Basic Linear Algebra
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Lecture - 37
Abstract Vector Spaces - I

Right so let us begin with today's topic. So this is going to be abstraction of linear algebra. So the concepts that we have studied namely Rn's and so on, they are going to be at abstract now. The reason being these concepts are used in other branches of mathematics as well as other branches in engineering and sciences. So let us begin with structure called vector space.

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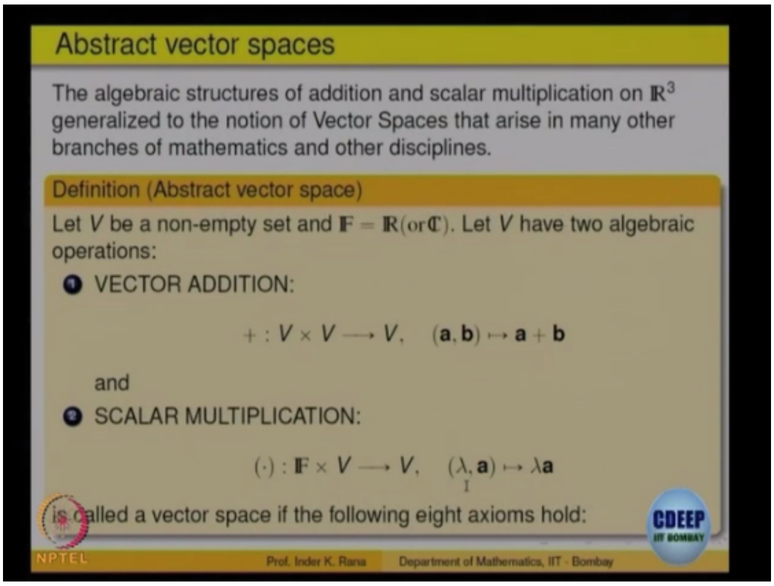

So the prototype for a vector space is Rn. So where there are vectors, you can add vectors, you can scalar multiply vectors with various properties. So a vector space V is a non-empty set and will denote by F the real numbers or the complex numbers. So the set, on the set V there are two algebraic operations defined. One is called addition, so for elements a and b, so we look at V x V.

So an element in V x V in ordered pair so a, b that goes to an element in V which is denoted by a+b. So using the binary operation of addition defined on the set V and there is an operation called scalar multiplication which is a function form the underlying field F which is R or C x V to V. So given a scalar and given a vector, we will call it as a vector alpha, a. It gives you something called the product alpha, a sorry lambda, a.

So these are two operations defined on vector space, one is addition, other is we call it as scalar multiplication.

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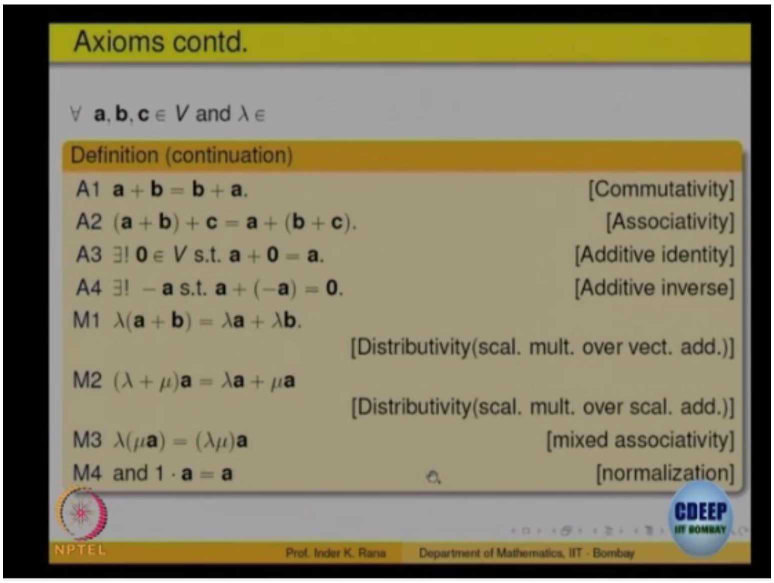

These are properties that the vectors in R3, R4, Rn have, addition is commutative, it is associative, there is a say weak element 0 in V a zero vector such that a+0 is a and that is same as 0+a because of commutativity and for every a there is this negative of a, so that a+-a is=0 and scalar multiplication distributes over addition so lambda scalar multiplied a+b is same as lambda a+lambda b.

And similarly if you add two scalars and multiply with a then this lambda a+mu of a. So this is regarding addition and here is there is multiplication of scalars. So that says lambda multiplied with mu of a is same as lambda mu multiplied first and multiplied with a. So normal properties that we know of scalar multiplication in R3, so these are made as properties defining properties of scalar multiplication and the abstract vector space.

And finally 1 times a the scalar 1 in the field is=a, so that is a normalization one calls.

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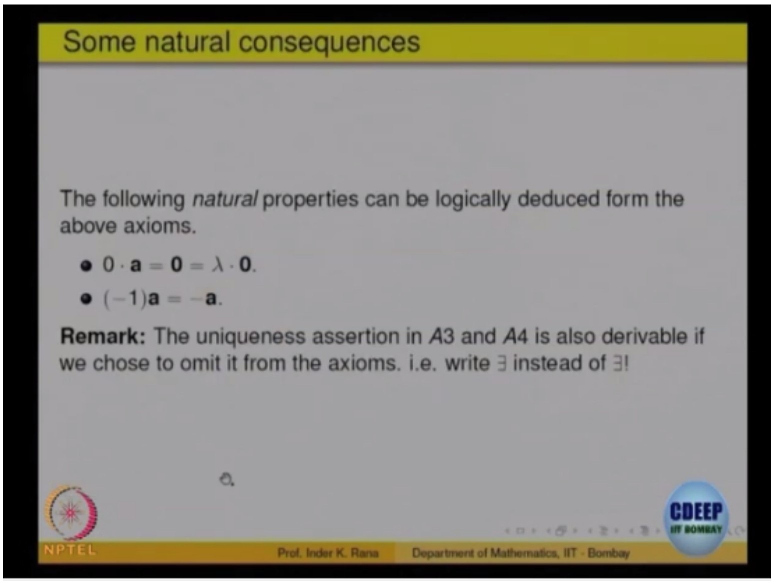

We have said here that these are unique 0 and –a but those can be proved using the properties and one can also prove obvious facts that 0 times a is 0 is lambda times 0 and -1 multiplied by so this is additive inverse in the field and that is same as – of a. So these are some obvious properties that one can prove using these axioms itself. So we will not prove that. We will just assume because we do not have that much time.

But it is a nice thing to reduce these properties from the basic axioms of addition and distributive properties right.

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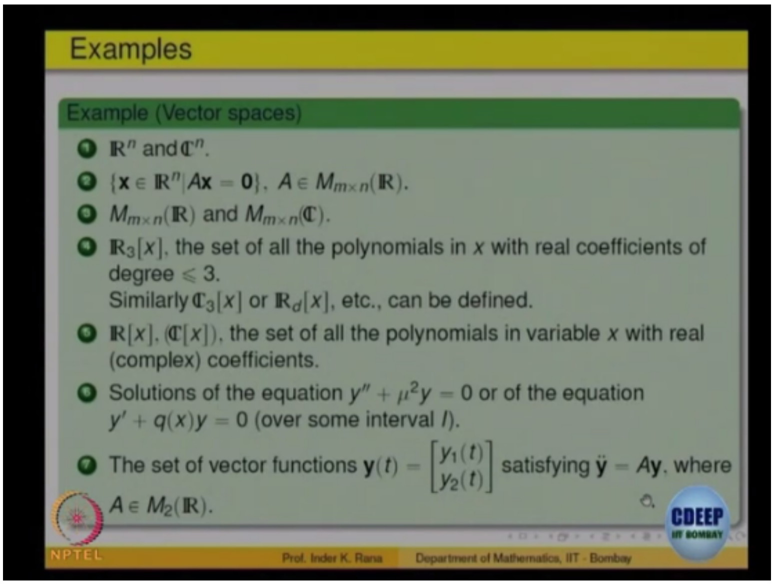

So let us look at some examples of vector spaces. Rn and Cn are the prototypes of vector spaces. So you can take vectors Rn that is n components right. So two vectors you can add them component wise and scalar multiple is multiplying each component by that scalar and similarly Cn. So Rn is a vector space over the reals. Cn can be treated as a vector space over reals as well as a vector space over complexes.

So the underlying scalars you can choose whichever you like. So that will give you two different objects. One will be Cn as a vector space over R; another will be Cn as a vector space over the complex numbers itself. This also we had studied that look at the null space of a matrix A. So given a matrix, look at the null space, that is Ax is=0, all solutions you saw that that forms a subspace of Rn, we call that as a subspace, so which is a vector space in itself.

You can also look at all matrices m x n, real matrices or complex matrices. So that is the set V. You can add two matrices right; you can take a scalar multiple of a matrix that will have all those properties that you have been actually using them. So that this is a vector space, all m x n matrices form a vector space over the field of real numbers and complex numbers right, all m x n with complex entries.

Again, this can be treated as a vector space over R or over C both, both are possible. Let us look at something which you might not have come across. We will look at all polynomials of degree<or=3. So look at all polynomials of degree<or=3 with coefficients being real coefficients right. So what does it look like? It is expression of the form A0+A1x+A2x square+A3x cube right where x is a variable right where A0, A1, A3 are scalars, real scalars.

You can also think of complex polynomials over the complex numbers where the coefficients are complex numbers. Why this forms a vector space? Given two polynomials right of degree<or=3 you can add them, you will again get a polynomial of degree<or=3, you can multiply a polynomial by a scalar. So only the coefficients of the powers will change right but the degree will still remain<or=3.

And it has all those properties that of commutativity, associativity and so on. So these are yeah **“Professor - student conversation starts.”** Yeah, you can have anything. So that is the next thing. See this you can have Rd polynomial have to be<or=any number d right. So all will be different vector spaces, examples are different right. The constant polynomial there is real number itself right. R1 is the real number itself, you can have R2x right. **“Professor - student conversation ends.”**

All linear polynomials, all quadratic, all cubic right and so on. So they all are different vector spaces okay. You can also think of all polynomials, all real polynomials, whatever the degree, put them all together in a box. So what is this Rx? It is nothing but the union of R and x where n goes from 1 to infinity, take any polynomial or any degree so that is an element of Rn. So how do you add them? How do you add a cubic and a polynomial of degree 5?

You can treat cubic also as a right polynomial but not of degree 5, it is polynomial of degree 3 only but the coefficient A4 and A5 are 0 right. So you can treat as a polynomial right with x5 but then you can add them. So basically the idea is given any two polynomials of any degree, different degree you can add them by adding coefficients of like powers right that is what we have been doing all along actually right.

So that addition becomes commutative, associative and so on. So that is the space of all real polynomials or you can also have complex polynomials and the coefficients are complex numbers. So these are examples. This is something that you will come across in your differential equation scores. Look at the solutions of the differential equation y double dash+y square y=0 or y dash+q x*y=0. Look at so what do you mean by solutions of this equation?

We are looking at a function y which satisfy the properties double derivative+y square y=0 right. if we have two solutions y1 and y2 and if we add them that will again be a solution of the same equation right. If y1 satisfies this, y2 satisfy this, y1+y2 also will satisfy the same equation. So addition is defined, scalar multiple again will define, it is something as homogenous, right side is 0 so that is not going to change right.

So this is normally one calls the homogenous differential equation but will study these things in differential equations or solutions of this type okay. So this is a second degree differential equation, this is what is called the first degree differential equation. So if you have already done a course in differential equations, you might have already come across. If not when you do it, you will come across these things.

You can also look at this in the vector form, so this is a vector that means it is y1 t and y2 t, these are the two components of this and you can have satisfying the equation y dot dot. What is y dot dot? So that is equal to the second derivative of y1 and second derivative of y2. So the notation for that is=Ay. So solutions of this also will form a vector space. So that is one of the reasons why these vector spaces are important, abstract vector spaces because you find applications in study of differential equations, so these are some of the examples.

So let us go and start further, what did we think do in when we have subspaces of Rn? We started looking at given a subset of vectors, we started looking at linear combinations of them, we started looking at generating a subspace with them, then started looking at when is a minimal set of generators right, we call that as a basis and so on. So same things are possible here in abstract vector spaces also.

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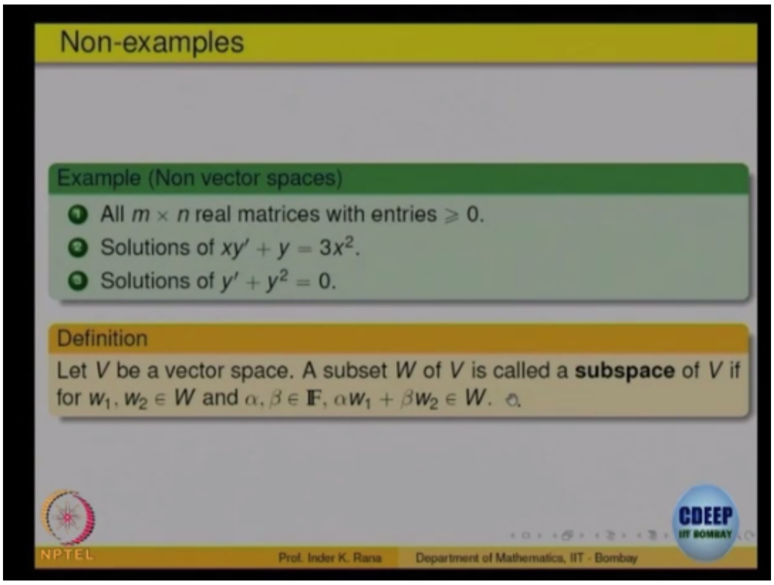

But before that let us just look at if you look at all real matrices the entry is bigger than or equal to 0. Does this follow vector space another addition of matrices? We got all real matrices m x n okay such that all the entries are bigger than or equal to 0. Do you think this forms a vector space? Of course, if we add we will get again a matrix of the same type but if we multiply by a scalar you may not get the matrix of the same type where the scalar is -1.

If all entries are positive, will you multiply by -1, you get all entries negative right. So that will not be an element, so scalar multiplication, the usual scalar multiplication does not work, may be some other scalar multiplication you can define. So that says that for a vector space, it is a set which is important, it is important also what is the operation of addition defined on it, what is the operation of scalar multiplication defined on it right.

It should have all those properties and whether it is over real numbers or it is over complex, so you can think it like 4 things in a vector space, a set, addition, scalar multiplication and the underlying scalars. All these 4 put together characterize a vector space. If any one of this changes, then you get a different example of a vector space okay. For example, look at solutions of this xy dash+y=3x square.

If you take y1 and y2 to be solutions of this differential equation that means what, that means xy1 dash+y1 is=3x square and y2 with another solution, so xy2 dash+y2 is=so can we say y1+y2 also satisfies this? Because if I put y1 and y2; however, this will give you two derivatives, this will be y1+y2 but right hand side you will get, you will not get 3x square. When you add those two equations, we will get 6x square right.

So it has only (()) (14:00) 0, it became a right vector space. So this is not, so the usual addition of functions and scalar multiplication will not give this as a vector space right. Similarly, of y dash+y square, so these are something which are nonlinear. So this is non-homogenous if the right hand side is not 0 and here it is y square coming so it is nonlinear part coming right.

So that is why they told bigger subspace, so study of nonlinear differential equation and non-homogenous differential equations will be a part of your course on differential equations right. So let us define the concept of the subspace. Given a vector space and given a subset right, we can think of creating subset itself as a vector space but that is possible only when given two elements of the subspace, you add you again get an element of that subset right.

And scalar multiple also should be part of it, so a subset is called a subspace if given two elements w1 and w2 and two scalars if you take a linear combination of that that should be again inside the same set right. So in that ways a subset which is closed under linear operations of addition and scalar multiplication can be treated as a vector space in its own right, so we call that a subspace right with the same addition and scalar multiplication taken from the original one okay.

So for example okay, so let us look at all polynomials. We had a vector space of all polynomials. Look at polynomials of degree<or=5. That is the subset of the space of all polynomials and if I take any two polynomials of degree<or=5 and add again get a

polynomial of degree<or=5. A scalar multiple again has the same property if it is a polynomial of degree<or=5, scalar multiple again is.

So you can treat polynomials of degree<or=5 as a subspace of vector space of all polynomials. So each polynomial of any degree, so <or=n right can be treated as a subspace of all polynomial. Quadratics can be thought of right as polynomials of the degree<or=2 as a subspace of cubics polynomial<or=3 right. That is also possible, so all these are subspaces.

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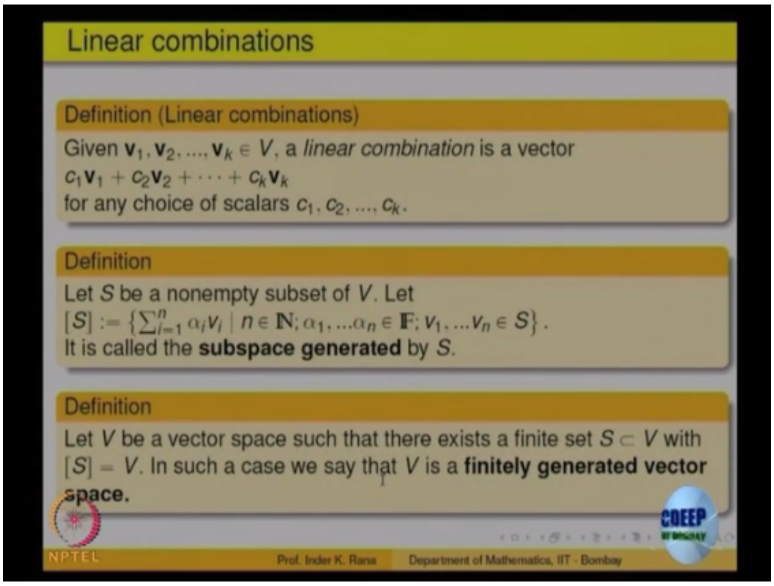

So now question is how do you describe a vector space? So let us look at what is called linear combinations. So once additions is there, scalar multiplication is there, given elements v1, v2, vk and given scalars c1, c2, ck, you can form this new element right. So this is called a linear combination of the vectors v1, v2, vk. Keep in your mind the prototype of Rn, something similar happening but we cannot say that V is Rn or Cn or anything.

V is a subset, for example you can take polynomials v1 could be a quadratic, v2 could be a cubic and so on, you can add them, will again get a polynomial right. So given elements of a vector space, you can form and scalars you can form a linear combination of that. So that is this element and we can put together all the linear combinations in a box and we will call that as a span of that set right like so given a set S subset of V.

Look at all sigma alpha i vi where this n is n could be any number, could be invertible 3, 4 and you can choose any finite number of v1, v2, vn in S, take same number of scalars in F and form the linear combinations. So what is this called? So this will be a finite linear combination right. This is the linear combination and we are defining only finite linear because there is infinite we do know how to add them anyway.

In the vector space, you cannot, there is no concept of adding infinite number of them right, only finite number is possible because you can add again and again till the finite state. So look at all linear combinations of elements of S okay where n is varying, alpha i's are varying and vi’s varying, so all possible linear combinations call that as the square bracket as one control easily.

And if you take two linear combinations then some is again a linear combination, scalar multiplication is again a linear combination. So this is a subspace and this is called the subspace generated by S right. So given a set S in a vector space what is the subspace generated by it? It is the set of all possible finite linear combinations of elements of S right. That is called the subspace generated.

So here comes if S itself is a finite set such that the space generated by S is V. We say it is a finitely generated vector space. If V is a vector space and you have order finite subset of it, so that when I take linear combinations of elements of that finite set that gives me everything in V right, every vector in V can be obtained as a linear combination of elements of that set S and if that S is finite, we say our vector space V is finitely generated right.

Finite number of elements of V give everything to add linear combinations. If it is not, then the set is non-finitely generated vector space. That is where abstract vector spaces will start differing from Rn and subspaces of Rn. So let us look at some examples of this.

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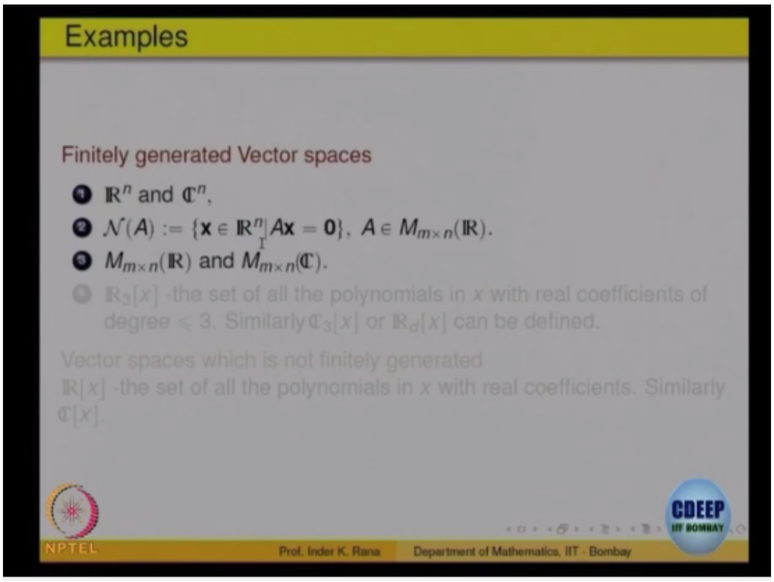

Rn and Cn are the finitely generated. Hence we know that standard basis is there 1 0 0 0 1 0 right. That is a standard basis for Rn similarly for Cn. Null space, that is a clear subspace of Rn. So that is also finitely generated and actually that became an important thing in our solving system of linear equations. How to find the basis for N of A? What is the dimension of N of A?

We call that as a nullity and finding a basis for N of A right that is the solution space of a homogenous system. All matrices do you think all matrices form a finitely generated that means what? Every matrix m x n can be obtained as a linear combination of some finite number of matrices. Let us look 2 x 2 for example, A, B, C, D right. I can write this as take the first element as A, all remaining 3 are 0.

Next, B all these 3 remaining elements 0, so we can take 4 matrices where first write one element is the A B C or D and the remaining 3 are 0. If I add these 4 matrices, I get the matrix A B C D and A can be taken out. This is A times 1 0 0 0 B times 0 1 0 0 and similarly other two. So basically what we are saying is we will look at a matrix m x n, it is nothing but R to the power m x n.

Think it now. How many elements are there in a matrix of order m x n? Is m times n right, instead of only for convenience we are adding first time elements as first row right, second as second row, third row, fourth row, mth row right but if I write everything as a 1 string first row right and then the next row and then the next row but written as one long string of elements. How many elements will be there? m x n.

So I can identify a matrix of order m x n with R to the power m x n and what is the basis for R to the power m x n, standard basis? 1 at one place and remaining all are 0 that is what we are saying. There is a basis for matrices also essentially right and actually it is not just for the sake of (()) (23:16) basis, that is how the computers stores your entries of a matrix. Computer does not know what is a row, what is a column right.

A machine only knows a string of elements that you are giving to store. You say 5 as a first element, so it has binary without storing, so it will store 5. Next element, next element, next when one row is over, you have to give some command to say that now you are starting a new counting right. That is possible to a computer. So first n elements is first row, next n is second row, next n is third row and so on.

So that is how the computer stores the data in it right. So that is a way we are saying we can find a basis for m matrices of all that m x n also.