Basic Linear Algebra
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Lecture - 41
Inner Product Spaces - II

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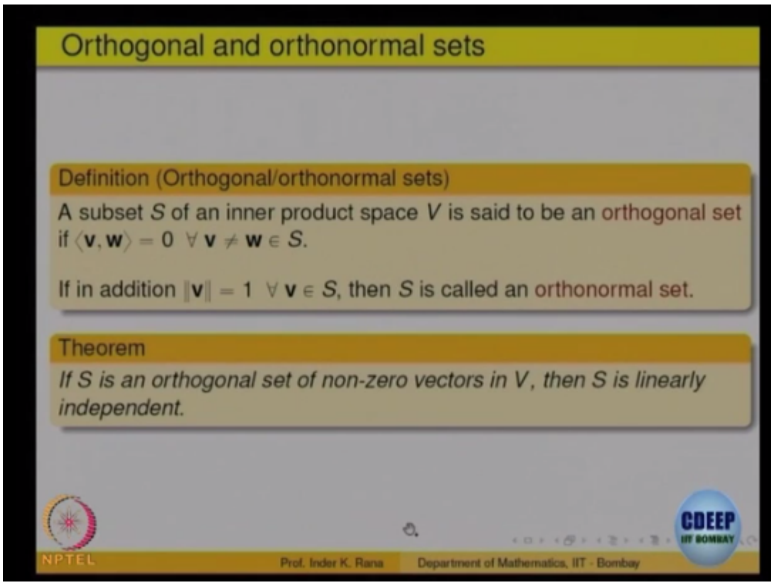

So once we have the notion of perpendicularity, we can talk about orthogonal sets. A set of vectors in an inner product space, we will call it as orthogonal, if any two of them are perpendicular to each other right, dot product is=0. Whatever be the definition of dot product, once there is a dot product right we can define perpendicularity with respect to that dot product okay.

And will say it is orthonormal if each vector has got length 1 right like infinite dimensional spaces okay. So here is same theorem works here also which same proof works that if a set of vector is orthogonal then it is also linearly independent. Same proof which was there earlier, no change at all right. So will not go to the proof again, so same proof works.

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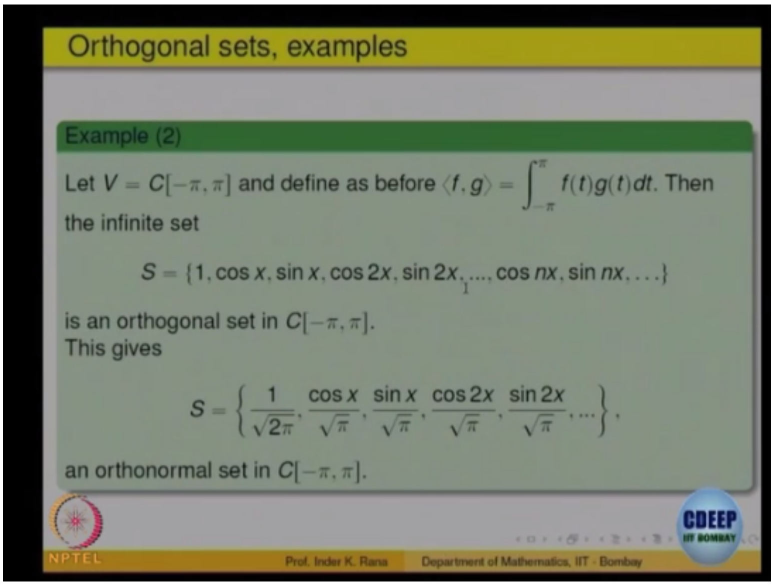

Now let us look at a particular case because you mentioned sin and cos this is nice here, look at space of continuous functions for –pi to pi. So a is –pi, b is pi okay and so what will be the dot product? –pi to pi f t g t dt right. It is okay right. On this, let us look at this set, so what is the set I am looking at? Constant function 1, function cos x with domain as –pi to pi sin x cos 2x sin 2x and so on cos nx sin nx. So 1 cos of function is cos nx and 1, 2, 3, 4, so on and similarly sin nx and the constant function 1.

So this collection of functions is a orthogonal set with respect to this inner product. Why is that orthogonal? What is the integral of cos x between –pi to pi? Where is integral of cos x? It is a periodic function right 0 right and if you take cos square right dot product you want to take, so what is the norm of this? Norm will be integral of cos square, so that will not be 0, that will be integral –pi to pi of cos square theta right.

But if you look at cos and sin, the product and integrate between –pi to pi what is that integral? Check out that is 0 right and similarly when you say integral of cos x is 0, so that is 1 times cos x integral is 0. So that we got 1 and cos x are perpendicular to each other. So nearly cos x is perpendicular to sin x right. So all so you can easily check when n is not equal to m, cos nx*sin nx right integral is=0.

So any two of them are orthogonal to each other and integral of mod of this square is not equal to 0 right. So this set is an orthogonal set. In the space of continuous functions, this is a orthogonal set. Why orthogonal? Dot product of any two of them is 0. That means what, so that means integral of take one function, take another function, that integral of the product is=0, so what will be the function look like?

It is cos nx sin nx right or sin nx and to sin m, n and m are different even then the integral is=0 if n is not equal to m right. So this is an orthogonal set and I can make it orthonormal. So how to make it orthonormal? By dividing by the norm of each one of them and integral of each one of them comes out to be equal to pi. So that is so this gives you a orthonormal set in C 0, 1 okay.

Just checking, nothing more than that and this is a very important orthonormal set. When you do some I think already if you have done something in differential equations or partial differential equations, it will come something called Fourier series right. So there what is Fourier series? That is expanding a function in terms of cos and sin. These are precisely the basis you can think of.

These are orthogonal set in terms of which you try to expand every function right. So this will come back to you when you study Fourier series problem okay. So this is important there. So important thing is this is a orthonormal set okay and this is an infinite orthonormal set that any two of them are perpendicular to each other. So this is also any two which are mutually orthogonal also linearly independent.

So this is also a linearly independent set in particular, in the space c –pi to pi. So that is an infinite dimensional vector space on which there is an inner product and there is a notion, there is example of an infinite orthonormal set okay right.

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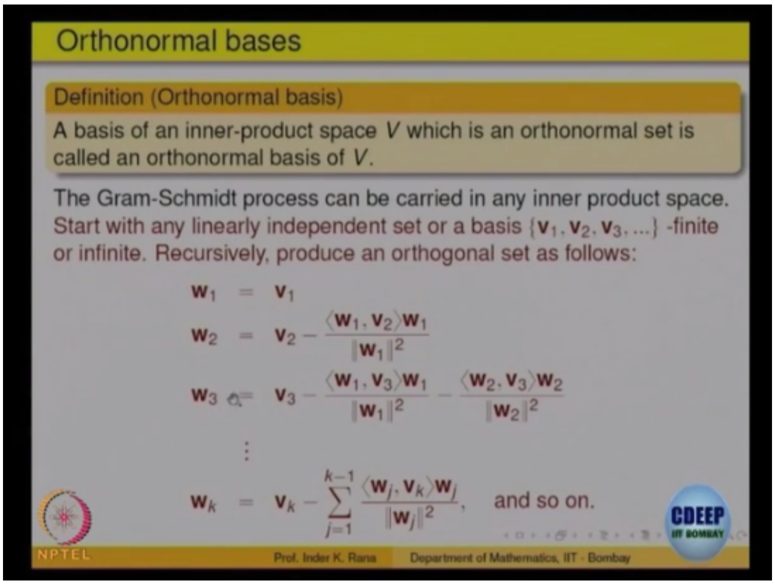

Now if you recall, we have done in a finite dimensional vector space given a basis you can make an orthonormal basis out of it right. So what all the process of converting a basis to an orthonormal basis? That was basically using what is called the Gram-Schmidt process right. So at every stage remove the projection on the previously defined, so I am just recalling, so given vectors okay.

Any finite number of vectors, so this set maybe finite or infinite but look at v1 right that is nonzero, so define w1 to be equal to v1. So what is w2? So in that process I am describing the Gram-Schmidt process again from w2, what is w2? Remove the projection of w1 on v2. So this projection you remove. So this will be perpendicular w1 and w2 will be perpendicular. What is w3?

Fromv3 the next one remove the projection on the previously defined vectors okay, the Gram-Schmidt process. So if you go on doing that process up to any k, you can remove the previous projection. So what you will get, you will get vectors w1, w2 and wk which are mutually orthogonal and the space spend by them is same as the space spend by w1, w2 up to v1 v2 up to vk right.

In finite dimensional that is what we have done. So same process we can carry over, if it is a sequence of if there is say infinite set but the problem is you can carry on given any n-1 have been defined you can define the next one okay. So we will get for any k you will get a orthonormal set which will span the same thing but I cannot say that if I define for all w1, w2, wn their space span will be same as v1, v2, vk, I do not know that, you understand what I am saying.

Given an infinite sequence up to finite stage, you can apply Gram-Schmidt process, will get a w1, space span by v1, v2, vk is same as space span by this but I cannot say space span by all of v1, v2, v infinity will be same as w1's, you do not have a process of saying that alright, up to any finite stage is okay.

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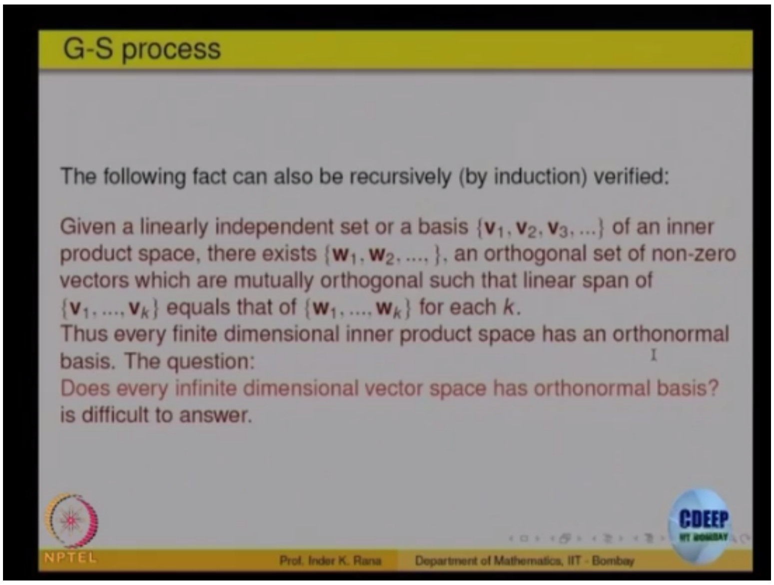

So what was the advantage of the Gram-Schmidt process? It gave you a way of finding orthonormal basis for a finite dimensional vector space. Start with a basis, convert it into an orthonormal basis. So you can say for a finite dimensional vector space right every finite dimensional inner product space has got an orthonormal basis because there will be some basis and I can convert that into orthonormal but is it same theorem true for infinite dimension?

Can you say that for an infinite dimensional inner product space there is an orthonormal basis? That is a difficult question to answer right and that leads to something called the study of spaces called Hilbert spaces. So will not go into that okay, will not bother about that, those of you go on to take some minor courses later on in mathematics in something called finite element methods are such things you will find this coming there.

So this subject this leads to a subject called study of topic called functional analysis. So C a, b is a function space right, space of functions. So what space of functions as a vector space has got a basis, notion of a magnitude and notion of an angle and existence of orthonormal basis, so such things fall under the study of a subject called functional analysis. So if you happen to take that, this will be a starting point for their okay. So will not go into that because that requires a lot of more machinery to understand okay.

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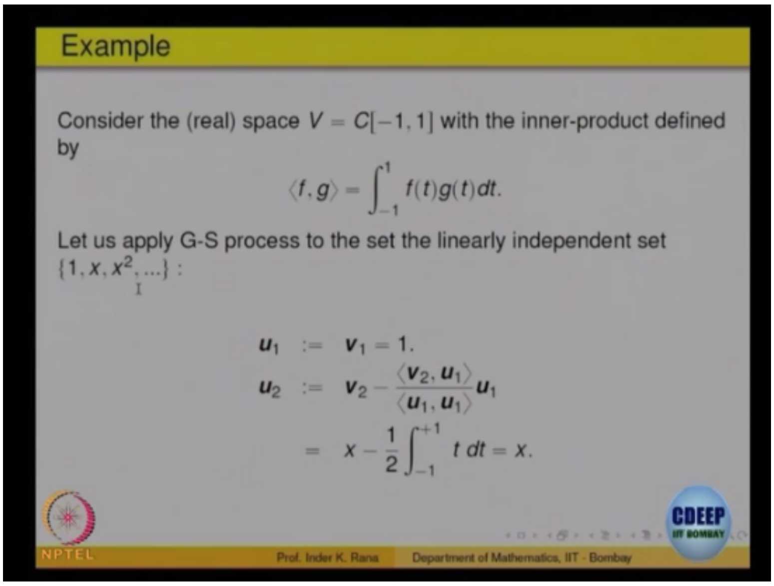

So let us do one example. The same one we have been doing, so let us take C of -1 to 1 right. That is a vector space and this is an inner product on that and we had the orthonormal basis sorry we had the linearly independent set 1, x, x square and so on remember. For any space of polynomials right 1, x, x square up to x (()) (11:05) is an independent set. So if I take this infinite sequence, any finite number of them is linearly independent.

So I can ask apply the Gram-Schmidt process to it, this set so what will be u1, that vector is same as v1 okay, that is the constant function 1, so this is 1 okay. First one is v1=w1 right. First one as it is, so what is the next one, u2 from v2 remove the projection right, so what is the dot product here? What is v1 and v2, v2, v1? Sorry, v2, u1 that is the integral of v2 u1 dx right n is 1 to 1 and this is a norm into u.

So if you compute that that comes out to be half of integral-1 to 1 dt okay and that right that integral you can compute that is obvious, so this is equal to x. So this part is 0 right. This (()) (12:11) integrally is 0, -1 to 1 the function is odd t okay, so that is 0 so it is x. So u1 is=1, u2=x, so the first two are perpendicular to each other. Gram-Schmidt does not give you anything new. Let us look at the next one x square what happens.

So what will be the next one u3, from u3 you have to remove the projections right on u1 and u2 from v3, that is x square. So let us do that.

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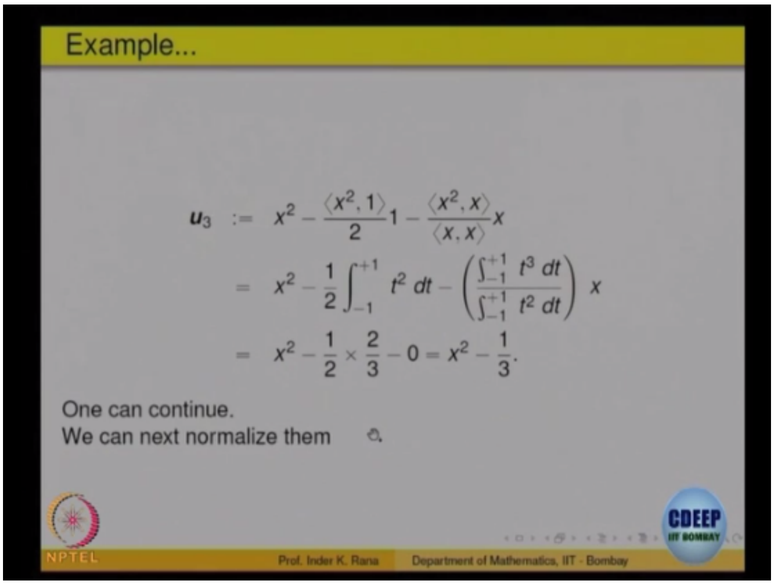

So x square remove the projection onto 1, remove the projection on to x and write everything in terms of the dot product that is integrals. So those are the integrals and you simplify, that comes out to be x square-1/3. So the Gram-Schmidt process gives you u3 to be equal to from x square you have to remove -1/3. Then, u1, u2, u3 are perpendicular to each other and you can go on doing this process for that, more integrals will be computed.

And then but these are not orthonormal, you have to orthonormalize them right. They are orthogonal only. So you can divide by the norm normalize them.

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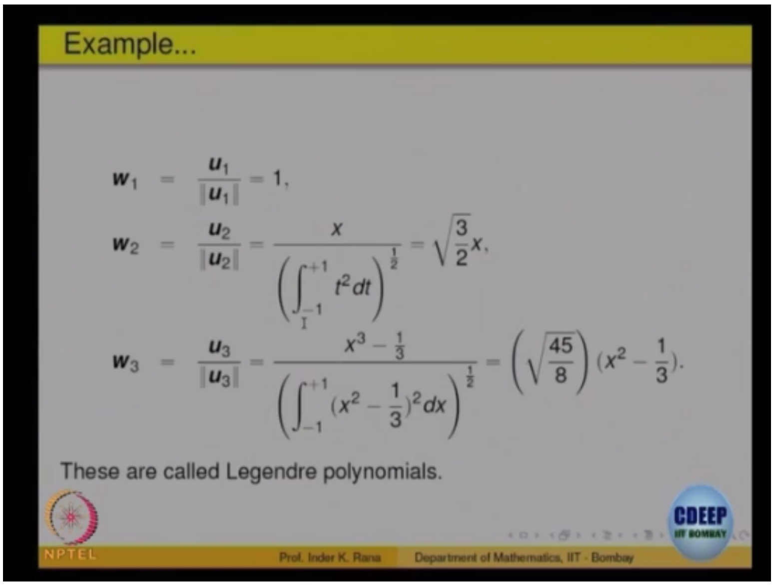

So w1 will be u1/this so that is equal to this u2 okay by the norm of that so it is square root 3/2 and similarly u3 is=this right. Just dividing by the integral of mod square normalize. So what we are doing is we are starting with that polynomials 1, x, x square, x cube and so on and Gram-Schmidt process we are orthonormalizing them. We get another set of polynomials.

Those polynomials are perpendicular to each other and has got norm=1. They are very special polynomials. They will come back to you some time again in differential equations, most probably called Legendre polynomials. So this is a construction of Legendre polynomials. How they are constructed? You take the standard polynomials 1, x, x square, x cube and so on and apply Gram-Schmidt process and orthonormalize that right. So that is useful there.

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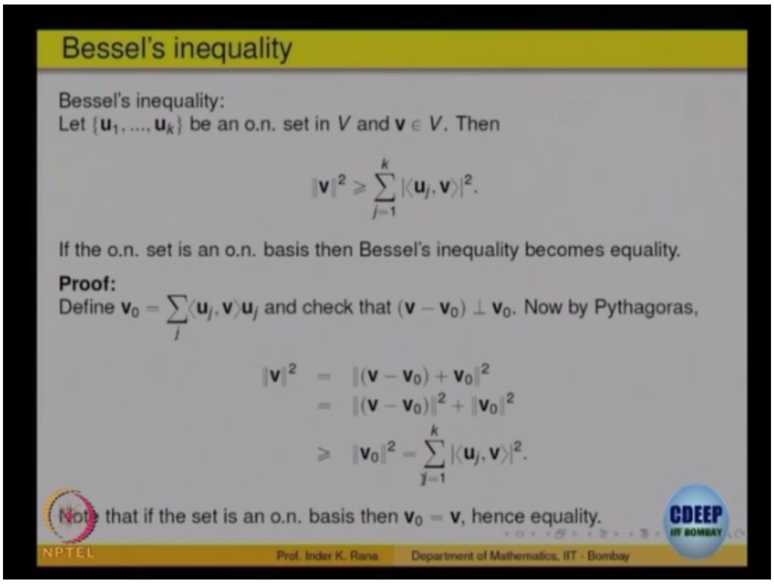

Now we can prove Bessel's inequality but that is same as what we did for finite dimensional spaces because that is only valid for 1 to k right but you can generalize this okay. So let us keep it finite only. So finite same proof works, only thing is there is a notion of dot product now, so same properties you can use and prove Bessel's inequality. So basically what we are saying is you can extend the notion of inner product from on any vector space.

So what is the definition taken? The definition is you take the properties of a dot product in Rn and transport it as a definition on any vector space right. So that is one, so once it has a notion of dot product, it gives you the notion of perpendicularity. Once there is a notion of perpendicularity, you can talk about orthonormalization process, Gram-Schmidt process.

But that works only for finite number of them that does not work for any induction, you can stop, go up to any stage, any n but you cannot just say that will get a sequence w1, w2, wn dot infinite sequence you will get. That will be orthogonal also right, any finite number of any two of them are perpendicular to each other but whether the space spent by this is same as the space span by up to finite is okay, v1, v2, vn and w1, w2, wn will give the same vector space, span will be same.

But all of them infinite sequence will give the same space that is not true actually, we need something more okay. So that is the study of other space. Now I think I will just probably say something, what we do in finite dimensional spaces? We looked at what are called maps which are isometries. Once there is the notion of inner product, you can define the notion of an isometry right.

What is isometry? If you take v and w, t is a map between two vector spaces or inner product spaces and that if you take inner product of v and w that is same as inner product of tv and tw right. So they preserve the notion of angle and distances, same you can define for infinite dimensional kind of things but analysis of them becomes much more difficult. For finite dimension, we proved what is called the spectral theorem right.

Corresponding thing is true but one has to do a lot of work and that again goes to the field of functional analysis. We looked at what are called orthogonal transformations between vector spaces and so on. So again I am just saying this is a beginning of another topic called functional analysis right, so which some of you will come across probably. So we will stop here with the course okay.

Northing more in the course because all mostly we have dealt with finite dimensional spaces only right and their properties, so with that we end the course.