Basic Linear Algebra
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Lecture – 39
Abstract Vector Spaces III

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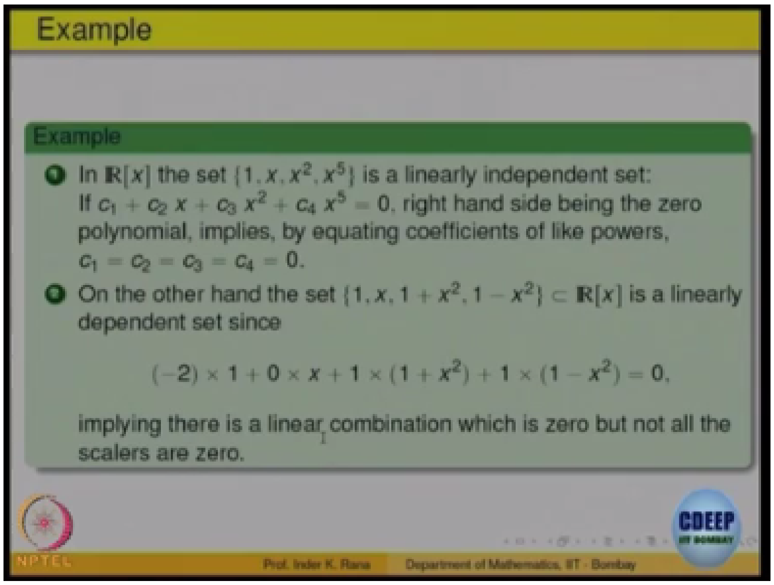

In this example, it is 1, x, x2, x5, if we just now what we worked out for was R5, right, polynomials of degree < or = 5; 3, same proof will work for 1, x, x square, x cube, x4, x5 also, right, so if I take all the powers 1 , x, x square, x cube, x4, x5, they are linearly independent, this is only a subset of it; this is only a subset, so this is general fact; if you are given a set which is linearly independent, every subset has to be linearly independent, obviously, right.

And if a set is linearly dependent, then a bigger set will be linearly dependent, right because there is a linear combination is to 0 here, so you can make, think it as a linear combination in the bigger set also, right, so anyway let us check it again; so, if this is = 0, right, c1 times 1 + c2 times x + c3 times x square + c4 times x5 = 0, same proof I am repeating. If this is = 0 means what; these are zero polynomial on the right hand side.

So, like powers, coefficient must be equal and that means what; c1 must be 0, c2 must be 0, c3 must be 0 and c4 must be 0, same reason, right. Let us look at this, 1; so I am looking at all polynomials now, I am looking at the polynomial 1, x, 1 + x square and 1 – x square, it is a linearly dependent set, why dependent; if I want to show it is dependent, what should I do so? Some linear combination is 0, where not all the coefficients are; so here is one produce -2 times 1 + 0 times x + 1 times 1 + x square + 1 times 1 – x square = 0.

And obviously here 1 is coming, - 2 is coming that is not 0, so that is a linearly dependent set, okay, right, so that is how we will check something is independent or dependent in an abstract vector space.

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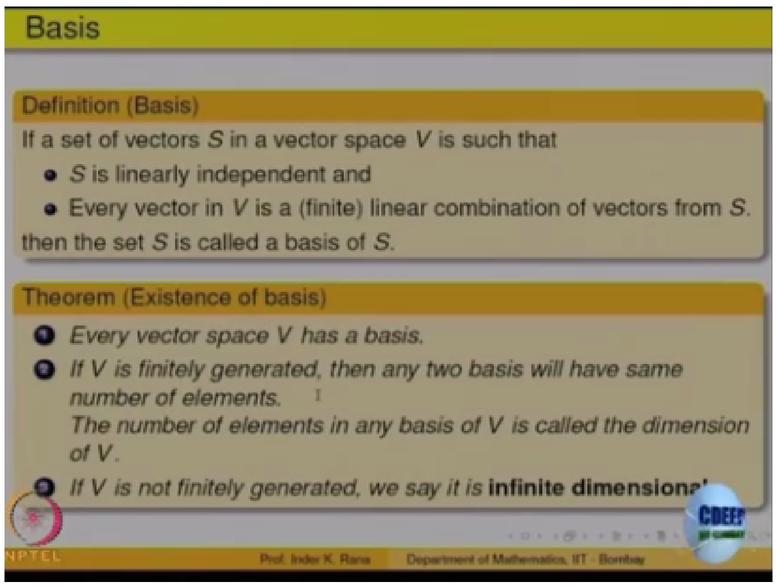

Now, we have defined independence, we have defined what is generation, so something which is independent and generates, we call that as a basis of that vector space. So, what is the basis? It is a linearly independent set and every vector, right, V is a linear combination of course finite only, right, so S may not be finite but it is a linearly independent set, okay, and set in vector V is a linear combination of some finite number of; the finite number may change, right.

So, but S may not be finite, keep in mind, now things are becoming different from the normal; in normal and here when we had sub spaces of Rn, everything was finite, right, so no problem. Here what we are saying is a subset S which may or may not be finite is called the basis, if it is linearly independent; one and every vector is a linear combination of elements of S, okay. So, here is the theorem, which will not be proved it that every vector space has a basis.

For a finite, right in Rn, we know every subspace will be having at the most n vectors, right, so there are generators; maximum number is n, so how do you get a basis there? They remove which are; you have got a set which is generating, so remove those elements which can be obtained by being the combination of the remaining, throw that out, throw out, whatever is left in the end that will be independent and generating, so that will give you a basis.

Or what you can do is; start with the ones are non-zero element and go on adding something more, right so that they become; remain independent and generates everything, so that is another way of getting (()) (04:50) but here then it is say in Rx or vector spaces, which are infinite dimensional, oh sorry, I am not defined dimensional, so now let us now go into that which are not finitely generated, how do show the basis of this, right.

So, there vector spaces is not given, basis is not given, nothing is given, so how to show, you do not know what that addition is; what is the multiplication and so on, a general theorem. Now, you see the abstraction coming in, so one proves the theorem that whatever be the vector space, whatever be V, whatever be addition, scalar multiplication, you can always find a basis, so basis; the idea is or say of finitely generated.

If this vector space is 1 0 that means there is some element in it which is other than 0, we have only 0, then nothing to show anywhere, is zero dimension you can; zero element is there, if there is 1 non zero element in that whatever it may be, start with that so, you have got one set right, which is linearly independent but it may not generate everything, so you want to enlarge it, add one more but how long you can go on adding?

If it is not finitely generated probably, you will go on doing it, when do you stop, how do you stop? So, the concept of infinite comes into picture, right, so it relates to some basic things in set theory which will I cannot go into it. So, this is the theorem which relates with some basic concepts in set theory, okay, so this theorem can be proved that every vector space but again, there are issues there because one uses some examine set theory called well ordering principle, which some mathematicians, some mathematicians do not accept, this is exam.

If you assume that exam, you can prove this theorem, if you do not assume, you cannot prove or disprove this theorem, so there is something new coming to you in mathematics that there are theorems which are dependent on what examine set theory you are starting, this is like a game, mathematics is like a game being played, it is like a football game or cricket game, right, different set of rules, something else may come out.

So, there are various formulations exams of set theory in which what is something called well ordering principle is not a part of standard set theory, okay, 90% of the mathematicians are assumed that exam and go ahead, you can prove these theorem, for others unless you construct 1, right, a set which is generating it; I would not believe it kind of thing, right, so anyway, so we will assume it that every vector space has a basis.

Once that is done, if it is finite, so there are 2 possibilities, it is finitely generated, it is infinitely generated. If it is finitely generated, then one can prove with your that any 2 basis will have the same number of elements, if it is finitely generated, right that means, there is a finite set S which will generate, which is linearly independent and will generate, right, so finitely generated implies that any 2 basis will has the same number of elements.

And that is called the dimension of that finitely generated vector space, if it is infinitely generated, we say it has an infinite dimension; it is infinite dimensional vector space. So, in vector spaces, there are finite dimensional, there are infinite dimensional, right, so let us; because this is the here itself, let us show that Rx okay, so let us look at examples.

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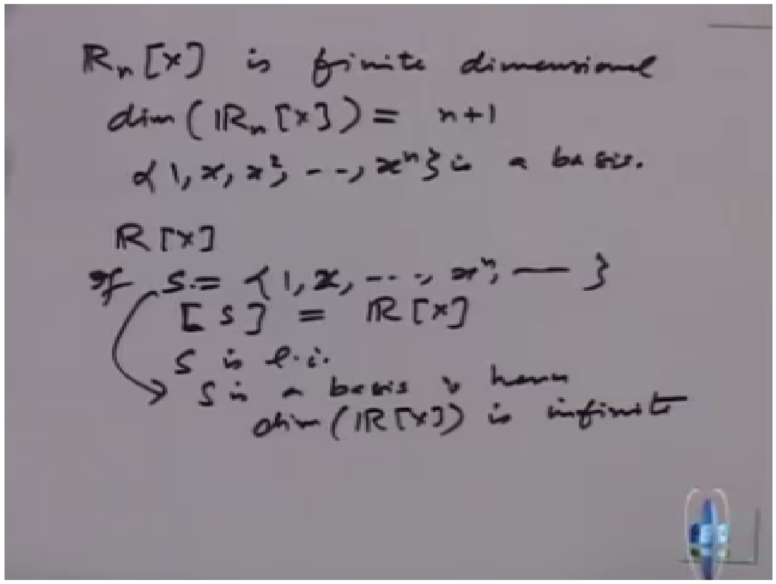

So, Rn X is finite dimensional, dimension of this; what is the dimension of this? You got a generator, right for R3, what was generating set; 1, x, x square, x cube, how many are there; 4 of them and they were linearly independent, so polynomials of degree < or = 3 is the finitely generated vector space of dimension 4, so this is of dimension n + 1, so what will be a basis? 1, x, x square, x to the power n is a basis, right.

Let us look at RX for this; this was the generating set, right, so if this is so then LS, oh this span of S = RX and we showed S is linearly independent also, just now we showed, right, which is linearly independent, is it okay for everybody it is linearly independent? Yes, because if I take a linear combination, right, just now we showed; alpha 1 p1 + alpha 2 pk = 0, a left hand side will be a polynomial which is 0, so all scalars must be = 0, okay, so this is also a basis, so this S is the basis.

And hence the dimension of RX is infinite, so that is a infinite dimension vector space, is it okay because we have got a set of generators which is not finite and they are linearly independent in generate also, so something that generates and is linearly independent is a basis, so this is an infinite; as an infinite set which is linearly independent and it generates the whole of RX, so it is the basis.

And hence, dimension of RX is infinite, you can call it infinite; infinite is not; what is infinite; again a question, its dimension is 3, 4, 5, 10, 1 million, what do you mean by saying dimension equal infinite? Dimension is the number essentially, you can call, on the right hand side, you are writing the word infinite equal, so you cannot write the word equal to infinity because infinity is not defined as the number as such.

So, it is better to so say it is infinitely generating, right or it is infinite dimensional whichever way you want to call it, okay.

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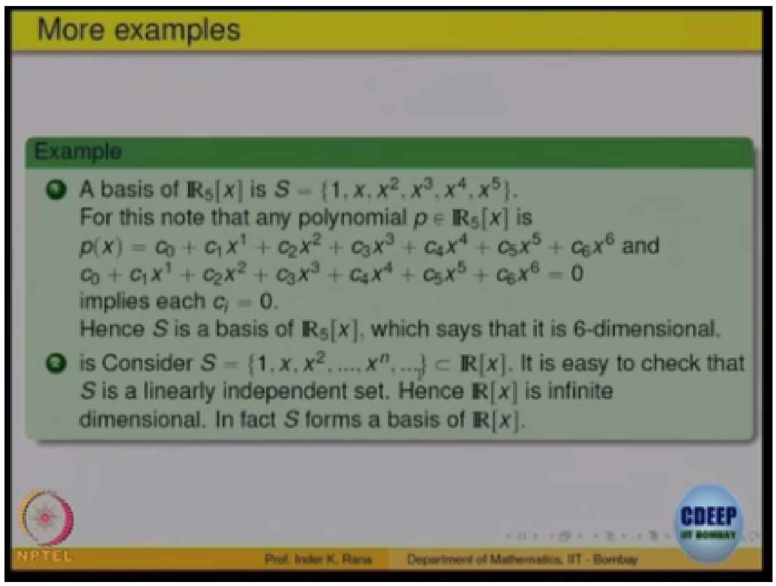

So, this is polynomial degree < or = 5, it is; right, so its dimension = 6, right, 6 elements constant okay and RX itself is infinite dimensional, right that set forms the basis for a; okay, so dimension is clear for a vector space, okay.

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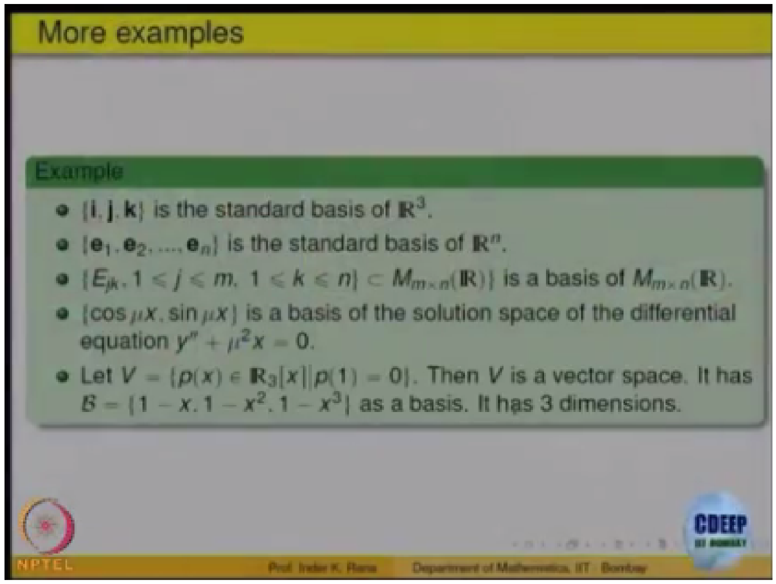

More examples; we have all come across this RF3, there are finitely generated i, j, k, standard basis, 1 0 0 0 1 0 1 0 0 1, in Rn the same thing, the same thing you can write as such, e1 is 1 0 0, e2 is 0 1 0 0 and so on that is the standard basis we know. For matrices, Ejk; what is Ejk? In that matrix, jkth element is 1; everything else is 0, right. So, how many such matrices are there? N cross n, right they are linearly independent and generate everything.

In the basical saying it is nothing but R to the power mn, right centre basis of that, written as matrices, nothing more than that. Let us look at this thing will come back to in differential equations just want to; if you look at this equation; y double dah + Mu square X = 0, is given that cos Mu X and sin Mu x form a basis of the solution space. What is solution mean? If I y = cos Mu X or y = Mu; sin Mu X as a functions.

Then they will satisfy this equation, right that is okay, you can just check, okay because when we differentiate, Mu will come out, it will become sin, once again differentiate, it become sine again and so on, so they form one will be + sign and other will be giving a – sign, so that is solution there, both are solutions, right, so and these homogeneous, right hand side, 0, so we just know seen that all; we have 2 solution are there; their linear combination also is there, solution.

So, look at the space generated by these 2 elements, all linear combinations that is the solution space for this, okay, right that is finite dimensional, so every solution is obtained as cos and Mu, linear combination of them, are they independent? That is the only question then they will form a basis then, form a basis if they are independent, so that means what? If I take alpha times cos Mu X + beta times sin Mu X = 0 that should say alpha = 0 and beta = 0, does that happen?

Because again, when you say right hand side = 0 that is the function, right that means, this equation =; this is, left hand side is a function which is = 0, for every value of X that is how you should understand that. If it is true for every value of X, I can put any particular value, so X = 0, so alpha cos, Mu X + beta Mu; beta sin Mu X = 0, for X = 0, what does it give me? Alpha cos 0 = 0, right because sin 0 is 0.

So, what does that mean? Alpha = 0, right, cos 0 is 1, alpha times 1 + 0 = 0, so alpha = 1, similarly, you can take the value say = pi/2, right, then cos 0; that term vanishes, you will get only beta times sin, right or pi/2 / Mu value you can change, right, it is not so important, so sin can be made as 1 for particular value, cos can be made as 0 and other way round, so beta will be 0, so they are linearly independent.

So, this will come across in when you studying solutions of a differential equations; this will come back to you, how do you describe all possible solutions of a differential equation; you will look at the homogenous point of it, you will find basis for homogenous part will give you a vector space and you want to generate the vector space by describing the minimum number of independent solutions, so this kind of things will come there, okay.

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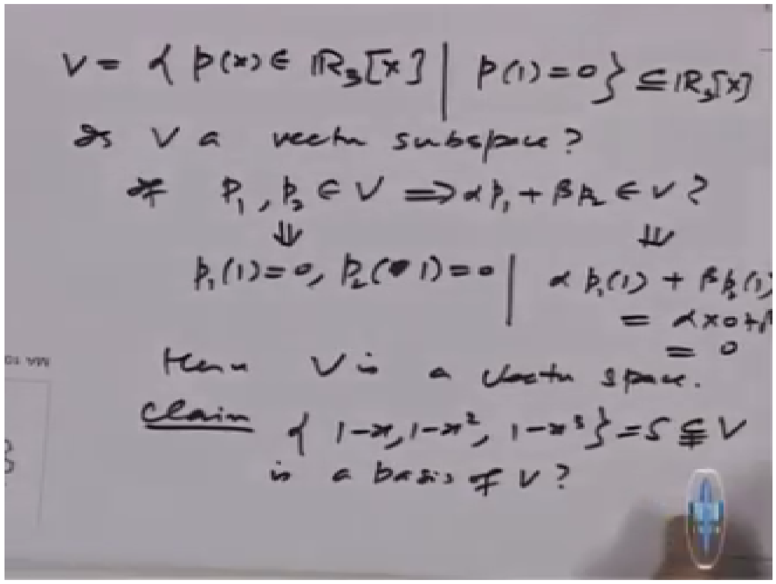

Let us look at one more example, let us look at all polynomials of degree < or = to 3 with p1 = 0, let us discuss this, so I am looking at this space is V, all polynomials, px belonging to R3 such that P of 1, the value of 1 is 0, first of all is V; so this is a subset of R3X, right, so is V a vector space or vector subspace, first of all you have to check whether this is subset or not that means what?

That means if pa1 and p2 belong to V, does this imply alpha times p1 + beta times p2 belonging to; right if that is the case then okay, so what does this mean? That means p1 of 1 = 0, p2 of 1 = 0 that is given to me, what does this mean if I have to check? That means, alpha p1 of 1 + beta p2 of 1 that should be = 0, what is this =? This is 0, this is 0, alpha times 0 + beta times 0 that is = 0, right.

So, hence V is a vector space, in itself it is a vector space, it is a subspace of it, for the claim is; look at the set 1 – x, 1 – x square and 1 – x cube that is a set as is a subset of V, of course, !=, right, claim that this is a basis; this is the basis of V. so, what is to be shown? First of all I do not know whether it is a subset or not, I should verify that also, right, so let us look at. In this if I put x = 1, this is 0, so p1 is 0, p1 of 1 is 0, p2 of 1 that is 0, p3 of 1 that also = 0.

So, p1, p2, p3; these 3 polynomials are; right in V, so that is not a problem, so it is a subset, so claim is that is a basis. So, what is the first thing we should check, what are the 2 things to be check?

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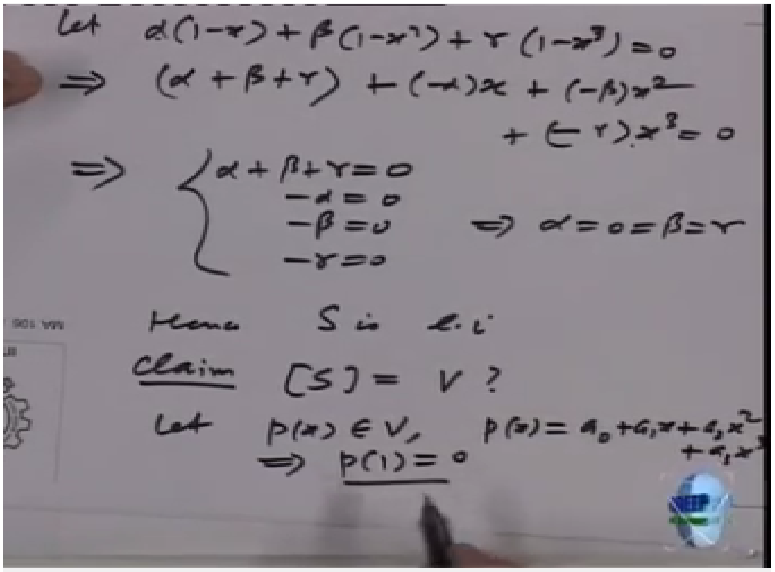

One; this set is has linearly independent and second; it generates V, these 2 things to be checked, so let us check one by one both of them, so independence, so let alpha times 1 – x + beta times 1 – x square + gamma times 1 – x cube = 0, right, let us say that = 0, we have to show that alpha = 0, beta = 0 and gamma = 0, so what does this mean? Implies so, this will give me alpha + beta + gamma that is the constant part of the left hand side + this is -, so – alpha x, there is no x here, okay + - beta times x square – gamma times x cube = 0.

What I have done; I just express left hand side as a polynomial nothing more than because one polynomial = 0 polynomial, I have to see what are the coefficients of the left hand side, so what does that imply? Alpha + beta + gamma = 0, coefficient of x – alpha = 0, coefficient of x square – beta = 0, coefficient of gamma – gamma = 0, right, comparing coefficients of like terms, so implies alpha = 0 = beta = gamma, is okay, just using the fact that one polynomial = 0 means, coefficient of each term must be each power must be 0.

So that so hence, S is linearly independent, what is the next thing to show? It generates, so claim S = V, so let us take a polynomial, let px belong to V, implies p of 1 = 0, right and let us write Px is the polynomial, so it will be of the form a0 + a1x + b is of cubic, right, so a2 x square + a3 x cube, is that okay, is some polynomial with some coefficients a0, we do not know what they are, we have to find them and express this with this property.

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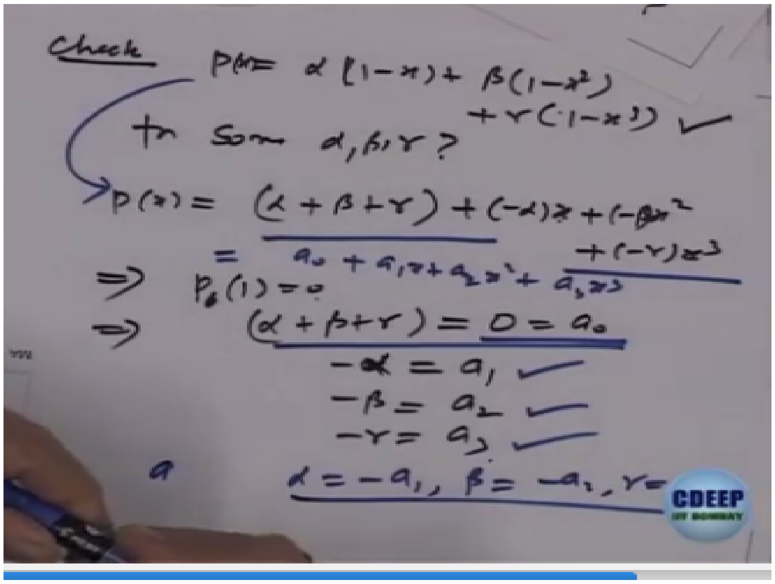

I should check, check p = alpha times 1 – x, so p of x, 1 - x + beta times 1 x square + gamma tines for some alpha, beta and gamma, we do not know what are they, every polynomial in V should be express as a linear combination, I have to find a possible alpha, beta and gamma, right so what does it mean? If I have to find this; when I find alpha, beta and gamma, what is the right hand side, so what do I want; px = alpha + beta + gamma.

I am rewriting that thing again, - alpha; + - alpha x, so what is the next one? + - beta x square + - gamma x cube that is px, right, is it okay, if; I have to find alpha, beta and gamma, I do not know what they are get but if this is so, this must be equal to this, so that means what, now, what is the property of p given to me? P was in V, so p of 1 of V is 0, so p1; p of 1 = 0, what does it imply? Alpha + beta = gamma = 0, what a coefficient of x here? -x; - alpha.

And what is the coefficient of x in p, right, sorry, that should be 0, p1 of V that means a0 should be =; say p1 of; p of 1 is 0, this goes off a0 = 0, so this is = a0, is it okay, yes, where the property, what is the coefficient of x here; a1, what is the coefficient of here; - alpha = a1, – beta = a2, - gamma = a3, is that okay. See, if px has to be a linear combination, then this must be true, right, so when we write this as a polynomial, this is nothing but px = this polynomial but this polynomial is also equal to; we have assumed is a0 + a1x + a2 of x square + a3 of x cube, right.

So, coefficients of like powers must be equal, so alpha + beta + gamma should be a0 but the property is p of 1 = 0, so I get this equation, coefficient of x – alpha, what is the coefficient of x here? a1, so a1 must be = this, a2 = this, this is a3, so that means what I have found px; for px, I have found alpha, beta and gamma in terms of; see, p is given to me right, the polynomial is given to me that means, a0; a1, a2, a3 are given to me.

I have to find alpha, beta and gamma in terms of a0, a1, a2 and a3 that also problem, right, then it will be a linear combination, so what I found is a0 = 0, okay, a1 = - alpha, a2 = -beta and a3 that means alpha = -a1, beta = -a2 and gamma = - a3, right, with that so a0 = 0, anyway that is 0, so that gives me the values. So, what I have proved is; that let us come back to the example, so this vector space, V1 = 0 is a subspace; one and second; these elements belong to V, this is a subset of V, this is linearly independent and every element is a linear combination of that right.

And all this is becoming possible to check because everything is finite dimensional, so we are able to bring everything to a system of equations basically, right. For example, here, it can be 3 equations because of variable, right, you can solve them, so that is the advantage of finite dimensional vector spaces, you can do computations with them, okay.

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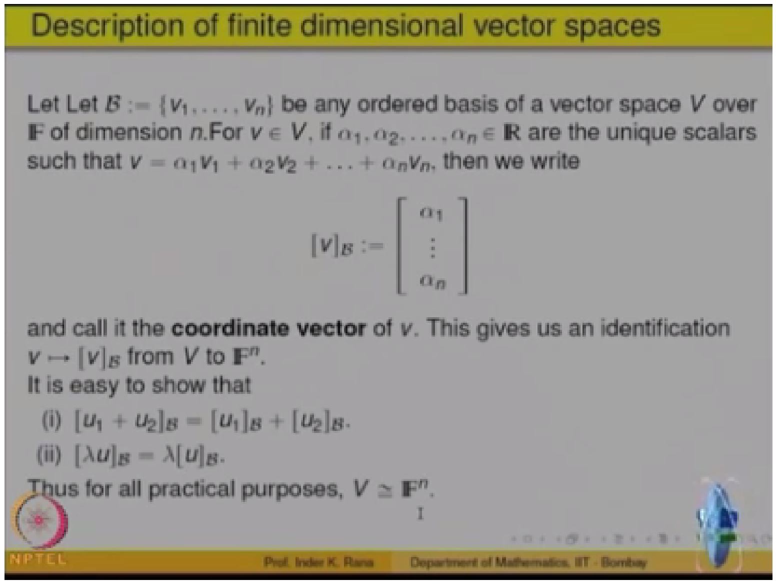

So, let us just; for every vector in finite dimension; what is the advantage of finite dimension, we will close there, that means what; there is a ordered basis, finite dimension means that is the finite basis for them, right. Even a basis, every element will have a combination, right, every vector v is a linear combination of them, so alpha 1, alpha 2 and alpha n are known that means this vector, which is in V which is a abstract vector space, I do not know what is that it will be a polynomial, it will be a functions or anything.

But if it is finite dimensional, I can associate with that vector, a vector in Rn, if there is an ordered basis n vectors in that right, is that clear. For example, Rn the polynomials, right of degree n, what are the basis? 1, x, x to the power n, so for a polynomial p with those coefficient, what is the corresponding vector; a0, a1, a2 and an, it is the vector of length and +1, column vector; door numbers.

Doubt as a polynomial, you get a vector associated with which is the number, right, which is vector, numbers are there, so that is under one type, this is what is called as a coordinate of that vector, this will be abstract and this can be put on a computer, so computation becomes possible and this association that means for every vector V in the vector space were associating a vector in Rn, if the basis is Rn, right, there is n value forms in the basis.

And this association is very nice, it preserves everything that means if I take 2 vectors and take the coordinates that is same as coordinates of the individual and added and coordinates of the scalar multiple is that that means what? Essentially, it says for all practical purpose is your vector space V which is finite dimensional of the dimension n, can be recognised as F to the power n, this is the complex c to the power n, if real R to the power n.

So that is why finite dimensional abstract vector spaces are not difficult to handle because you can transform everything to Rn or cn by are this association, by picking a basis which is ordered basis and writing the coordinates of that. What becomes difficulties, when they are infinite dimensional right, okay, after defining vector spaces, what did we do; after I will dimension, basis and everything, next thing was looking at maps between vector spaces, transformations, right.

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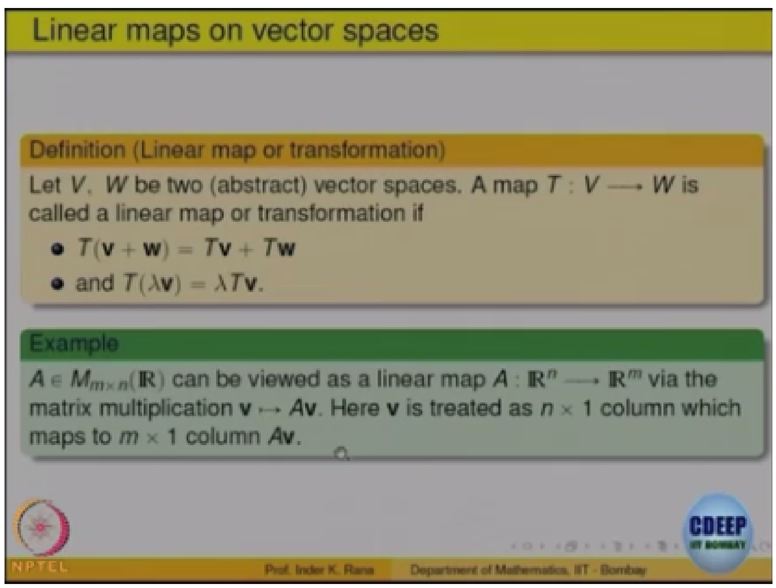

So, let us do that thing now, so let us look at 2 abstract vector spaces; V and W, there will be arbitrary vector spaces, right then a map between V and W is called a linear map, if it takes a linear structure here to the linear structure there that means T of; if you add take addition in the vector space v and take their images that gives you the image as some addition there in W, so this addition is in W, this addition is in V, T of v + w = Tv + Tw and similarly, scalar multiple of that.

So, it preserves the linear structure that is all essentially, we are saying, right, it gives to you regard to the linear structure, addition goes to addition of the images, scalar multiple goes to the scalar multiple of the image and finite dimension we have seen everything was collateral with a matrix, right, the matrix multiplication gave you a linear map and the same is possible here.

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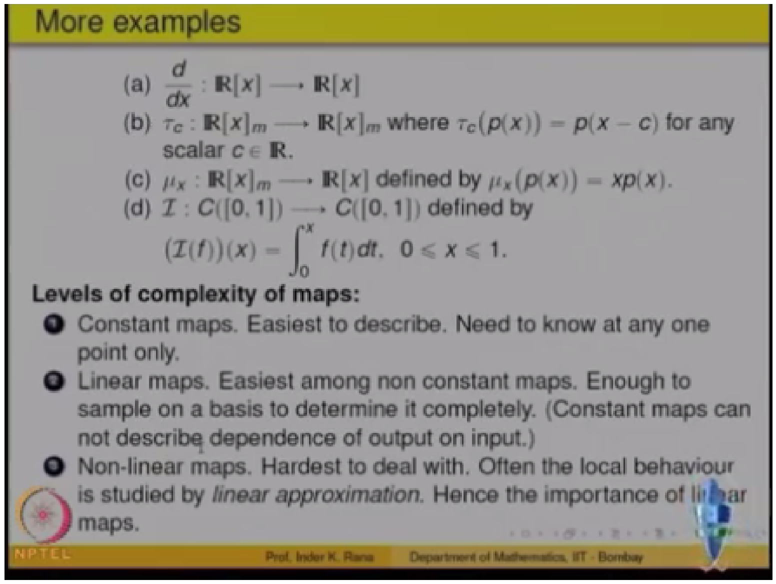

But let us give an examples first, look at all polynomial and look at the derivative, take a matrix p, take a linear polynomial p and look at its derivation, so is a map, e goes to p dash, it is the map from polynomials to polynomials, it is a map from Rx to Rx, right, derivative of polynomial is again a polynomial yes, calculus, okay, so it is a map from polynomials to polynomials and it is a linear map, what is the derivative of p + q?

Your addition formula of derivatives says it is a derivative of p + derivative of q, what is the derivative of alpha times p? again, alpha comes out is derivative of p, so differentiation is a linear operation, be it polynomials or be it, you can take a space of all differentiable functions that also is a vector space and that also it will be a linear map, okay, so derivative is a linear map and that is why they become important.

Look at the polynomial when map, T of c or tau of c that is p of; you take x and translate x by c, scalar, okay, in that way you translate and take the image that is a map, is it linear or not? So, what are you doing? P is the polynomial, p goes to q, what is q of x; q of x is p of x – c, right. If you take 2 polynomials; p1 and p2, add them, p1 + p2, what is the image of p1 that is q1 that is p of; p1 of x – c, what is the image of p2; that is q2 that is p of; p2 of x – c, what is the addition?

That is same as q1 + q2, so that is the linear map, translation in the domain is a linear map, okay translation in the range that is not linear, it is not mentioned I think, now, we do not have that. What about multiplying by x? take a polynomial px and multiply by x, is that a linear operation; x times p1 + p2 is same as x times p1 + x times p2, right and scalar also comes out, so that is linear, okay.

This is another important linear operation that is integration, like differentiation, integral of F + g is the integral of F + integral of g, integral of alpha F, alpha comes out, right, so differentiation and integration both are linear operations on respective spaces, here the functions; what are the functions; which should be integrable, right then only it make sense function. So, for example, you can take continuous function, all continuous functions are integrable, so it is a linear map.

And these are the why; these are the maps, the linear and the swipe, they become important in higher mathematics, okay. Now, just wanted to say a few things; constant maps are the easiest ones to integrate, nothing changes, everything goes to a constant. Next are what are called; the next map what are the easiest ones; constant + x linear maps, so linear maps are the next which are easiest to handle.

And the next one are the non-linear ones which are most difficult to handle, so most of the mathematics; applied mathematics is going from non-linear to linear, you make approximations, try to convert whatever is non-linearity approximated by linearity, all of your numerical analysis will be that only, mostly, numerical linear algebra, numerical analysis all will be trying to convert everything to linearity because linearity is easier to handle and apply.

For example, if you take derivative, right, the curve is there and derivative is linear approximation nearby it will becomes linear that is why linear approximation, tangential line approximation has become important in calculus and they are the ones which are used in your calculators. How do you think punch a key on the calculator, what is the value of sin 7.5, how does that give you immediately some value?

What it does that the background, there is an algorithm, we says replace this by tangent line approximation linear and calculate and give me the value, so all algorithms, most of them are based on yet to be slightly better, quadratic operation and so on but linear is the simplest one, so these are linear maps and of course, if it is a finite dimensional vector spaces, right and every linear map will give rise to a matrix, ordered basis, like we have done it earlier, same, right, so they are easier to handle.

So, we will stop here today. Next time, what we will do is; see how the notion of (()) (37:24) right is taken over to an abstract vector spaces, perpendicularity, okay.