Basic Linear Algebra
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Lecture - 35
Diagonalization and its Applications -II

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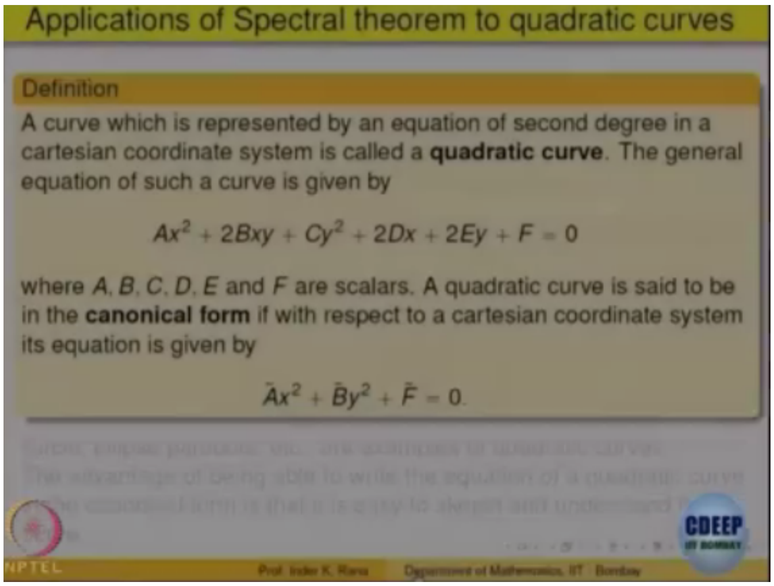

Here is another application which is very useful. Let us look at what is called quadratic curve. A expression of this form Ax square + Bxy + Cy square + Dx + 2Ey, so what would it look like? It is a polynomial in sort of X and Y, right but the highest power is 2. So it is called a quadratic curve. The idea is if you try to plot X and Y in the plane it will give you some points, right. Look at X and Y whether they satisfy this equation.

For example, let us look at some special examples. What is equation of the circle? X square +Y square= some R square that is a quadratic curve were some of these coefficients are 0, a parabola, Y square=Ax, right or a hyperbola or ellipse X square/A square – Y square/B square = 1, okay. All are quadratic curves, right. All conic sections come as quadratic curves. So the question is if you have given how do I visualize this, what is this term? I want to visualize this term, right. So there are two simplifications one can do.

One can; supposing your equation is such that terms Xy right and Ey and Xy and EXy sorry product terms and Xy terms are 0, supposing the equation looks like this, okay that means this B is 0 and D is 0 and E is 0 supposing, a special type of, okay. Then that form is called canonical form of the quadratic form. So first step is how to bring a general equation of a quadratic curve to canonical form, can we give some method or transforming because then it looks slightly simpler to handle. Okay. And that is done by shift of origin.

If your just shift the origin, so let us do that process. Okay. So it is easier to sketch and understand if the curve is given in the canonical form. Okay, so let us see we will see how it is done. Okay.

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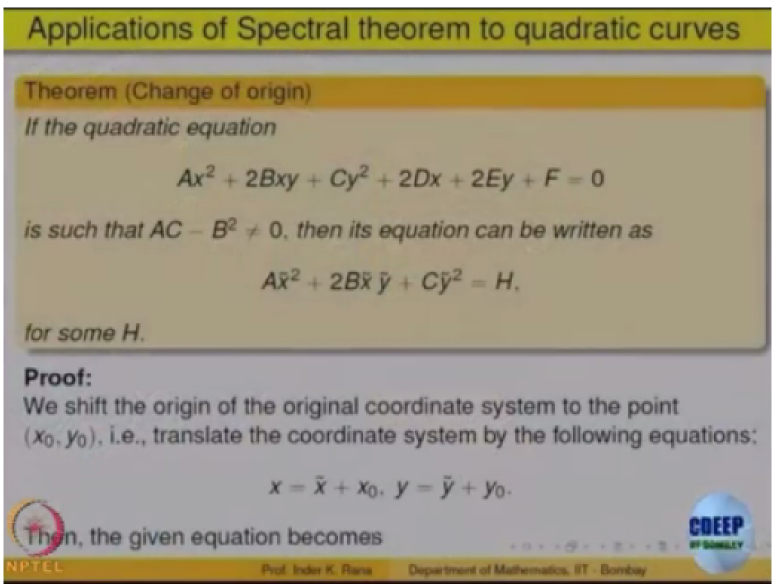

So this is a quadratic, we want to transform this to this form, right. So to do that what we do is we shift the origin. How do you shift the origin? Earlier origin in somewhere you translate right, you translate it. So what is the shift of origin, is done as follows, you right x=x tilde + some x0 and Y tilde as Y as y tilde + x0 that means what you are defining a new variable X tilde and Y tilde. What is X tilde? It is X-X0 = Y=Y tilde is Y-Y0 we are shifting, right.

Every point is shifted by X0, Y0. So what is the aim of doing this, that means if I put these values in this equation those terms which I want to make 0 they should vanish, right that is the criteria. By this substitution, I want to choose X0 Y0 in such a way that when I put those values in the equation the terms of XY, X and Y they vanish. So let us put that condition. So put these values in the given equation.

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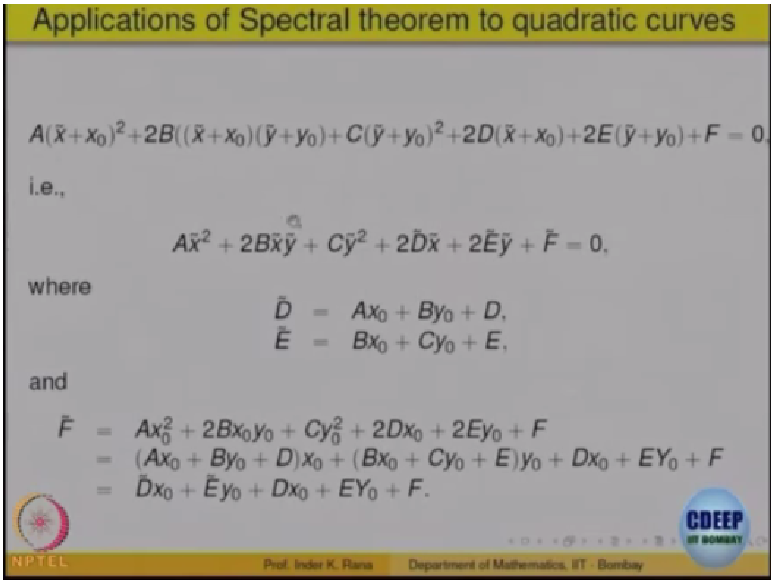

So that is the equation Ax square + 2B XY + right that is =0 that is original one in terms of X tilde and Y tilde. So once you just simplify it become Ax tilde square + 2B X tilde Y tilde + this =0. So what do you want now? In this transform equation, I do not want a D to up here right, I do not want E to up here, right this, I want them to be equal to; I want them 0. That means what, is what, this is D tilde this, is E tilde. I want them to be equal to 0.

So let us see the condition that put X0 and Y0. So compute all these three quantities in terms of X0 and Y0 in terms of the original E, F and D right. Those are already given to us. So this equation A, B, C, D, E, F all are given to us. Our aim is to find X0 Y0, such that those terms vanish, right so then this term is vanish. So we have found those values in terms of X0 Y0 E and F. So what is required? This should be equal to 0, this should be equal to 0 and this should be 0, let us put those values.

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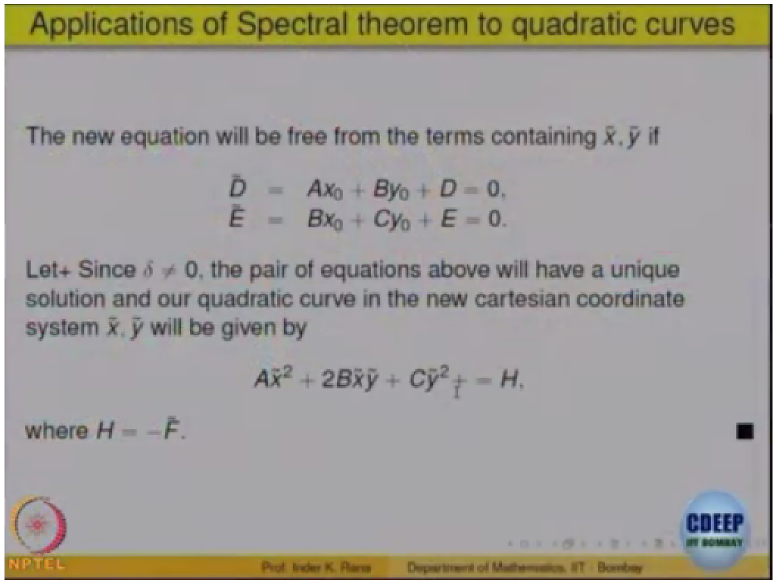

So it says, D tilde and E tilde to be 0. That is a value of D tilde in the previous one, that is the there is a value of E tilde right that is the value of A tilde, so that should be equal to 0. So that X0 Y0 are given to us, right. Sorry, X0 Y0 is to be found A, B, C, D, E they are all given to us. What is this, this is the system of equations; two equations and two variables so which want to solve, okay. So if you going to solve that, when is this system solvable? We want a unique solution, right? We want only one. So, what is the two equations and two variables?

What is the condition? The determinant should not be equal to 0, right. Is it clear? Go back 2/2, right determinant should not be equal to 0, rank should be full. So AC-B square right that would be determinant of that matrix AX= right that is not be 0, so that should not be equal to 0. So we will have a unique solution in that case, right. 2/2 determinant eminent matrix should not be equal to, the determinant of the coefficient matrix should not be 0, so you get in this form.

So if it is not 0 then you will find a unique solution take that choice of X0 Y0 your equation becomes of this form, right it becomes to canonical form. So that is normally we have been doing in our school also, whenever you want to solve some problem in quadric geometry or geometry, right. So let us assume the origin is air that is very common. Why is that done, because something becomes simpler, something vanish, right computations are simpler.

So this is the first what one does is, by shift of origin you can bring your quadratic curve to canonical form. Okay.

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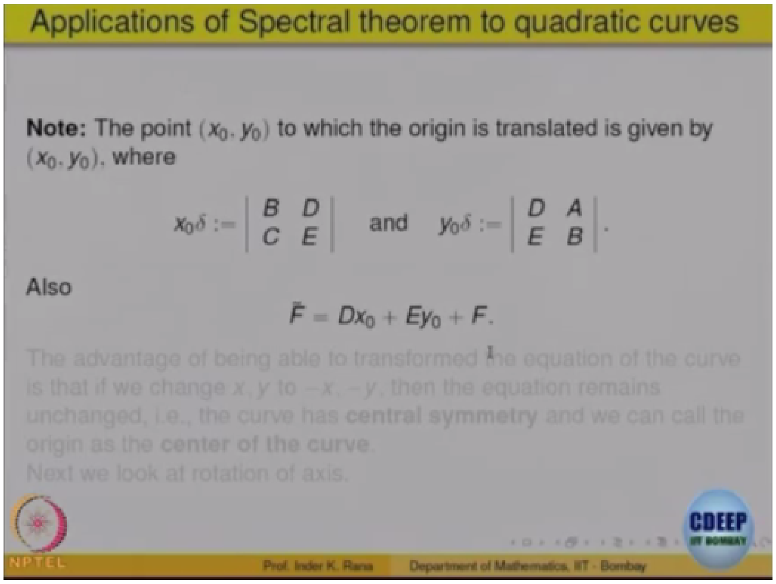

So that is your X0 delta so; and F is given by this. So what is the advantage of canonical form? X square + Y square = something right that is canonical form of a curve. What is the advantage? It is symmetry here, if we change X2-X and Y2-Y the curve remains the same, right. What about ellipse? That also has a same propitiatory, so canonical forms have central symmetry also, so that is one advantage.

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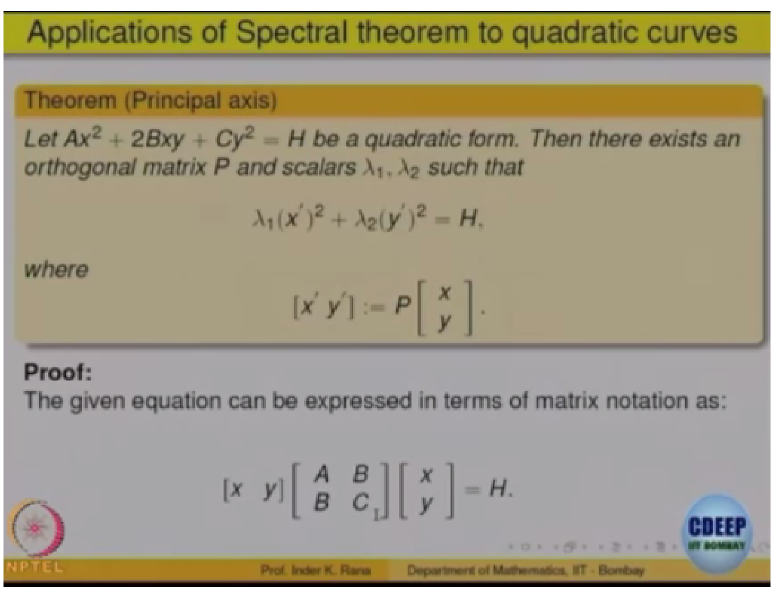

So now let us look at, so what we will do is we will assume you have got already shifted the origin and you have got a quadratic curve which is in that form, Ax square + 2Bxy + Cy square = some constant H. Such things are normally called quadratic forms. So canonical form of a quadratic is given names, right all such expressions are called quadratic form, okay. Just to say that in the canonical form.

The claim is that given in the quadratic form there exists an orthogonal matrix P and scalars lambda 1 and lambda 2 such that this equation in a new variable looks like lambda 1 X dash square + lambda 2 Y dash square = H. Where is X dash Y dash? That is related to the old variable xy/ that matrix P. So a change of variable what is I am doing. Xy is original variable, right. So what we are doing is you can find a matrix P which is a orthogonal matrix and define new variables as X dash Y dash by multiplying Xy/p.

In that variable this equation looks like this only, right. So now let us try to understand geometry of this (()) (10:28). You are given a Ax square + 2By + Cy square that is some kind of a curve. What does changing the variable by orthogonal means, what is the orthogonal transformation say in R2? What are orthogonal transformations in R2? It Determinant equal to 1, right. Orthogonal means P transverse P=identity so determinant is is +-1, right.

Basically these are all rotations. Orthogonal means they are rotations only, okay or reflection against a line through the origin. So what will what does a rotation do, so to the normal coordinate axes? What will it change it to? If I apply a; I am applying a change of variable in the plane, a point XY goes to X dash Y dash where P is orthogonal. So the original, if two lines are perpendicular passing to the origin, say the coordinate axes, what will happen to them?

Will only be rotating, right they will go to some other two orthogonal lines, right. So this means what this means that by a rotation you can change your coordinate axes to the original coordinate axes that ordinary X and Y, right. So basically what we are doing is we are trying to bring a geometry, given a quadratic curve one is translate it, right once you have translate it then you are going to rotate it to look at a normal standard form of some conic or quadratic form, right.

So I am changing the geometry of this P is yes that you are rotating points in the plane essentially, right is the orthogonal transformation. So will see application, will see exact example. So how do you do that, that is a question and that is the application of spectral theorem. So what do you do is a following. So this given equation Ax square + 2Bxy + Cy square = H. We rewrite in the matrix form. How do you right it in the matrix form?

Imagine X and Y are the variables, then what is this? It is just this equation. Rho vector xy apply to the matrix with rho's AB and BC and the column vector xy. What is this product? This is precisely Ax square + 2Bxy square + Cy square, right and that is equal to H. So this is 1 cross 2, 2 cross 2, 2 cross 1 so it will be a scalar and that is equal to H. So this quadratic form is written in terms of matrices where X and Y are treated as components of the vector XY

Now advantage of writing this is, look at this matrix. It is becomes the real symmetric matrix AB BC it is a real symmetric matrix. So once it is real symmetric it will be diagnosable by a orthogonal matrix. So align a matrix P say that P this matrix P inverse is diagonal. And if you put that value here you will see the new variables coming.

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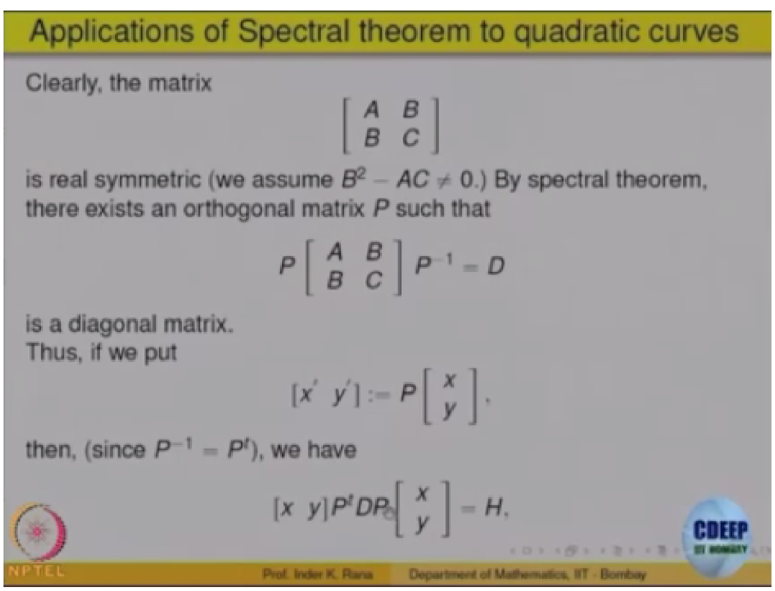

So of course we are assuming that B square – AC is not equal to 0 in the original one. And P at matrix P inverse is equal to diagonal that is the diagonal matrix. So if I define, right. So if I define X dash Y dash as Pxy then what is the equation P inverse is P transpose again that is useful for real symmetric, so I get x y P transpose DP right that is equal to A, right this is equal to A. From here what is this matrix, this matrix is take P on the other side P inverse on the other side right. So what is this matrix that is = P transpose, because this will be P inverse here.

P inverse is P transpose. So P transpose DP, right. This matrix A B B C this matrix what is this, multiply by P inverse on both sides it is P inverse DP, right. But P inverse is P transpose, so it is precisely P transpose DP. This is your matrix and XY was already there, so that is equal to H. So what is this thing now XY P transpose. It is P apply to XY transpose, right. DP P xy = H. So if I call this P of xy as a new variable that is what we have done X dash Y dash. So what is this equal to?

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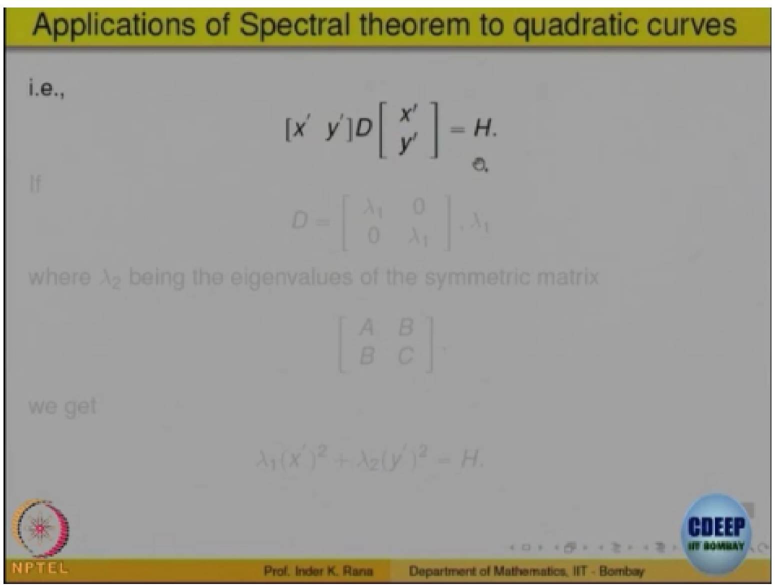

So this is precisely equal to X dash Y dash D x dash y dash equal to H. What we have done is the 2/2 matrix which came as a real symmetric matrix from the quadratic curve, we have taken the diagonalizable form of that and put that value and that will give a new variable. So where D is the lambda 1, lambda 2 are eigenvalues, so this is the form we get, right. So what is the change of variable that I am doing that the change of variable we are doing is precisely given here, right.

That matrix P apply to XY is a new variable X dash and Y dash. So first one was shifting the origin right that made all other term vanish XY and those terms vanish right it become a canonical form. In the canonical form now you rotate it right expressing that quadratic form matrix as a symmetric matrix, right. So let us look at an example of this. Is it clear what we are doing, right? Basically, applying spectral theorem, real symmetric matrices to visualize a general quadratic curve right how does it look like.

So it says, to visualize it first shift the origin so that the term XY and terms vanished becomes to canonical form and then you rotate you will get your standard form of a conic which is easy to visualize. Okay.

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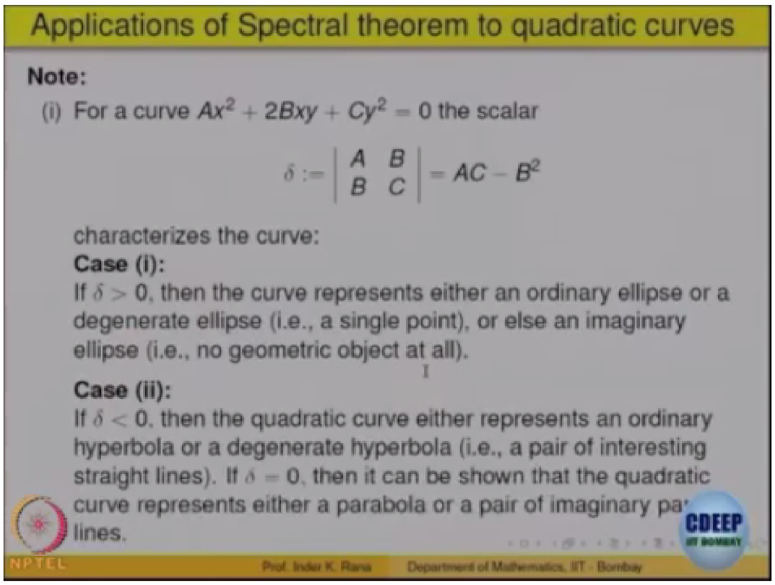

So of course in that, right this delta is important, right that should not be equal to 0. So how does this delta characterize, what kind of a curve I have got, it should not be 0. So if it is positive okay so it depends on if it is positive essentially it represents a ellipse okay. And if it is negative that essentially represents a hyperbola. So those delta also characterize it what is going the shape of, so that is not really important from this point of view.

But that becomes the important, one study is called differential geometry, right where you study what are curves, what are surfaces, how do you characterize surfaces and so on. Curves and surfaces. So this is essentially.

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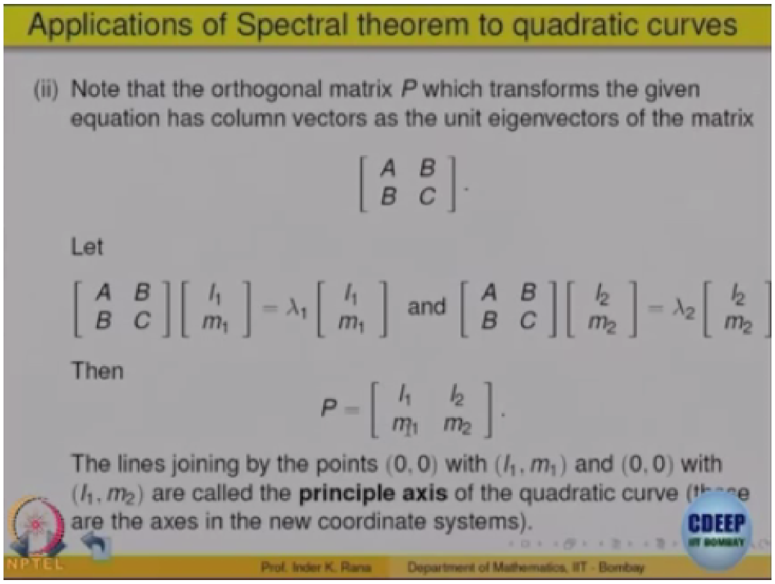

And you can find out actually if; there are two Eigen vectors for this, right. So let us say the first Eigen vector is L1 M1. The second Eigen vector is L2 M2. So this is your matrix P is going to be this, right which is going to give you rotation, right. And what are this L1 M1? If you join L1 M1 that is what going to give you one coordinates axes and the other one is joining L2 M2 because they are orthogonal right. So this gives you two axes or when you shifted the origin right you can get a new point and you when you rotate, right you get the two axes.

So that point where you have shifted you essentially what is call the center of the curve. And these are the axes which are new axes, so this theorem normally is also called the principal axes theorem. If you join L1 M1 with origin you get one line, L2 M2 you get another line right that is going to be the axes for the canonical form, okay. So let me just look at one example to illustrate this idea.

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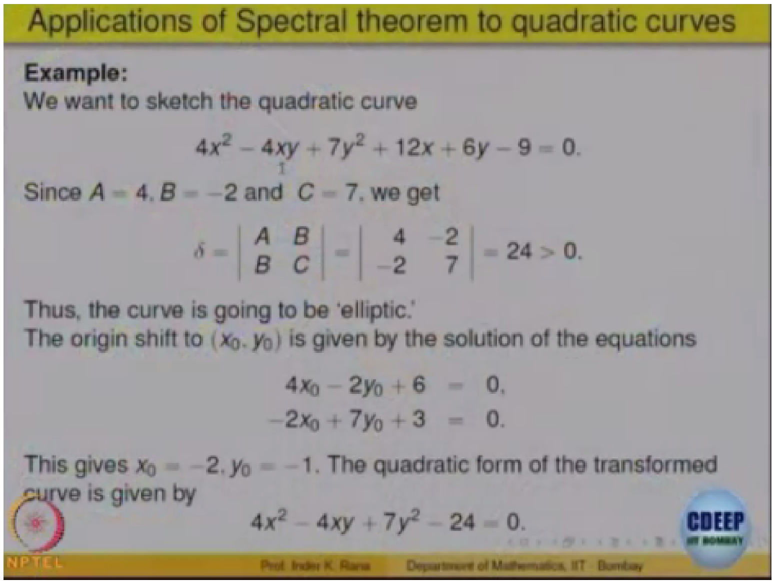

Let us look for 4x square – 4xy + 7y square + 12x + 6y – 9 = 0. What curve is this we want to know? So what are you going to do? First shift the origin, right and then rotate it, right. So anyway for delta for this because ABC are not going to change when you shift or you do, right they are same as it is. So you can find delta in the beginning itself is bigger than 0 so it is going to be some kind of a ellipse by looking at A B and C, right and looking at delta if it is AB BC.

You can see that this is going to look like a ellipse. So first of all what is the shift of origin? Where do we shift that origin to? So remember that equations Ex naught + that thing, so these are the two equations, what is for shift operation is given by like do you know how these two equations come? That are shift of origin, right. Putting X shifting X to tilde + X0 Y2 and making the other terms vanish those two equations which came by putting those 0.

So these two give you, once you solve those two equations because we already assume that determinant is not equal to 0, unique solution will exist and the unique solution is given by X0=2-2 and Y0=-1. So that is where the origin should be shifted to. Okay. And that is what we call as a center of a curve. So origin 0 0 should come to -2 and -1 and then I should rotate it now there. So for the rotation what is the canonical form of this? 4x square – 4xy + 7y square, right.

And once you change that, right that comes out to be 24 equal to 0, okay that H, right so that is; so what is the matrix for this now? This is A, this is 2B and that is equal to C, right.

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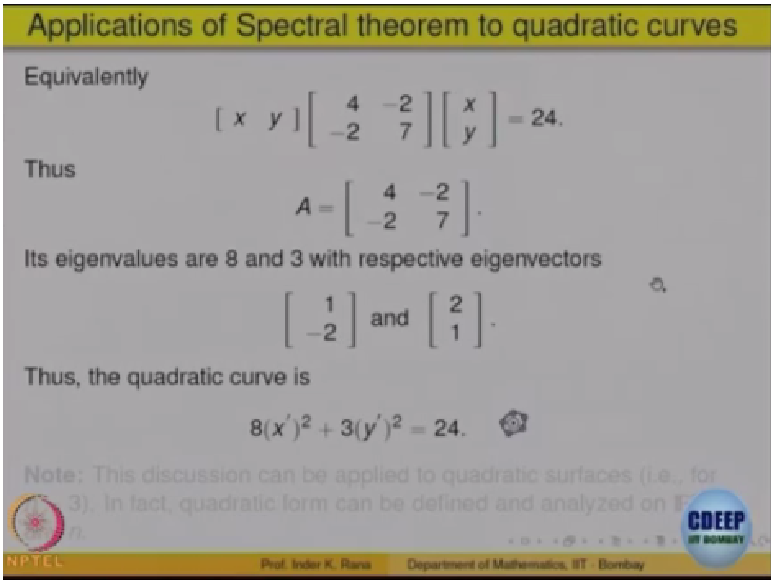

So write down the matrix, so that is a matrix. So we want x y apply to this; xy=24 that is the form of the canonical form returning as matrix multiplication. So this is the matrix A so we want to find the corresponding matrix P which will diagonalize it now. Because this is a real symmetric matrix, right. Orthogonal matrix will exist. So its eigenvalues are 8 and 3 one can check. I am doing the computation here. 2/2 is a quadratic you can solve it. eigenvalues are 8 and 3 and these two are corresponding eigenvectors, okay.

Now are they orthogonal? They are orthogonal, right? Because for real symmetric distinct eigen vector, eigenvalue is corresponding to distinct eigenvalue have to be. So seems okay fine, right. So what will be your curve? What is the transformed curve? Lambda 1, x dash square + Lambda 2 right, so that is your curve in the transformed form, right in the new variable x dash y dash, there what is x dash?

At P you look at this matrix P, first row and second apply to xy=x dash y dash that is your new, right new variables. Is it okay? So I just written directly.

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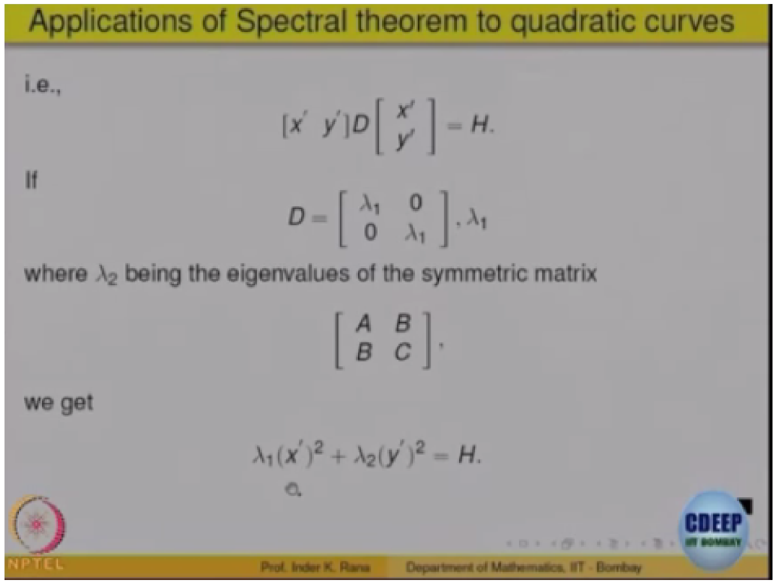

I have taken the last, see this is the form, transform form will be this. Lambda 1 first eigenvalue, Lambda 2 with a second eigenvalue = H. So that once you write, so your equation comes out to be H was 24 so this is your new form. Let us try to visualize it. Okay. So what is this? This is the ellipse anyway, right, clear it is a ellipse, right. We are not finding eigen vector because we are not really interested in finding.

Once we have formed the formula that x dash is nothing but P of xy, right x dash, y dash is P of that, right. So you can do that and you will get this equation again as Lambda of; because we have done it in the theorem that computation, right. So we are not really interested in P we are interested in is what is the new equation looks like, right as part to visualize it.

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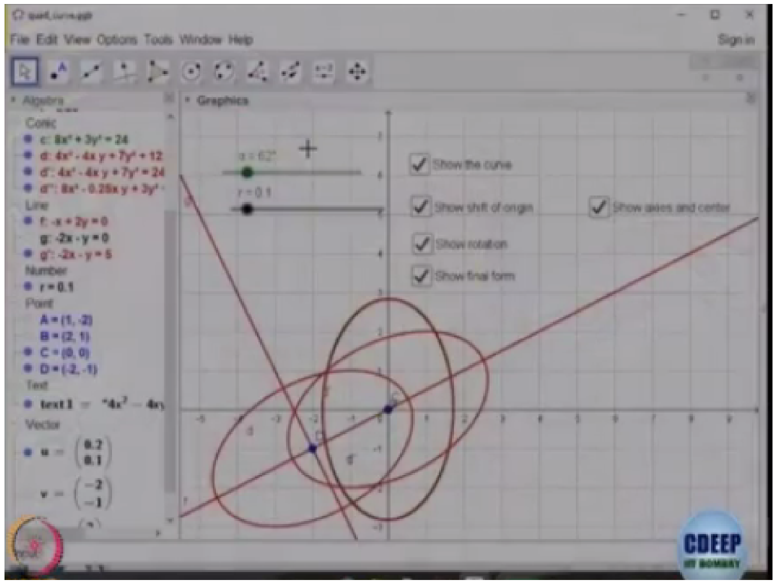

So that is that original one, right. That is original quadratic curve. If I plot it as it is xy in the xy plane what does it look like? This will look like this. I have to move it now slightly. This is what it looks like if I actually plot it, okay. But if I; is only after plotting it looks like a ellipse to us now, right. But if you do not plot it we do not know what is kind of a ellipse it is. So shift the origin, so let us shift the origin, if you shift the origin so this will go here.

So this will become; this is the new shifting the origin translation only, right. If you want to see how does this translation come about; but one can actually show that it is translating and sort of emerged thing slowly it is going and translating. So this is the one, what will I do. I should rotate it by matrix, so here is the rotation you rotate it. So now it looks like perfectly nice ellipse eigen is organized as we draw it kind of thing. So that is how visualization is obtained okay.

So this is the final form, okay. And this is the, these are the axes right which we have'; origin x-axes, y-axes have been transform by that orthogonal transformation to this. That is what it look like. You can try to do it; try to draw it you will understand everything. I am using a software called GeoGebra which his freely available, free to download okay. I have just sat down and plotted all these things and rotate it.

All things are available here, how to translate, how to right. Everything is there. So use that if you want to sort of have a change of scenario. Paint with software and try to understand mathematics. Okay. So that is an application of spectral theorem.