Basic Linear Algebra
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Lecture - 32
Diagonalization and Real Symmetric Matrices - II

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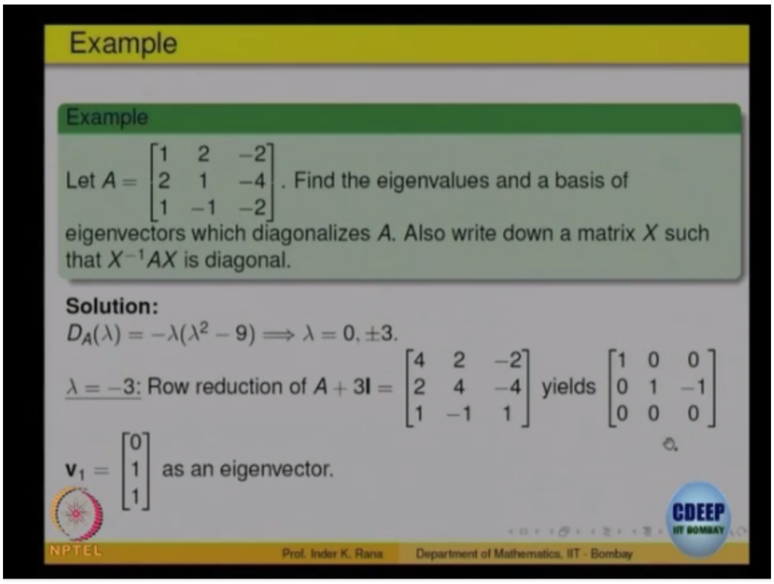

So let us look at one example to completely understand the process. So we are given this matrix A okay, so you want to find the eigenvalues and a basis consisting of eigenvectors to diagonalize this matrix A if possible right. So what is the first step? Find the eigenvalues. So look at determinant of A-lambda I so A-so along the diagonal you are subtracting lambda, so here it will be 1-lambda 1-lambda -2-lambda everything else remains the same.

Find determinant of that, so just written the answer here, it comes out –lambda times lambda square-9 that means lambda is=0 and lambda=+3 and -3. So lucky here we have got 3 distinct eigenvalues right. So obviously because of the 3 distinct eigenvalues and matrices of order of 3 x 3 and each eigenvalue is going to give you a eigenvector and eigenvectors corresponding to distinct eigenvalues are linearly independent.

So for each eigenvalue will find a vector which is eigenvector, put them together, they will form a basis of consisting of eigenvectors. So that also will give me the matrix P which is going to diagonalize right because we are lucky here we have got 3 eigenvalues for 3 x 3 matrix. So let us just go through the process. For lambda=-3, so what will be the matrix we will be looking at A-lambda I, so lambda is -3 so it becomes A+3I right, so that is the matrix.

And how do you find the null space or nullity? You have to reduce it to the row echelon form, so you are given here. The row echelon form is given by this okay. So once that is the row echelon form, what is the rank of the matrix? The rank of the matrix is 2. So what is the nullity? 1 right. So the eigenspace or the null space for this has only got dimension as 1 so how do you find that?

So you will find from here right X2-X3 is=0 right, so X2 is=X3. So you can give any value you like right. For the vector which does not have the pivot right that what does that give me X3=0 anyway, sorry X1, the first one is X1, X1 is=0. So 0 1 1 is an eigenvector, is clear. Once you have gotten the reduced row echelon form, writing down the basis for the eigen-subspace or basis for the null space is quite easy.

So for the eigenvalue lambda=-3 0 1 1 is an eigenvector right or you can also say the null space is spanned by this vector, scalar times this also will be an eigenvector right. So for lambda=-3 we have found.

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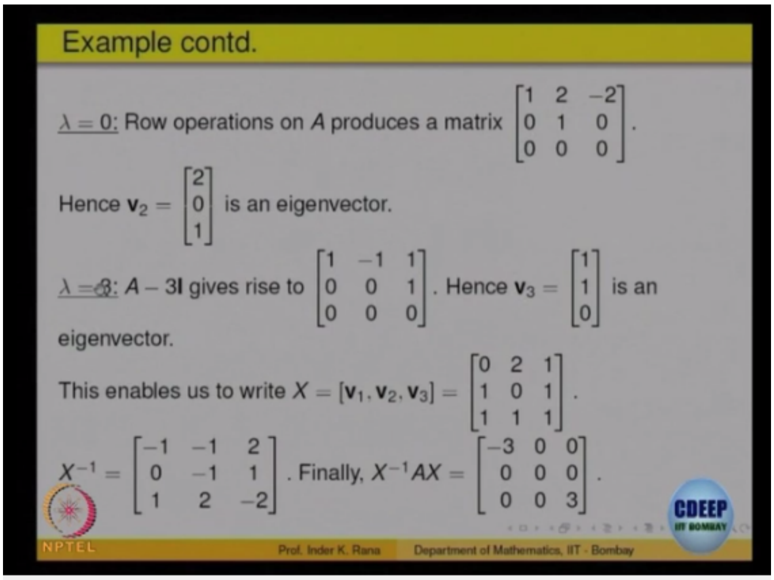

Let us go and do for lambda=0, so what will be lambda=0? A-lambda I that is the matrix A itself right. So that is the matrix and if you make it row operations, it becomes likes this okay. So each again and again we have to look at A-lambda I, reduce it to row echelon form and see what is the dimension and find the basis. So again the rank is=nonzero number of rows is 2, so nullity is 1, so dimension is=1 right.

I will have only one eigenvector which spans this. So how do you find that? What is X2 when the second equation X2+0X1 X2 X3=0, so what is X2? 0, so you get 0 here. What is the first equation? X1+2X1 2X2-2X3=0. Which one gets the arbitrary value? X3, everything else is determinant of that. So X3 you can give the value 1 just to be simple right. So give it 1 and compute X2 is 0, compute X1, so you get the vector right.

Pivotal variables are computed in terms of non-pivotal variables. So arbitrary values are given to the non-pivotal variables. So here X1+0X2 right, so we got X2=0 that is computing because there are right and the first one the non-pivotal variable X2 is already determinant, the non-pivotal one is X3 which can be given arbitrary value. So that arbitrary value we have given it as 1 and found everything in terms of that, so that is one eigenvector.

For the eigenvalue lambda=0, so let us do the same for lambda=3, so lambda=3 A-lambda I so that is A-3I right. Again same process, so if you so this is already in that, so if you do the row echelon form, this comes out to be this. So what is the rank again? Rank again is 2, nullity is again 1. See we have got 3 distinct eigenvalues, so rank of each null space is going to be 1 anyway.

It cannot be more right because each vector is going to be there and they are going to be linearly independent. So if you are getting something else then you are making a mistake in your computations. So again so for here you get X3 is=the second equation gives you X3=0 so that is=0. In the first equation, X1-X2+X3=0, X3 is given the arbitrary value, X3 has been determinant, X2 is given the arbitrary value, so X2, X3 are known, X1 can be computed from that right.

Because this is X3 is=0, so X3 is known and from the first equation X3 is known, X2 is given as the arbitrary value that is non-pivotal variable and X1 is determinant in terms of X2 and X3. So that gives you 1 1 0 okay. So once you have got the 3 eigenvectors right, you can write them as the columns, so that is your matrix P or X whichever way you want to call it okay, so the first eigenvector, second eigenvector and the third eigenvector okay.

As the columns, I think there is a mistake here right. This v3 this should be 0 here right, so once that is done, you check that X inverse AX is equal to this or simply AX is=X times diagonal. If you do not want to compute X inverse, you can just verify that okay. So it is clear? How do you find given a matrix to see if it is eigen variable or not, first step find eigenvalues, characteristic polynomial and see what are the roots.

If all roots are distinct, you are lucky, will find eigenvectors for each, write down the columns of the matrix as eigenvectors and you get the matrix P which is invertible which will diagonalize it. If not, you have to see whether you are getting as many independent eigenvectors. The nullity is same as the algebraic multiplicity of that eigenvalue. If that is the case, again you are lucky (()) (08:42) right. If not, it is not diagonalizable right okay.

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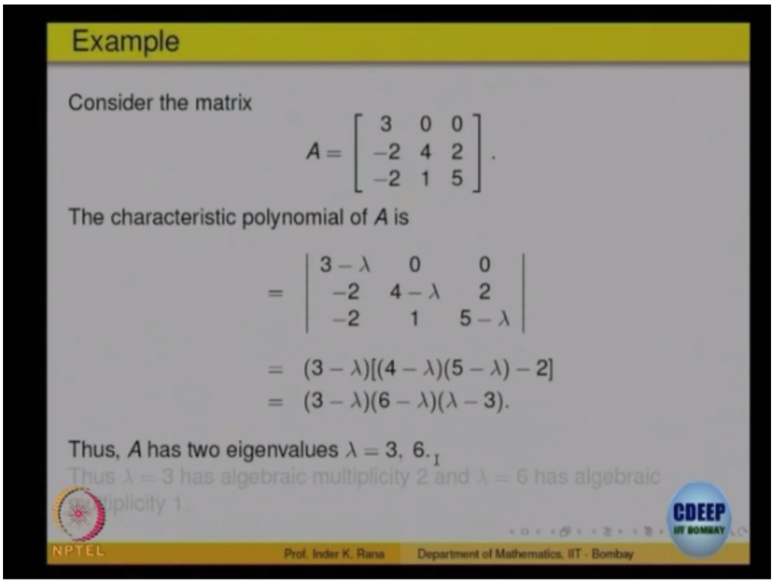

So another example okay. I want to do many examples, so that it is very clear what we are doing, so A is the matrix given like this 3 0 0 -2 4 2 and -2 1 5. So how you will write the characteristic polynomial? Diagonal entries becomes 3-lambda, 4-lambda and 5-lambda, everything else remains the same A-lambda I right. So write that, find determinant of that matrix okay, so you can expand by any row or column.

But because here are 0s coming, so it is good to expand it by the first row itself. So 3-lambda multiplied by, this goes, this multiplied by this, so that -2 right and simplify, it comes out to be this. So that means what? If that is the characteristic polynomial factorize, that means lambda=3 is an eigenvalue and lambda=6 is another eigenvalue. So this matrix has got only 2 eigenvalues, it is 3/3, it has got only 2 eigenvalues and the root lambda=3 is repeated right.

That means the eigenvalue lambda=3 has got algebraic multiplicity as 2. Lambda=6 has got algebraic multiplicity=1. Total of algebraic multiplicity has to be equal to the order anyway because they are roots of that degree right okay. So that has algebraic multiplicity 1. So for lambda=6 there is no problem because only one solution is going to be there which is going to span right, dimension is going to be 1, so you can do that process.

We will have to see for lambda=3 what happens? What is the null space? We have to find the geometric multiplicity of the eigenvalue lambda=3. If it is 2, then you are lucky, then you can diagonalize. If not, then it is not diagonalizing, so let us check what happens.

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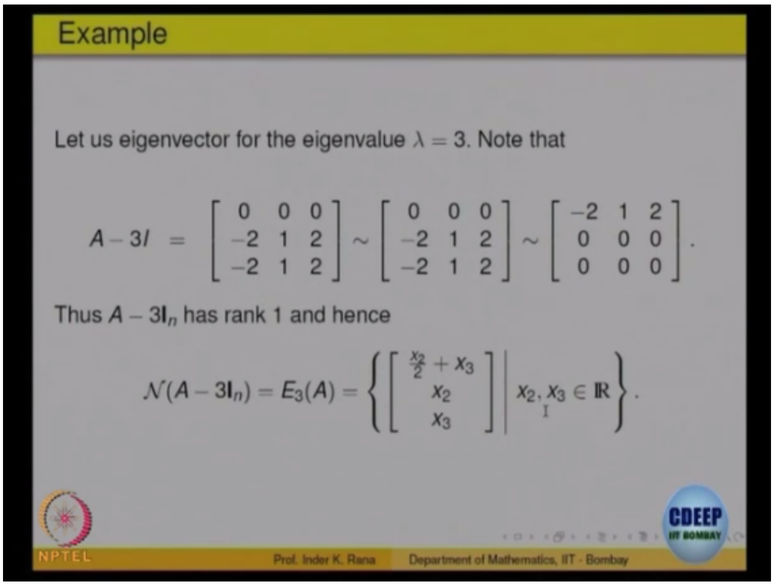

So lambda=3, so A-3I so that is this matrix and you reduce it to row echelon form that comes out to be this right. Again and again since beginning, the only thing which is playing a role is how to make a matrix in the row echelon form or reduced row echelon form. If you know that, there is nothing much in the course as such other than understanding the concepts okay. So what is the dimension of this? Sorry, what is the rank?

What is the rank of this matrix A-3I? Rank is 1 because only one nonzero row right. So that means what nullity is=3-1 that is 2, so we are lucky. Nullity is 2 that means we should be able to find two linearly independent eigenvectors for this eigenvalue lambda=3 because the reduced row echelon form or row echelon form says its nullity is 2 right. So how do you find that? What is the process of finding all solutions are linearly independent solutions for the null space?

What is the pivotal variable? 1 X1, X2 and X3 will get arbitrary values. So the idea is you give X2 and X3 arbitrary values, so that they become independent right. So one choice is give X2=1, X3=0 one choice and find X1 and the other choice is give X2 0 and X3=1 and find out X1 in terms of that. Those will automatically become linearly independent, so that is what we do. So rank is 1 okay. So what is the null space?

You can give X2 and X3 arbitrary values right that is what we said X2 and X3 arbitrary values and find X1 in terms of X2 and X3. So that is how you write, it consist of all vectors of this form where X2 and X3 are arbitrary right. That is writing the null space and to find the basis, so this gives you the dimension is=2.

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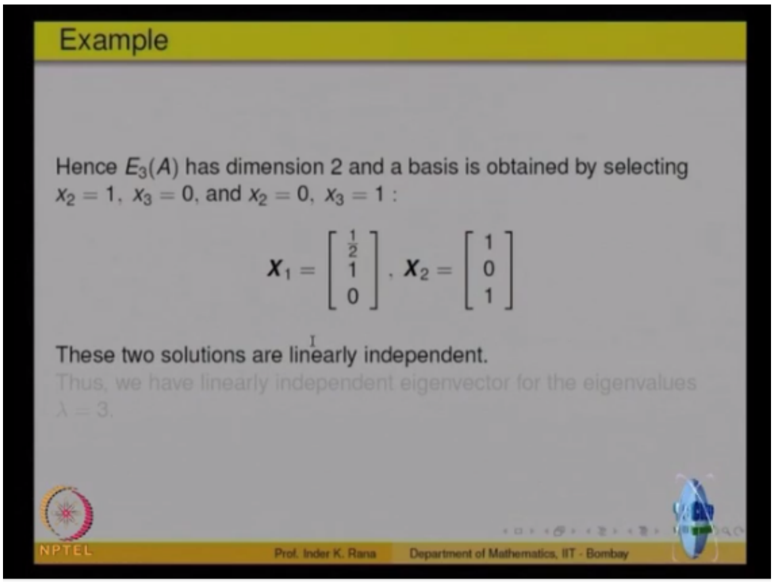

And find the basis, so you give first X2=1, X3=0 and second choice X2=0 and X3=1. So once you do that, you will get X1 and X2, you get two eigenvectors for the same eigenvalue lambda is=3 and they will automatically be linearly independent, you can just check anyway but it is not the issue at all because you have already done these kind of things, so these two are linearly independent.

So you have got two linearly independent eigenvectors for the same eigenvalue lambda. For lambda=6, we are going to get anyway one eigenvector, so you got 3 eigenvectors which are linearly independent, so matrix will be diagonalizable okay. So let us compute that.

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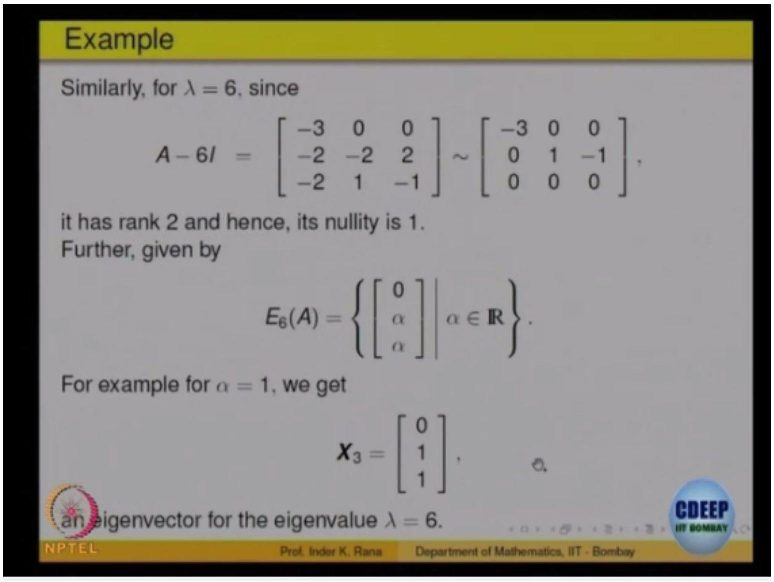

So A-6I, so that gives you this. So its rank is=2 as it should be because it is an eigenvalue and multiplicity 1, it cannot be more than 1. It has to be something at least 1 okay, so and what how will you get the solution? X2-X3 is =0, X3 will be given as the arbitrary value and X2 determinant in terms of that right and those two values when they put here you get the value of X1 but anyway X1 gives you X1 everything=0, so X1 is 0 anyway from there right.

The first equation gives you X1=0, so you get the eigen-subspace corresponding to that is 0, alpha alpha when you put that value okay. Alpha is equal to right, is it clear? X2-X3 is 0, so X2 is=X3 right. So you put the value=alpha you got X2=alpha, so you can make a choice, you can take any alpha you like right, the span right. So one eigenvector which will solve you, is one-dimensional so any vector nonzero vector will span, so will have to take alpha=1, so 0 1 1 is an eigenvector for the eigenvalue lambda=6 right.

So once that is obtained, we have got 3 eigenvectos corresponding to 2 eigenvalues but they are all linearly independent.

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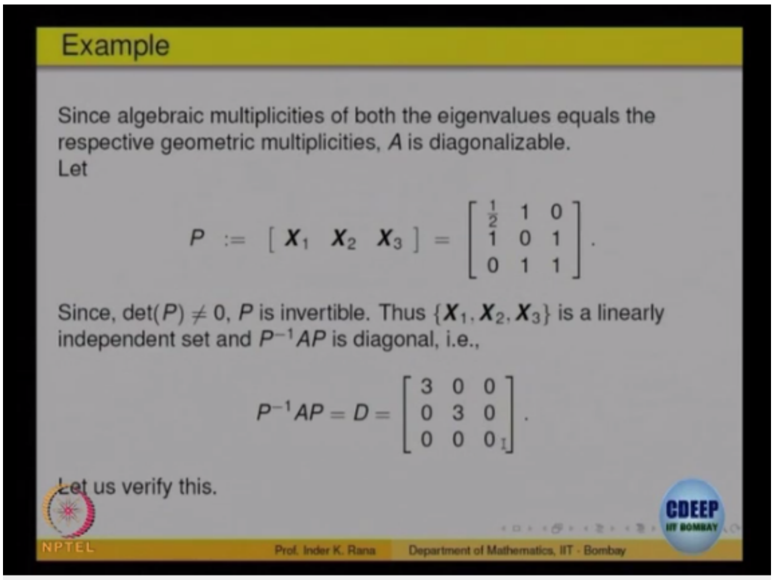

So we get the matrix the first one eigenvector, the second and the third, put them together you get the matrix P, determinant of this is not 0 because they are linearly independent, you can check also, so it is invertible and P inverse AP should be equal to diagonal, this is again a mistake here. This should be 3 3 n, there should be 6, the third eigenvalue right.

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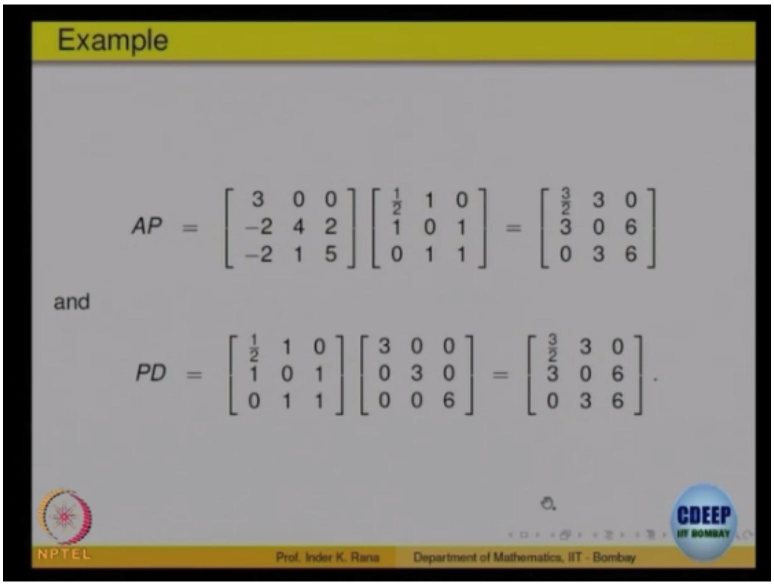

So check, so you can check by computing P inverse, you can compute P inverse AP and see it comes out diagonal or A*P is same as P*the diagonal matrix right 3 3 6, so either way you should check. So that is verification of that, actually you are getting a diagonal matrix. Is it clear diagonalization process? Everything clear to everybody? Yes, okay.