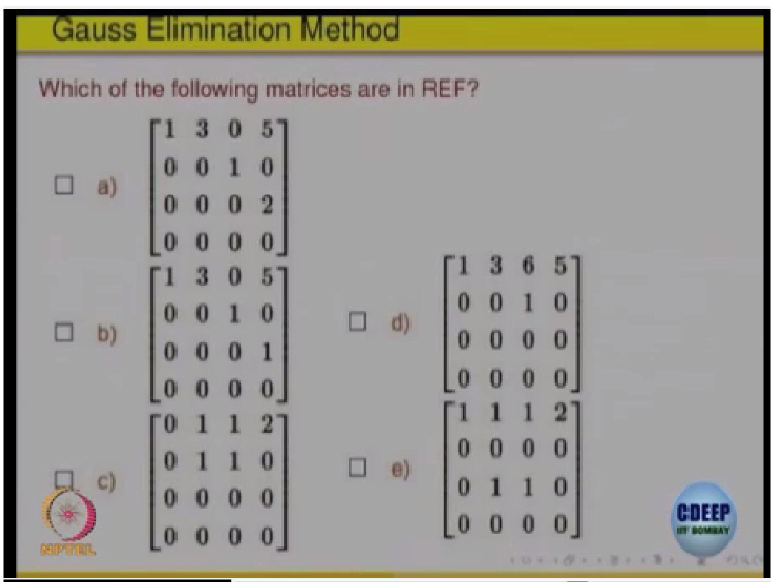

So a says, right. So there is a non-0 entry here that is at the first place. Second one, it comes at the third place, third column and this one comes at the fourth and bottom is 0. So first matrix is in row echelon form. Let us look at the second one? 1 here; in the first row, it is, first entry at the first column. Second at and third also on the third. So p1 1, p2 3, p3 is 4. What is r? r is 3. There are 3 non-0 rows, right.

For r1, row 1, the non-0 entry is coming at p1 which is the first column. r2, it is coming at column number, right, 1 2 3, third. And in the third row, it is coming at the fourth place. So p3 is 4, right. This also is in the row echelon form. What about this one? Obviously not because here the non-0 entry is coming at the second in the first row, also on the third, so it is not. What about the last one, this d?

It is because there are only 2 non-0 rows. Bottom rows are 0. In the first one, is coming at 1, column 1 and in the second one, second row, it is coming at the third place. That is the right side of the first one, okay. The last one? That is not because there is no non-0 entry here that should have been at the bottom, right but that is all. You can bring it to, we are not saying, using as such. This is not in row, it can be brought, okay.

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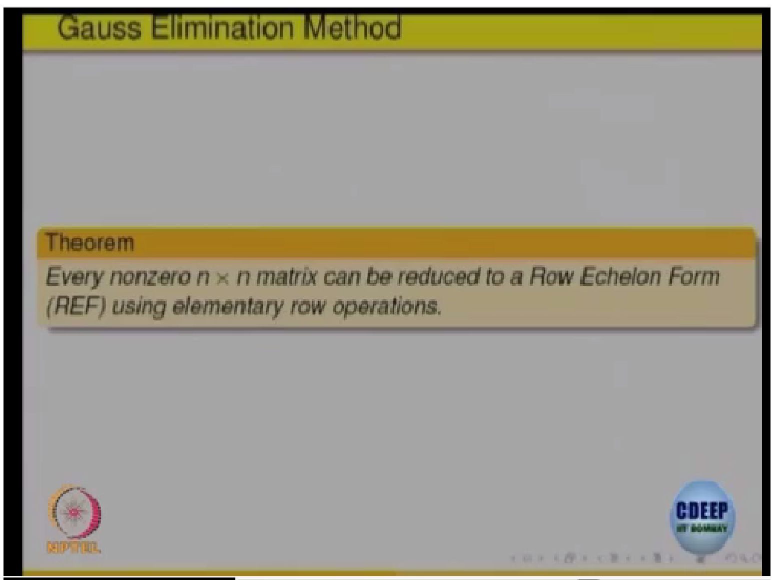

So let us look at, so this is a theorem which can be written down, the proof can be written down. Actually when we do the changing a matrix to the row echelon form, we will see that the proof automatically will be a part of it. There sometimes, you say there exists, you actually construct that thing and that shows that it exists also, right. So there are some proofs in mathematics which are purely existential proofs where you cannot show that object but you can prove it exists, right.

There are other proofs in mathematics where you actually construct and show that it exists. The difference is (()) (06:35) my pet quote saying that the god exists, right. Some people believe god exists because of some reasons. But you cannot produce god in front of you, right. So there are existential proofs which purely say, right. You might have seen in calculus that a criteria for convergence of a limit.

Every equations sequence has a limit. It does not tell you where it is. It says it exists, right. But definition of convergence says you can locate the point which is the limit. So that is the difference between existence and proving actually it exists, okay. You will come across these things in many parts of mathematics. So this term says every non-0, why it should be n*n. It should be m*n.

There is a typo here. Every matrix, right, of any order m*n, there is no relation between m and n, okay. There is a typo. It should be m, m*n matrix can be reduced to a row echelon form. The word a is used indicating that row can be more than 1, row echelon form of a matrix, using purely elementary row operations, okay. So the row echelon form we will shorten it as REF. Row operations we will write as row operations. Is the theorem, we will see how the proof also goes as we do something, okay.

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Why the word echelon comes? If you look at the English dictionary, what is echelon? So go back and look at English dictionary. What does the word echelon means? From where does this word suddenly drops in, in mathematics? Echelon means a particular arrangement formation of something. For example when the planes fly, they fly in a particular form. It is called the echelon form of the planes.

So one is behind the other, right. They do not, one position is slightly behind. So the position where the non-0 entry is coming is on the right side, that is the formation. So echelon word basically means formation, particular formation of something, okay. I just brought this picture now. Hopefully the echelon word will stay in your mind. It is formation of one going behind or staircase if you want to keep in mind that kind of a thing, okay.

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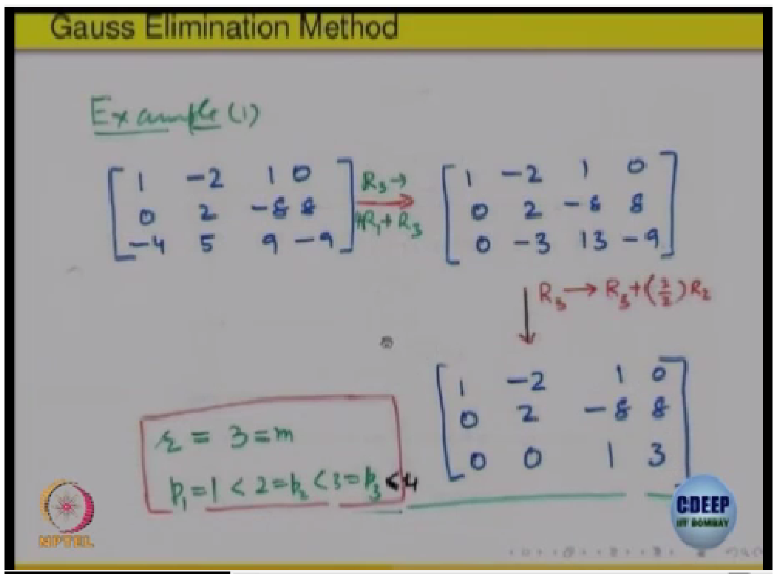

So let us look at an example how does one do that. So this is a matrix. First row is 1 -2 1, this is not 10, there are 4 entries in the first row and 3 rows are there. So it is a 3 rows, m*n, n is 3 rows, 4 columns are there, okay. So what do we want to do? Our aim is Gauss elimination method by row operations, right, I want to bring in 0's at the bottom and non-0 at the top in that fashion, right.

So what we do? Here it is staircase. The row R3, see this is how you write R3 changes to 4*R1+original R3. So we are doing a transformation here, row operation. We are multiplying the first row by 4 and adding it to R3. So R3 will change. So that is indicated with a change R3 to 4*R1+R3. Why that is done? Because we multiply this by 4 and add here. This will become 0. So I will get everything below as 0, right, non-0 should be shifted.

It is okay? So we do that. So you get this matrix, okay. First row remains as it is. Second row remains as it is. Only the third row has changed, right. Because of this row operation, okay. And now what I want? Here it is 0 0, here there is non-0 coming at third row. If possible, I would like to make it 0, right. So what should I do? I take this row, right. I can multiply it by, if I multiply by 3/2, then what will be the entry here, 1, right.

I want to add it, okay. So what shall I do? 3/2, 2 will cancel out. This will be 3 and added this will become 0. So I do that operation. R3 goes to R3+3/2*R2, right. So if that is done, I get this form. Now let us check. So at every check when you do operation with the intention of bringing 0 at the bottom and pushing non-0 to the right, okay. So this becomes 1 -2, first row remains as it is.

Second is remaining as it is. Only third has changed to this form. So all rows are non-0. So what is R? A number of non-0 rows. It is 3. So R is 3 which is the number of rows here. Number of columns is 4. So this is less than 4 anyway, right. What is the p1? The place where first non-0 entry is coming. So this is p1. p1 is 1, right. What is p2? In the second non-0 row, the non-0 entry is coming at the column number 2.

Value is 2 but the entry is at the column number 2, the entry is 2. So the p2=2. What is p3? That is equal to 3. So p3=3 which is less than 4 anyway, right. Clear? So this is in row echelon form. Remember I could have multiplied this row by 4, right. If I multiply this first row, this matrix, I take R1 and multiply by 4, what I will get? I get 4 here, -8 here, 4 here and 0. That is still is in an echelon form, right.

Still has the same property. So that shows that the row echelon form, okay, need not be unique. The entries are not important. It is how the entries are placed, that is important in the echelon form, right. At the bottom if there are 0's, all rows 0, should be at the bottom. The non-0 row should be coming in the increasing column order.

That is all, okay. And all should be obtained only using row, elementary row operations. That is the only condition. Those are the only tools given to you, right. Using those tools, you have to transform the given system into a new system, that is all, okay. So there is row echelon form. Let us go a step further.

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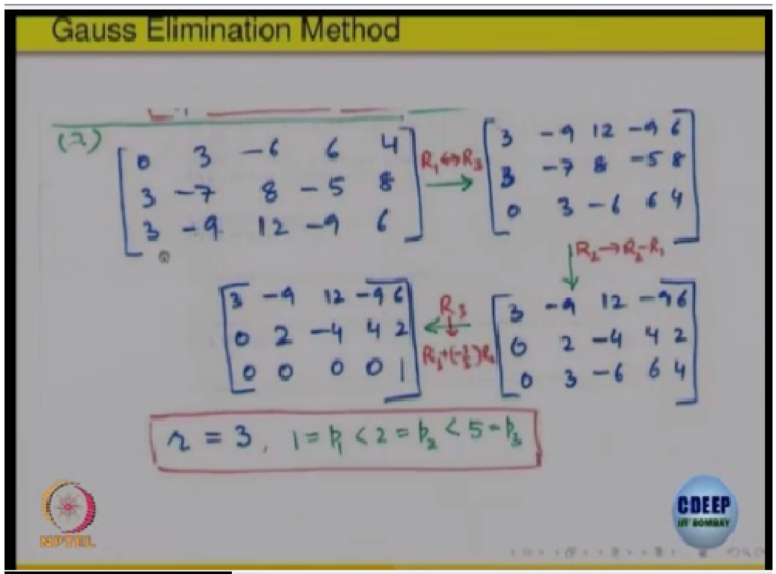

Let us look at 1 more example, okay. For example here, the first row is 0 3 -6 6 4 and so on. Now obviously I have got a 0 here, right. I would like to make bottom 0 at the bottom. So one first step should be to shift it to the bottom. That means R1 should be interchanged with R3, interchange R1 with R3. So that is written here. This arrow means interchange R1 and R3. R3 goes in place of R1, R1 goes in place of R3, you get this matrix, right.

So 0 has come here. Now if possible, I will like to bring 0's here also, right in the second. How can I bring it? There is 3 here, there is 3 here. Obvious thing is that from R2, subtract R1, right. From R2 subtract, that means R2 is going to change. R1 is not changing. R2 is going to change. R2-R1. So this will become 0. This -7-9, - becomes +, so 9-7 that is 2 and so on. So this changes. So a 0 comes here by that operation.

The idea was to make this place as 0. So the 0 is counted. So now we have gotten this stage, right. First non-0 entry coming here, okay. Non-0 entry coming on the right side, okay. But here it is coming on the same place. So I would like to make this also as 0. So how do I make this as 0? Again If I multiply by something and add here, I should get 0. So what should I multiply here by?

-3/2, add it here, then I will get 0. So this operation R3 goes to R3+-3/2*R2, so that comes to this form. And that is row thing form, right. Because there is no more rows left at the bottom. We are finished, right. So look at the example. So this is a, how many there? 1 2 3 4 5, so it is 3*5 matrix, number of rows is 3, number of columns is 5. But what is R? All rows are non-0, so R=3, right.

The number of non-0 rows are 3. All rows are non-0. In the first row, the non-0 entry is coming at first place, 1. p1 is 1, p2 is 2. And what is p3? That is the last one, that is 5. p3 is 5. So p1 p2 p3, okay. p1 at the first column, p2 at the second column, p3 at the last column, okay. These non-0 entries normally are called pivots, right.

So you can say that the row echelon form means that there is a R, so the bottom R rows are 0. In the non-0 row, the non-0 entry is pivot, right. The pivots come at in the column which are increasing order. That is all, right. You can understand that way also, right. So let us pick up one of them which is not say.

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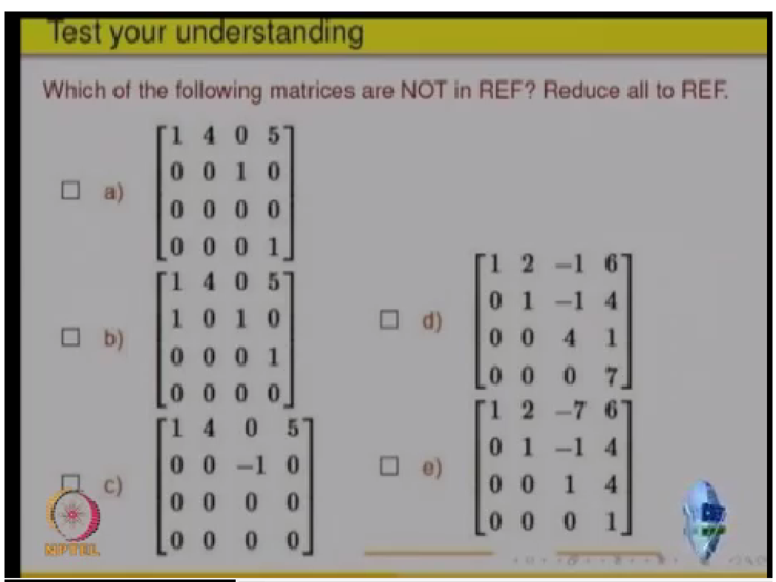

Which one? a is not, right. So if you want to make a in the row echelon form, what you should be doing? Interchange the R2 with the last one, right. R3 should go down anyway, 0 0 0. And 0 0 1, last entry has 1 should come up, right, before the R2, okay.

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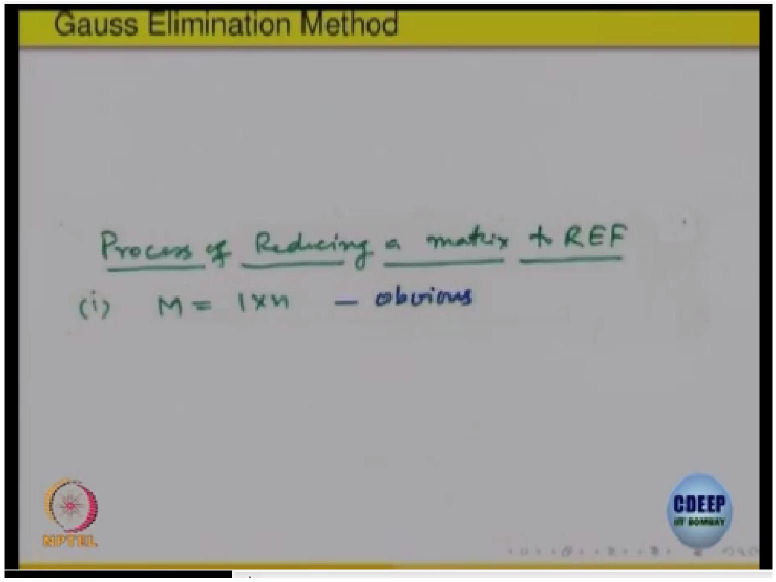

Now here is a proof of that theorem or a process also. You see the point is when we are saying it is a calculation of numbers only, the variables are not important. That means the numbers, the values of the entries of the matrix, we are going to multiply by something, add 1 row to another and do something. And here 2/3, 3/2, 4/5, you can do humanly. You can use calculators and do it.

But in applications, these things, the number of equations, you will be surprised can run into thousands. For example this weather prediction, how do they predict weather? How do they make model for economy? There are lot of equations coming. Lot of factors which are effecting the equations coming. And more often than not, to solve these equations one make them linear that is called approximation.

So you linearize the equations, the model, you have a model in which lot of equations are coming which are not linear. You linearize it. So you get an approximation kind of a thing, right. Because when a system is linear, it is easier to solve. We have methods of solving them. So how does one solve a system of linear equations is very important thing for applications point of view also.

And the idea is that even if the matrix is order or 1000*1000, 1000*2000, we can put it on a computer and we can ask the computer or a machine to do the calculations for us and make it in the row echelon form. But the question is, how does computer knows what to do, right. They only may have numbers. So that is why the matrices coming to the picture. The variables are gone. Only the entries of the matrices and that b1 b2 bm, the column that is important.

So and that we can feed a data to the computer, right. When you are saying interchange, we will see later on how those operations can be done by computers. But we have to give a definite method of doing it. See you may like to interchange R1 with R3 but somebody else may like to do it with R2 with R3 first, right.

There is no definite way of saying that okay this is the only way of changing a matrix to row echelon form. So can we give a definite way by which we will proceed so that the computer knows we know, everybody knows that this is the way we have done it. So that method is also the proof of this theorem that every matrix can be reduced to row echelon form.