Basic Linear Algebra
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Lecture - 33
Diagonalization and Real Symmetric Matrices - III

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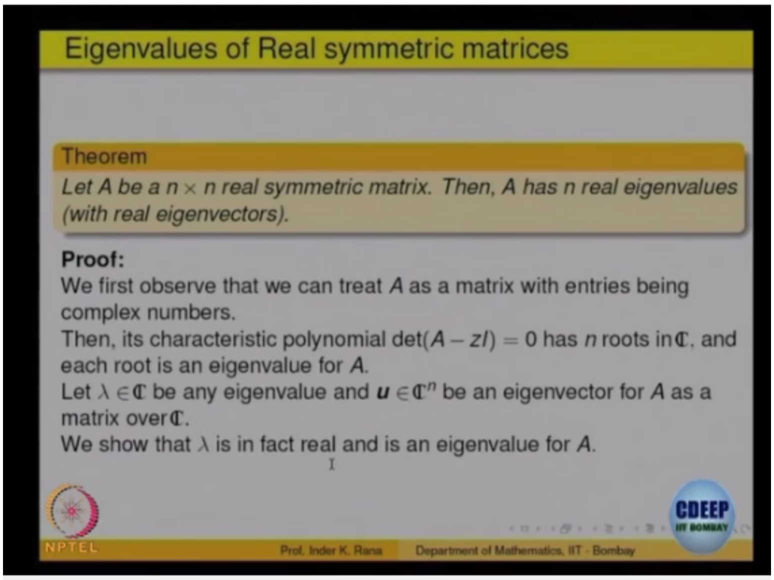

Now the problem comes, we are all along lucky here, in the earlier example we saw that eigenvalue lambda=1, the nullity was 1, so you could not find eigenvectors that was not diagonalizing. There is a class on matrices where everything goes very nicely and they are called the real symmetric matrices. So the theorem says if A is n x n real symmetric matrix, then A has n real eigenvalues.

See what are eigenvalues? That the roots of the characteristic polynomial right. Even if A is a real matrix, the characteristic polynomial will be a polynomial with the real coefficients. So it may not have as many real roots as is the degree but there will as many roots as the degree, they maybe complex right. So fundamental theorem of algebra says given a polynomial either real or complex does not matter what coefficients, it will have as many roots as the degree.

But some of the roots maybe complex roots and we are given a matrix which is real, so we are interested only in the real roots. So the characteristic polynomial of a real matrix may not have real roots at all, that is also possible. It may have real roots right but not all of them roots may be real, some roots may be complex right but here is a special type of matrices which are called the real symmetric matrices.

It says if A is a real symmetric matrix that means a matrix with real entries and what is the symmetric matrix? When to say matrix is symmetric? If A is=A transpose, so if it has that property then it says the theorem says for a real symmetric matrix it will have n eigenvalues that means the characteristic polynomial will have all roots real right. They may or may not be repeated, that is a different story.

They will have all and hence there are n eigenvalues that means there will be as many eigenvectors also, each eigenvalue will give you eigenvector right. So important thing is for a real symmetric matrix, all roots will be real okay. There will be as many eigenvalues as is the order of the matrix that is very important okay. So that is so how do you let us prove it. So we will start with the characteristic polynomial is a polynomial with real coefficients right.

So let us treat it as a polynomial with complex coefficients, the real number are part of complex numbers anyway right. So it will have roots which have probably complex and will say will show that each root of the characteristic polynomial has to be a real number right that is what we want to show. That means if lambda is the root of the characteristic polynomial then lambda is=lambda bar where lambda bar is a complex conjugate.

If we show that then we are through, so for that so let us assume characteristic polynomial will have n roots right in complex numbers. So let us take any lambda okay and see to be an eigenvalue and a vector u if the scalar lambda is complex when you write A-lambda I, it is not sure that you will get eigenvector with all real entries only right. If lambda is complex, it will turn out to be complex.

If lambda is real, then all row operations will give you only real only right. So if lambda is assumed to be a complex number, so we cannot say that eigenvector will be real. So by at least eigenvector will be in Cn with entries as complex numbers. So let us start with that. That means A applied to lambda is, A applied to u is=lambda*u right. So that is given too. So we will show that this lambda is in fact the real eigenvalue.

So we have to show lambda is=lambda bar right, that is what is to be shown using the fact that lambda is an eigenvalue. A applied to u is=lambda applied to u, only that fact is to be used. Now in complex numbers, how is the dot product we find? That is what is going to be important. In Rn, the dot product is sigma AI BI right, A dot B if A has got components a1, a2, an, B has got components b1, b2, bn then what is A dot B?

That is the sigma AI BI that is the complex then what is the definition of the dot product? Then just saying AI BI does not help right what you want has to do a modification is AI BI bar you have to put, you have to take the product AI BI bar because there are two reasons for that. One, when you take dot product of vector with itself you should get the magnitude, so if B is=A and their complex right then what is A dot A, will be sigma AI AI bar (()) (06:03) mod of AI square right.

So you get magnitude has to be real right, so if you do not put the bar you will get AI A AI which is again a complex number. So A dot A will be a complex number which does not make sense. Is it clear? That is why the definition of dot product in complex entries is defined as for a vector A with components a1, a2, an; B with components b1, b2, bn; A dot B is defined as sigma of AI*BI bar okay.

So that is the definition of the dot product and then it has all the properties that dot product has, so will be using that.

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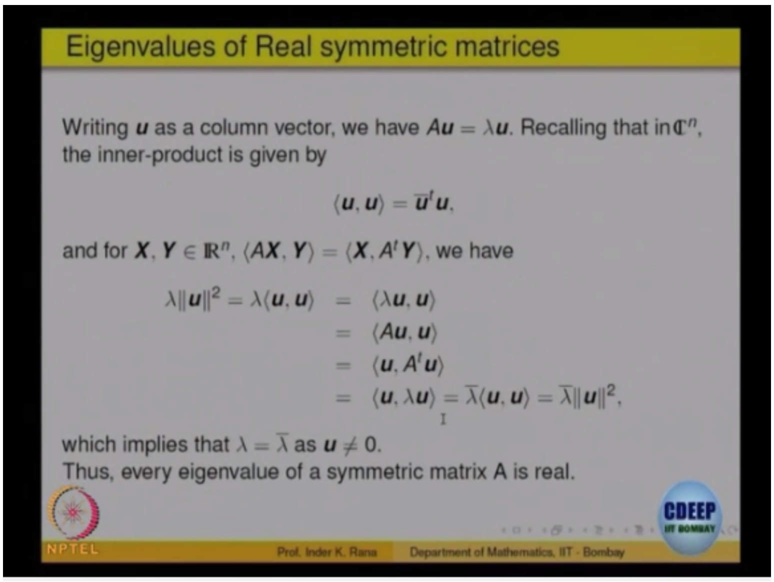

So here it is repeated again. So if you write u as a column vector, if you want to write in terms of matrix multiplication okay then the dot product u dot u is nothing but u bar right, why transpose? Because you are writing as a column, so write it as a row vector 1 x n*n x 1 the product should be a scalar right. So that is why this is in terms of matrix multiplication and you see the bar coming here right.

So the dot product is u bar transpose us matrix multiplication of if you are writing u as a column vector because reinterpreting the dot product in terms of matrix multiplication when the entries are complex. That is the definition also okay. So using that will proceed, so it is easy to see that with this definition if you take dot product of AX and Y right that is same as the dot product X with A transpose Y.

You can take A but that becomes transpose in the other variable okay. So that is easy to see from this definition. So with this understanding let us proceed. Lambda is an eigenvalue with eigenvector as u, so let us compute lambda times norm of u square. What is norm of u square? Dot product of u with itself okay. Now this lambda because of linearity, I can bring it inside here dot product, so it is lambda u, u right.

But lambda is an eigenvalue, u is an eigenvector, so what is this first term? That is A*u right and now this A I will take it on the other side okay, that will become as A transpose, so u times dot product with A transpose u but what is A transpose u, A is symmetric, keep in mind, till now you have not used that the fact that A is symmetric, so what is A transpose, that is A itself, so what is this term?

This is again lambda of u right, so that is again lambda of u but in the component dot product when this lambda comes out from the second component, it comes out as a bar right. So this comes out as a bar u, u. So you get lambda times magnitude of u square is lambda bar*magnitude of u square but u is an eigenvector, so it cannot be 0. So this magnitude cannot be 0, so you can cancel out both sides so lambda is=lambda bar right.

So it is a simple property of the dot product in complex numbers right. In Cn, the dot product is not just sigma AI BI, it is AI BI bar right, that only gives you that the property that in the second variable when you take lambda comes out as lambda bar, in the first variable it remains as lambda. So those are the properties of the dot product for Cn okay, so that gives you lambda should be equal to lambda bar.

So simply using the fact that only we have used symmetric here right, A transpose is=A this place we have used A is symmetric, otherwise it is just a definition of the dot product nothing more than that. So all the eigenvalues are real, so these are special class of matrices which says if A is real symmetric then be sure you are going to get as many eigenvalues as is the order of the matrix and all if the matrix is real you will get real eigenvalues right okay.

There is a corresponding real for complex also but will do it a bit later (()) (10:38). If A is complex, what is the corresponding thing for saying A is symmetric? It is not just A is=A transpose, it is A=A bar transpose you put in there, so that is definition once you call it as Hermitian matrices, that is the definition of. So a Hermitian matrix if it is real, it is a symmetry matrix right and for that will see what are the results, we are bit later probably.

So for real symmetric all eigenvalues, so eigenvalues will exist because fundamental theorem of calculus says eigenvalue should exist right and this theorem says all eigenvalues have to be real, so n x n real symmetric matrix will positively have n real eigenvalues and hence n real eigenvectors also okay.

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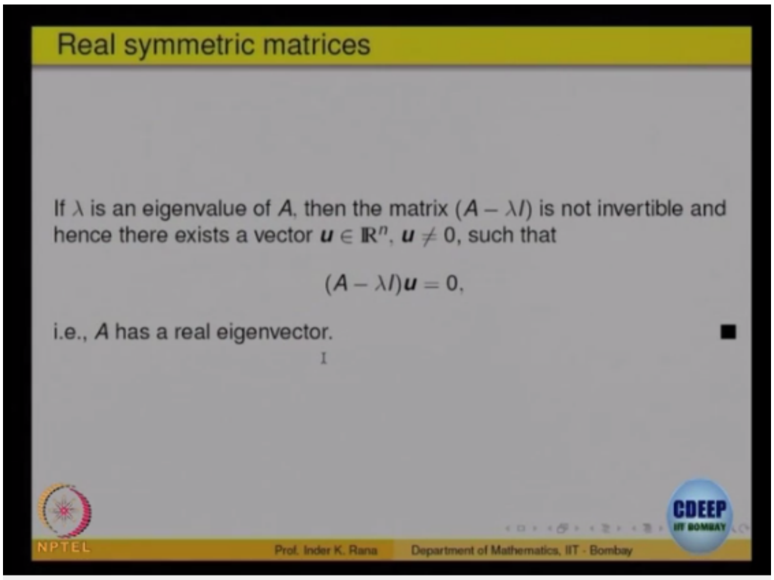

And the corresponding eigenvector when you find that will be a real eigenvector because you will be solving that homogenous system applications right okay.

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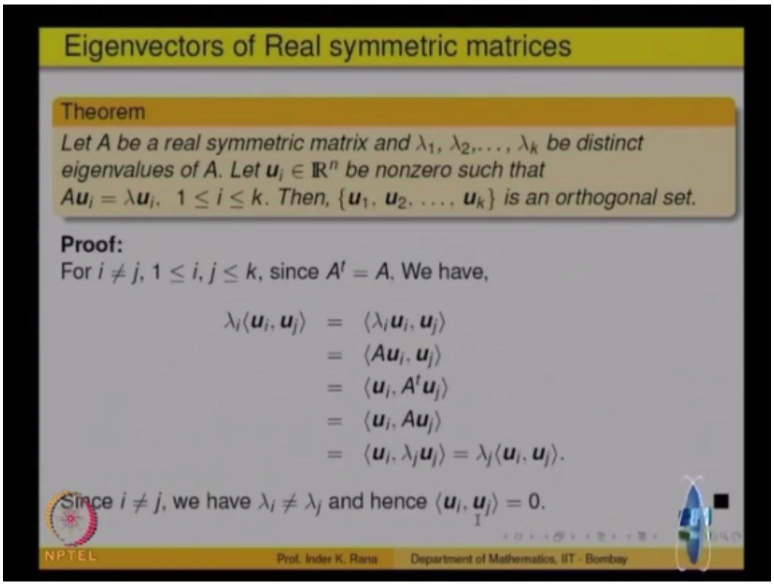

There is actually something more which happens, so I think that also is simple, let us prove it today itself. So if A is real symmetric and there are distinct eigenvalues, we have already shown that for any matrix if the eigenvalues are distinct then the corresponding eigenvectors are linearly independent but now it says something more. If lambda 1, lambda 2, lambda k are distinct, eigenvalues and pickup eigenvectors corresponding to them ui.

Then, they are not only independent, they form an orthogonal set, they are perpendicular also. So you get a special property that distinct eigenvalues give you eigenvectors which are linearly independent for any matrix but if a matrix is real symmetric then the eigenvectors are actually perpendicular to each other okay. So let us again the proof just realizes the fact that what is the dot product right.

So let us just look at. So take any two i!=j and we are given A=A transpose. So let us look at lambda i ui, uj. I am taking the dot product of ui with what is my aim? Aim is to show that dot product is=0, if I want to show them they are perpendicular ui must be perpendicular to uj means dot product is=0. So that is why I am taking ui uj dot product multiplying with y scalar lambda i corresponding eigenvalue, this lambda i I can take it inside.

So take it inside this lambda i ui right but what is lambda i ui? It is eigenvalue, so this is A to apply to right it is A applied to ui right. Now this A I will shift it to the other variable now, so I can shift it to other, it becomes A transpose but symmetric, so what is this A transpose? That is A itself, again the same idea is used. So A uj but what is A uj? That is lambda j because it is an eigenvector again.

So it is lambda j and this lambda j comes out right, it is lambda j again. Why lambda j not lambda is a bar because real symmetry, there is real now right. We have already proved. So and lambda i is an eigenvalue, so what does this imply? I can cancel out lambda i. I have already resumed right the distinct eigenvalues okay. So what does it imply? If i is not equal to j lambda is a distinct right.

So if I take it on the other side, lambda i applied to this is lambda j applied to this, taken on one side what is it? Lambda i-lambda j applied to ui uj is=0 but they all distinct right. So the lambda i-lambda j cannot be 0 so what should happen? Ui uj dot product must be equal to 0 right. Is it clear? Because these are distinct lambda i is not equal to lambda j, I can take them other side and divide by that because that is not 0.

So I get ui=uj that is=0 right. So we get that the eigenvectors corresponding to the eigenvalues of a real symmetric matrix if they are distinct eigenvalues and the corresponding eigenvectors are not only linearly independent actually they are orthogonal to each other right. So this we play a role later on because once supposing all the eigen for a real symmetric matrix and distinct eigenvalues exist right, will get eigenvectors corresponding to them, we know that.

See for a real symmetric matrix, eigenvalues will exist, they may be repeated but supposing you are lucky and you get distinct, all distinct right then the corresponding eigenvectors will be orthogonal to each other right. You will get as many, so you will get a basis consisting of eigenvectors which are orthogonal right and if you divide each one of them by their norm you get an orthonormal basis right.

So for an ordinary matrix if eigenvalues are distinct, you will get an invertible matrix right but if it is a real symmetric matrix and the eigenvalues are all distinct right where n eigenvalues which are all distinct, you will get a orthogonal matrix which will diagonalize it. So that is the advantage right. So I think will do that next time. Upshot of today’s thing is how to find eigenvalues, how to find eigenvectors.

If you are lucky, distinct eigenvalues you will get a matrix which will diagonalize it and important that happens because eigenvectors corresponding to distinct eigenvalues are linearly independent in general. When it is real symmetric, all eigenvalues will exist. There will be n real eigenvalues and eigenvectors corresponding to distinct eigenvalues will be not only linearly independent, they will be actually orthogonal to each other right okay. So let us stop here today.