Basic Linear Algebra
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Lecture - 36
Diagonalization and its Applications -III

So how did we; I am revising what we have done. How did we start off the course? We started looking at, how does linearity arise in real life. And that was Systems of Linear Equations. Trying to find solutions of Systems of Linear equations and the method given by Gauss that is eliminating one variable at a time we want to do formalize it. So to formalize that we brought in, what is called the Matrix representation of a system of linear equations.

That look like Ax=B, right. A is a M cross N matrix, right. M equations and N variables, so X is the vector which is components written as a column x1, x2, xn = B; b; b1, b2, bm. So that is Ax=B where A is M cross N, your variable X is a vector which is N cross 1, right; one column, N rows and one column and that is equal to b1, b2, bm; that is M rows and one column, right. So matrix form, so the question is how do you solve Ax=B, right.

We said the method of Gauss is nothing but reducing the matrix A to row equivalent form or specialize one called reduced row equivalent form. So what is the row equivalent form? Row equivalent form of a matrix is that by elementary row operations, right; elementary row operations are you can multiply a row by a nonzero scalar. You can add one row to another, or you can interchange two rows. So these are called elementary row operations.

By doing these operations alone you can bring your matrix to a special form. You are doing something on the rows and the row equivalent form is essentially says, in your matrix, right, there will be some bottom rows which are going to be 0 possibly, right. There are some nonzero rows and some rows which are identically 0. So suppose there are r rows which are nonzero they are at the top n-r are at the bottom, right. So it says, that, this r, we give it a name called the rank of the matrix, right.

And so there is a number r such that r between 1 and N such that bottom r, top r rows are nonzero bottom rows are 0. Now why a row is nonzero? Because somewhere a nonzero entry is coming, so how do we describe that? You look at a row, first row; first spot the nonzero entry. It comes at a column called P1. Look at the second row, the first nonzero entry comes at a place called P2 column number P2. So the condition is P2 should be bigger than P1.

Each next row nonzero entry should be coming in a column which is in the right side of the earlier one, right. It may not be immediate right it will be somewhere on the right. Okay. So that form of the matrix is called row equivalent form. And you can further do row operations, so that in that column one nonzero entry is there, right; there is a nonzero entry, below that everything is 0. So, that entry is called a pivot.

In that, everything on the top also you can make it 0 by elementary row operations. So in pivotal columns where the first nonzero entry of a particular row is coming, by row operations further you can make; that is the only nonzero entry everything else is 0 and that nonzero entry also you can divide by that constant and make it equal to one, right. So that is called the reduced row equivalent form of the matrix.

Then we looked and applications of this. One linear system is consistent or inconsistent of rank, so take the Ax=B, take A, augmented matrix AB, right; reduce to row equivalent form. If the rank of A right is same as the rank of B, then the system is consistent otherwise if it is bigger one more then it is not consistent. When it is consistent that means what there are r pivots among the r n variables there are r pivots, right the P1, P2, Pn are coming in the columns.

So those pivotal variables their values are obtained from the non-pivotal variables; by giving non-pivotal variables arbitrary values and backward substitution. So take at the bottom row, right, compute the pivotal variable in terms of non-pivotal ones. Give them arbitrary value, put back, put back and get your all solutions, right. That is how finding all possible solutions.

And then we also said that, solving a system Ax=B is essentially equivalent to solving the homogenous part of it, Ax=0. What is the difference? You find one particular solution of Ax=B somehow and find all solutions of Ax=B so general solution is obtained by taking one particular solution adding to it a solution of the homogenous system, right. That gives you all. That also brought, okay, we will come to it a little bit later.

But row equivalent form also gives you a method of checking whether a square matrix is invertible or not, right. So, if it is going to be invertible that means there should be unique solution for Ax=B. That is same as equivalent to saying rank should be full; and that is same as saying when you reduce row equivalent form that should be identity. So, a method of checking is take A M/N, put along with identity matrix N/N and do row operations to A and do the same operations to identity also and try to find out what is the row equivalent form of A.

If this becomes identity the next portion gives you the A inverse where the identity has changed to, right that becomes A inverse in finding what is, how to check whether something is invertible or not and same time computing the inverse also, so that is application of row equivalent form, another one. Then we looked at matrices and we looked at the row space and the column space, right. They are all; subspace is of Rn and Rm, right.

What is a subspace? It is a subset of Rn or Rm so then in itself it is closed under linear combinations, right that is a subspace, okay. For example, in R2 if you take a line passing through the origin that is a subspace of R2, right because if I take any two points and add them we will be somewhere on the line again, right scalar multiple also, so that is subspace. So Row space, column space and then we looked at what is the dimension of the row space, dimension of the column space and we said both are same that is same equal to the rank of the matrix, right.

And what is dimension of the space? That was involved what is called a basis and the number of elements in a basis. So we said every subspace of a vector of Rn or Rm will have a basis. What is a basis? You take linear combination of a subset B is called a basis. If you take all linear combinations of elements of that, that should give you all elements of the space and it should be minimal. You should not be able to reduce number of; you should not be able to remove some elements from that.

So it is a minimal set of generators equivalent to; it is a maximal linearly independent set. So those were equivalent ways of describing what is a basis and then we proved, we did not prove; we said the theorem, that any two basis has a same number of elements, vector subspace can have different basis; remember, r3 itself has different basis, right, we gave many examples. But the number of elements in a basis will always be same and that is called the dimension of the space, right.

So we had dimension of the column space, dimension of the row space and we also had the dimension of the null space Ax=0. That was called the nullity. So we had the theorem, Rank Nullity theorem, the rank + nullity= n. right. That was the rank nullity theorem we proved. And that had some applications namely every matrix or every, okay we came to linear transformations. Rank nullity theorem for linear transformation and the dimension of the row space plus dimension of the range space, right. That as a consequence, okay.

I think more systematically we looked at matrix multiplication as a linear map, A applied to X=Y that is a linear operation on the domain XY. So, you can call that as a linear map induced by a matrix and conversely every linear transformation from one vector space to another; of course vector space is subspace of Rn gave rise to a matrix representation of the linear transformation. So, how was that obtained? Say t is from v to w, v of dimension 2, w of dimension 3, right.

So v of dimension 2 it has a basis element of v1, v2, w dimension 3, it has basis elements w1, w2, w3. So take v1, take the image of v1 under t then go to some vector in w but that should be a linear combination of w1, w2, w3. So ev1 is a linear combination, so those scalars which are coming in the linear combination that give you the first row. E of v2, again a linear combination that gives you the second row, so take the basis elements in domain each one take the image and write it as a linear combination, you get the corresponding columns of the matrix representation.

So that is the matrix representation of a linear transformation. And then as a consequence of rank nullity theorem, we said t is 1, 1 is same as t is 1, 2 as maps. So that is what we proved, so that was linear transformations basis of linear transformations, rank, basis, dimension. Then we looked at generalizing the notion of angle in vector spaces, right because there is a notion of angle, dot product in Rn, so that we brought in the notion of perpendicularity, two vectors are perpendicular if the dot product is equal to 0.

So, we said given a basis, can we generate an Orthonormal basis out of it, right. A basis is linearly independent of vectors which generate everything. Orthonormal is, those every orthogonal collection is also linearly independent automatically, right. That is a simple property that we proved. And then we said that, every linearly independent need not be orthogonal, obviously.

So how do you get from linear independence to orthogonal that is Gramshin process? So given a linearly independent set; one by one iteratively you can generate a basis which is a orthonormal basis, right. So, once that is obtained, that means, what is the advantage of orthonormal basis? It says that coordinates of a vector, see every vector is a unique linear combination of the basis elements, right.

Every v is sigma alpha vi, where v and i are basis elements, this alpha i's are unique, right. That is what we called as the coordinates of the vector. These unique coordinates are immediately known if the orthonormal basis is given, because they are just a dot product. If you take the dot product on both sides of a particular vi the right side is vi vj, everything will vanish except vi, vi, right.

So that means, advantage of an orthonormal basis is that coordinates of a vector are immediately computable, right if your basis is orthonormal. That is why we prefer to have an orthonormal basis whenever possible and that is Gramshin process, right. And then we looked at a particular case that given a matrix how can we make it similar to a diagonal matrix? That means finding an invertible matrix P such that P inverse AP is diagonal. That was diagonalization process for the problem.

We said, this leads to computation of eigenvalues, computation of Eigen vectors and checking when are the Eigen vectors linearly independent and they forming basis of the underlying space if the matrix is M cross N, that should form a basis of Rn, right. So; computation of Eigen values that led to determinant of A- lambda i because A- lambda i you want a solution of that, a nonzero solution. That means A-lambda i should not be invertible, right. Then only it is possible.

If it is invertible, only unique solution for homogeneous system that is a tribal one. If you want a nonzero solution then the determinant of A-lambda i should be equal to 0, because it has to be singular. That means, solving the polynomial, finding roots of the polynomial determinant of A-lambda i. That gave you the Eigen values. How do you find the corresponding Eigen vectors? The solution of A- lambda i applied to X=0, you have to find x. So finding solution of the homogeneous system where a matrix is A- lambda i.

And for that we know, go back to row equivalent form and see what is the rank, what is the nullity, so null space you want; all possible solutions, right then find the dimensions of the null space, those many linearly independent Eigen vectors you can obtain. And we proved a theorem that a distinct Eigenvalues, Eigen vectors are linearly independent. The general matrix the Eigen vectors are corresponding to distinct Eigenvalues are linearly independent, right.

Now the question is only inside for a particular Eigenvalue, if you look at all Eigen vectors, right, they form a subspace. Okay. The question is, whether dimension of that subspace is same as the number of times a root is repeated or not? The number of times the root is repeated it is called algebraic multiplicity; the dimension of the null space of A- lambda i is called geometric multiplicity; whenever the two are equal your matrix will be diagnosable.

Because that says you will be able to find as many linearly independent Eigen vectors as the number of times Eigenvalue is repeated, right. So that was Eigen's ability of arbitrary matrix. That may or may not happen. So the condition is you should have Eigenvalue, each Eigenvalue; algebraic multiplicity should be equal to the geometric multiplicity, right. And there should be n Eigenvalues and that happens if your matrix is a real symmetric matrix.

For a real symmetric matrix, you are assured that there will be as many Eigenvalues as is the dimension that is M cross N where n is the dimension of the matrix. There will be n Eigenvalue even if an algebraic multiplicity of each will be equal to geometric multiplicity that is also part of the claim that is the part of the theorem. And thirdly, you can find a orthonormal basis for Eigen vectors. For any two distinct Eigenvalues, the Eigen vectors are perpendicular to each other.

But, inside there will only be linearly independent for each Eigenvalue; using Gramshin you can ortho-normalize it. So that is what we did today. Okay. So that is our 5 minutes; 10 minutes, I have revised the whole course. So next two lectures, what I am going to do is, I am going to look at abstract vector spaces; vector spaces other than subspaces Rn or Rm itself. They also come your courses later on. So we will spend some time on that. Okay.

So we will stop today. We will not go to them today itself. So, today's lecture ends here. Okay.