Basic Linear Algebra
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Lecture - 31
Diagonalization and Real Symmetric Matrices I

So let us begin today's lecture. We will start recalling what we have been doing. We have been looking at the definition of eigenvalues, eigenvectors and diagonalization of matrices.

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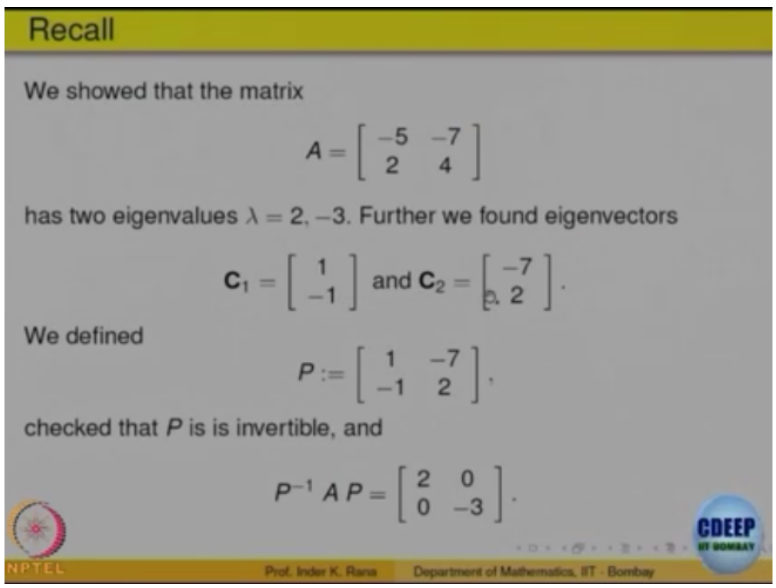

So let us just recall the example we have done. We had taken a matrix A as shown with first row -5, -7 and 2 and 4 and we showed that it has 2 Eigen values, lambda=2 and -3. For each one of this eigenvalues, we found eigenvectors. C1 is the eigenvector for the eigenvalue 2 and C2 is the eigenvector for the eigenvalue -3 and finding these eigenvectors is essentially solving a system of linear equations and this gave us the matrix P with first column as the first eigenvector.

Second column as the second eigenvector and this we showed that since these are eigenvectors corresponding to distinct eigenvalues, these 2 vectors are linearly independent and as a result, this matrix P has got full rank, so it is invertible and then we will check the property that P inverse AP is a diagonal matrix and the diagonal being the eigenvalues 2 and -3. So this was the illustration of how to find a matrix, which will diagonalise a given matrix A.

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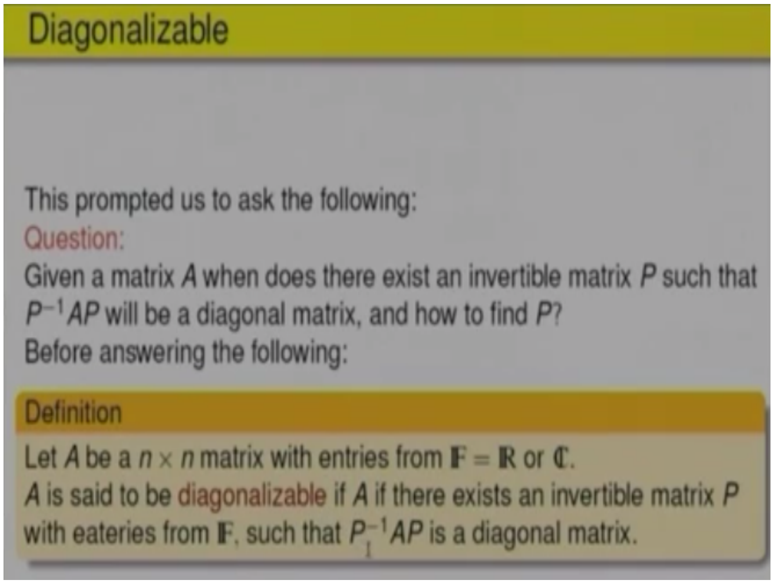

So this prompted us to define and ask the question, given a matrix A, when does there exist an invertible matrix P, such that P inverse AP is a diagonal matrix and how to find that P. So we made a definition that a matrix A is diagonalizable if there exists a matrix P, which is invertible such that P inverse AP is a diagonal matrix and we stated a theorem. Actually we proved it also partially, so let us just go through the proof again.

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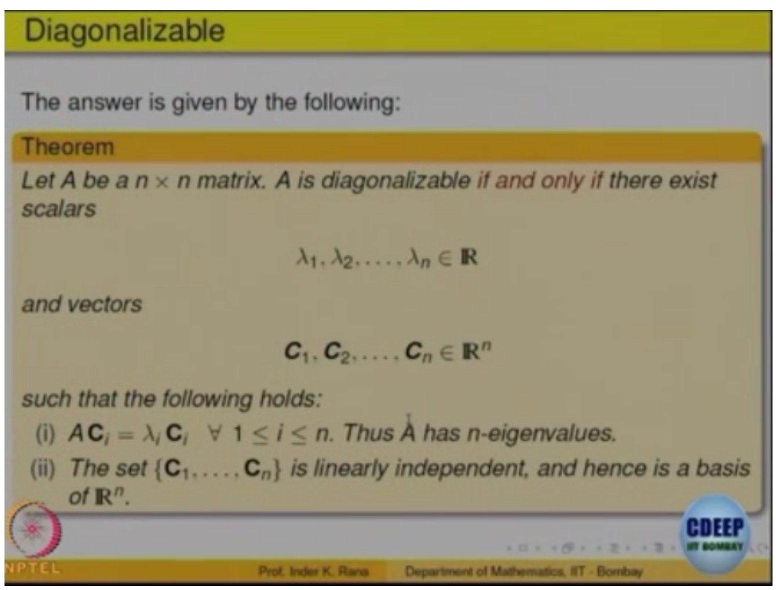

If A is nxn matrix, then A is diagonalizable if and only if, so this is an if and only if statement, there exist scalars lambda 1, lambda 2, lambda n and vectors C1, C2, Cn, such that the following holds 1, A applied to the vector Ci is lambda i Ci and this essentially says that the scalar lambda is an eigenvalue and Ci is an eigenvector for the eigenvalue and the diagonalizability is captured by the fact that the set of this eigenvectors C1, C2, Cn is a linearly independent set and hence form the basis for Rn.

There are n vectors, which are linearly independent and form a basis. So let us just run through the proof again.

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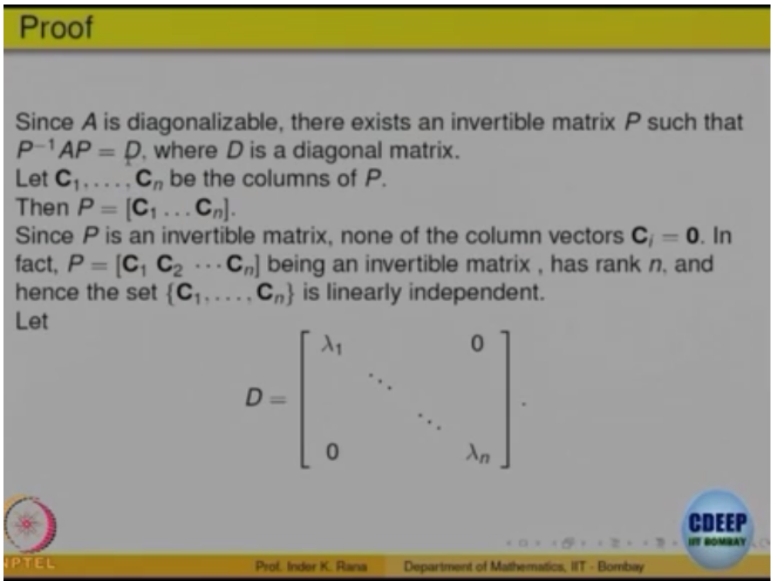

So suppose P is, A is diagonalizable, then there is a matrix P such that P inverse AP is the diagonal matrix. So there is an invertible P implies there is a matrix P with this property. So let us call C1, C2, Cn the columns of that matrix P, right. So P is given by C1, C2, Cn. Since P is invertible, then none of these vectors could be 0, because all the vectors are, they form a linearly independent set.

So none of them is 0 and in fact being invertible matrix as a full rank, so they are linearly independent. So let us define D to be the matrix with a diagonal entries as lambda 1, lambda n. Everything else is 0. So we are defining D as this and we want to check that these are the lambda 1, lambda 2, lambda n are eigenvalues. We want to check that. D is given to us. We are given P and we are given D. So D is given to us. This is D and C1, C2, Cn are the column vectors of P.

So we want to check that lambda 1, lambda 2, lambda n are the eigenvalues and C1 to Cn are the corresponding eigenvectors. So for that, we can also rewrite this equation P inverse AP=D, slightly differently.

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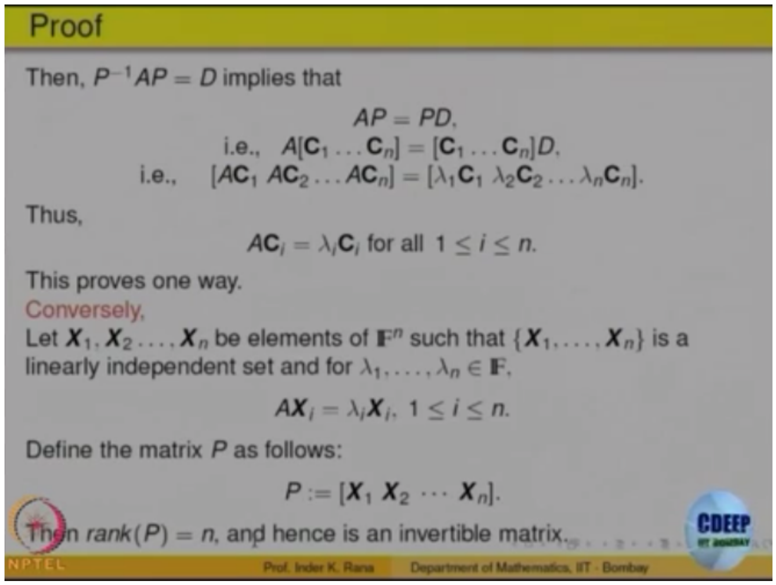

So the matrix multiplication, this implies if I multiply both on the left by P, then AP=P*D and let us write P and D in terms of the columns. So P is column as C1, C2, Cn. So this value is put here into D and what is this multiplication? That is precisely writing it as the columns being AC1, AC2, and ACn. This being the diagonal matrix is column where lambda 1C1, lambda 2C2, and so on, right. So this is just matrix multiplication, nothing more than that.

So this is equal to this, that means AC1=lambda 1C1, AC2 is lambda 2C2 and so on. So in general the jth entry ACj is lambda jCj and that precisely says that lambda j is an eigenvalue with Cj as the eigenvector. So one way is done. So conversely let us suppose that we have got elements, vectors X1, X2, Xn, says that is a linearly independent set and this lambda X are the corresponding eigenvalues. So AXi=lambda iXi, so that is given to us, write the property.

So what we want to show. We want to show that there is a matrix P where that P inverse AP is diagonal. So let us construct the matrix with these as the column vectors, right. So P is defined as, with these vectors as a column vectors, so these being linearly independent. So this is a linearly independent set, so this is invertible, right. So rank of P is n, it is an invertible matrix.

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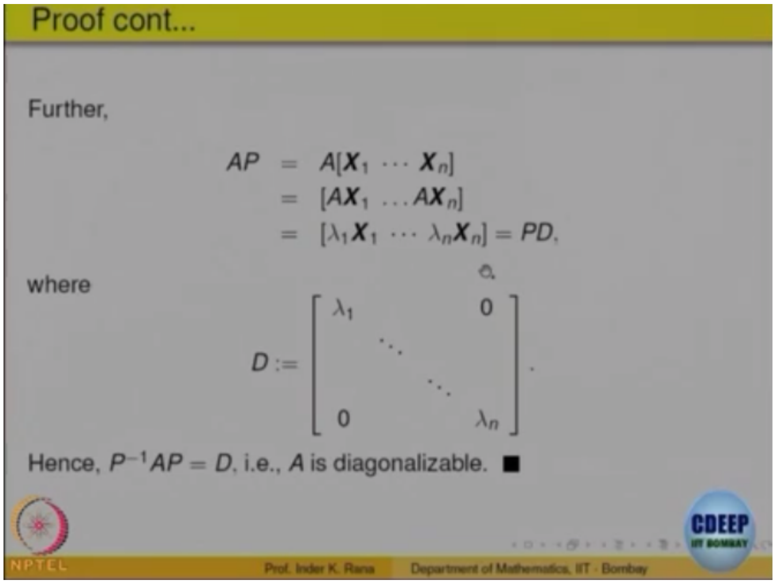

So we can compute now what is AP and that comes out to be right AP is right to the column A times the column vectors of P X1 up to Xn that is the same as AX1 to Xn and using the property that AXi is lambda iXi, this comes out and that is precisely P*D. So D, if you write D as a diagonal matrix lambda 1 to lambda n, then we get AP=P*D, P is invertible, so that can be written as P inverse AP, just writing.

There is nothing in the proof except that writing the matrix multiplication A*P as A* the column vectors and expressing it appropriately. So this proves the theorem that a matrix A is diagonalizable, if and only if there are n eigenvalues, right and there are eigenvectors corresponding to them form a linearly independent set. So that is the condition for diagonalization. So the question comes.

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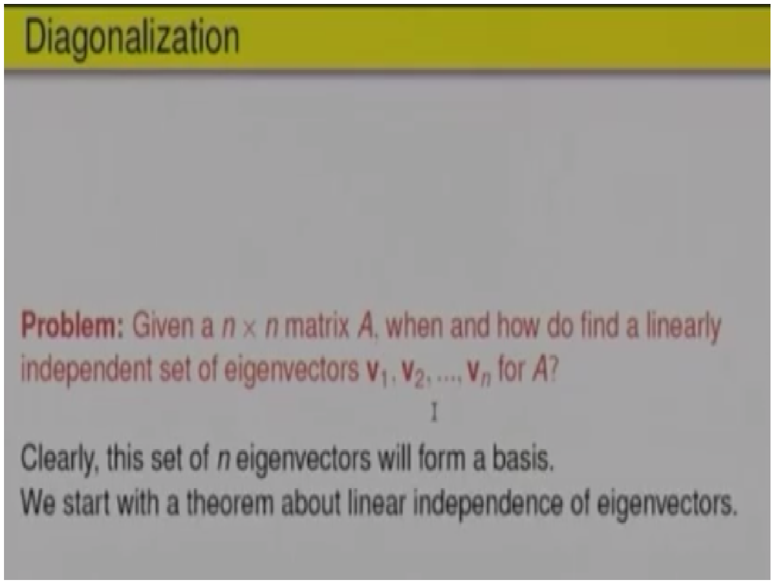

When they are linearly independent, everything is okay. So the problem is given nxn matrix, how do you find a linearly independent set of eigenvectors. So the question, does a given matrix have n eigenvalues 1 and for each eigenvalue, you will have an eigenvector with a form linearly independent set or not. If yes, the matrix is diagonalizable. If not, you cannot help it and it is not diagonalizable, right.

So basically what we are saying is, this n eigenvectors will form a basis, because this is an nxn, right. So saying that a matrix is diagonalizable is equivalent to saying finding n linearly independent eigenvectors for that matrix, right. Finding eigenvector means first you have to find the eigenvalues anyway. So that is a problem we want to take.

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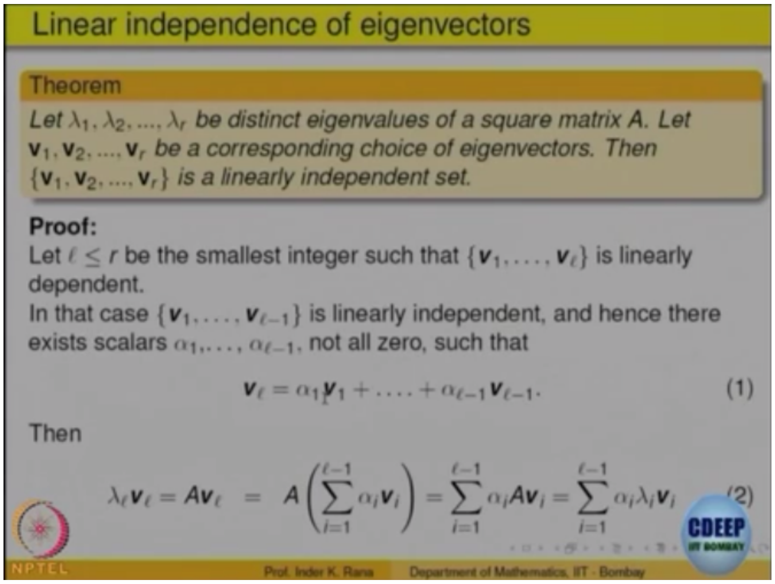

So first of all, there is an observation that supposing you have got lambda 1, lambda 2, lambda r are distinct eigenvalues of a square matrix. All the eigenvalues may not be distinct, right are the roots of the characteristic polynomial. So roots can repeat, right. So let us assume out of the n, lambda 1, lambda 2, lambda r are distinct eigenvalues of the given matrix and V1, V2, Vr are the corresponding eigenvectors, right.

Then the claim is this set is linearly independent. So what we are saying is even though all the eigenvalues need not be distinct, but if you collect the eigenvectors corresponding to distinct eigenvalues, they will form a linearly independent set, right. So at least there is some achievement. They may not be, all of them may not be distinct, but whatever are distinct, for each one of them, we find an eigenvector.

So that many linearly independent vectors we have got, okay. So let us prove this. So let us assume, see this number is r. There are r linearly independent, sorry r eigenvalues. So let us assume that l, there is a number l </= to this r, with the smallest integers is that V1, V2, Vl is linearly dependent set, right. Assume that there is a number l, such that V1, V2, Vl is linearly dependent, okay. Suppose such as thing exist, okay and let us pick up the smallest of them, right.

That means what, if l is the smallest, such that V1, V2, Vl is linearly dependent set, then lesser 1 will be independent, right. This is the smallest vector, which is linearly dependent. So if you take less, then that should be independent. So that means that, if you look at V1, V2, Vl-1, that is a linearly independent set, right. So that means what, so there is this scalars not all 0, such that, see this is linearly independent.

So Vl must be a linear combination of right, because that is the dependent. V1, V2, Vl is linearly dependent and V1, V2, Vl is independent. That means Vl must be a linear combination of the other 1. So let us write, so let us Vl be = alpha 1V1+alpha l-1 Vl-1, where all of them cannot be 0, right. Because if all of them are 0, then Vl will be 0, okay, that is an eigenvector. So eigenvector cannot be 0 right, Vl is eigenvector. So we assume that is an eigenvector.

So that cannot be 0. So at least 1 of them is non-zero. Alpha 1, alpha 2, alpha l-1, 1 of the scalars is not =0. So that is, so let us keep that. So let us multiply both sides by the scalar alpha l. Then what you will get. So alpha l Vl will be = lambda 1 alpha l, right, okay. So what is A (Vl)? See lambda l Vl, lambda l is an eigenvalue. So what is lambda Vl. That is =A(Vl), right? Because Vl is the eigenvector for the eigenvalue, right lambda l.

So this is because this is an eigenvalue and Vl is an eigenvector, but what is Vl from this equation, that is a linear combination. So we put this value here, okay and use linearity of A, so A*sum=sigma alpha iA(Vi), right. But A(Vi), again each Vi is again a eigenvector, so what is this equal to A(Vi), that is lambda i (Vi), right. Is it okay? So the first equation is because Vl is an eigenvalue and from 1, this value is put here, right A of the summation.

And then by linearity, this is sigma alpha iA(Vi) and each Vi being eigenvector, this is nothing but lambda i(Vi), okay. So let us keep this equation as 2. From this equation also, I can multiply both sides. If I multiply both sides by lambda l, what is lambda lVl? That is alpha 1 lambda lV1+alpha l-1 Vl-1, right. So I can multiply this equation also both sides by lambda l, I will get another value of lambda l Vl.

So these 2 must be equal then, after multiplication, right. So multiplying this by lambda Vl, it will get the next equation.

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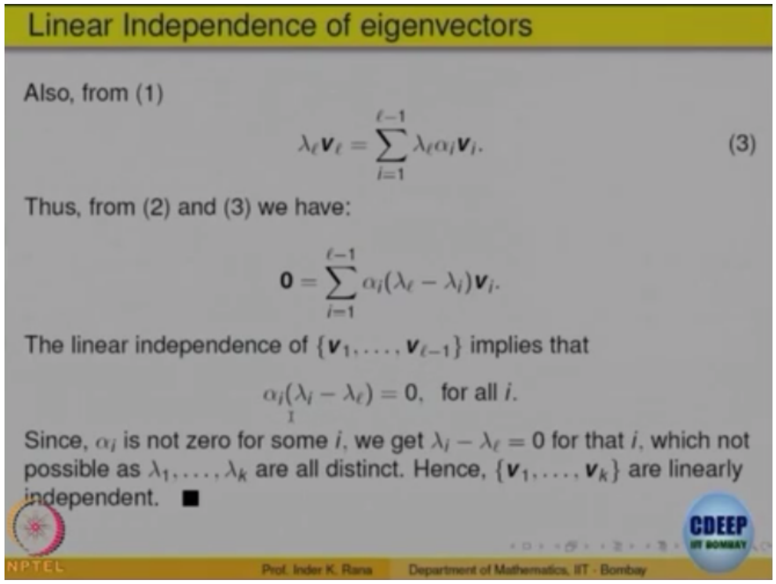

So lambda l Vl=that, so from the 2 and 3, if I subtract the 2 equations, right both give me value of lambda lVl, right. So this is lambda lVl as equation, this is 1 value and the other value is this, so subtract the 2, you get 0=alpha i, that is the scalar comes out lambda l-lambda i Vi, i=1 to l-1, right. Subtracting the 2 equations, that is all. Now I know that Vis are linearly independent and this is a linear combination, which is equal to 0. So that means what?

All the scalars must be equal to 0. So alpha i lambda l-lambda i must be equal to 0, but at least 1 of the alpha i is not equal to 0, right. That means for that particular i, lambda l-lambda i must be equal to 0. If that product has to be 0. That means for that particular i, lambda l will be equal to lambda i, but that is not possible because they are distinct. So our assumption must be wrong, right. So that is what we are saying. So by linear independence, all these scalars must be 0.

But for some alpha i is not 0, so that means for some i, lambda i must be equal to lambda l, which is a contraction, because all are given to be distinct. So that means we assume there is l, so that V1, V2, Vl is linearly dependent, that means that is nothing, does not exist. That means all of them are linearly independent.

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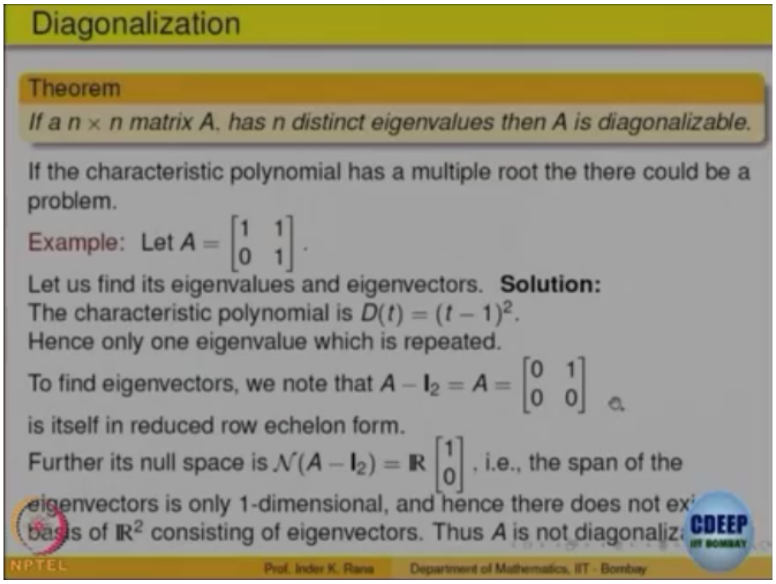

So the theorem says, that if lambda 1, lambda 2 are distinct eigenvalues of the square matrix, and you pick up and eigenvector for each 1 of this eigenvalues, V1, V2, Vr, then that is a linearly independent set. That means the eigenvectors corresponding to distinct eigenvalues are linearly independent, right. So that is it. So in case we have all eigenvalues are distinct, then you will get n linearly independent eigenvectors. You are very lucky, okay.

If not, then the problem can come. So the problem can come, if the characteristic polynomial has multiple roots, right, a root is repeated, right and for that particular eigenvalue, you may be able to find 1 or 2 eigenvectors. We do not know, right. So let us look at an example. So let us look at the example of matrix A. 1, 1, 0, 1, right. So how do find the eigenvalues? This is A, A-lambda i, determinate=0. So that will give you 1-lambda, 1, 0, 1-lambda.

Determinant of that will be t-1 square, if t is the variable you are introducing. If lambda, lambda-1, so that means what, so there is only 1 eigenvalue, right and that has got, it is repeating, right. So there is 1 eigenvalue t=1 and that is repeating, okay. So let us find the eigenvector correspond to that eigenvalue. So how do you find the eigenvector? So look at the eigenvalue, what is the eigenvalue, t=1. So A-lambda i, right. So what is A-lambda i? 1-1 that becomes 0, okay.

1-1 that becomes 0, so everything is 0 except the second entry in the first row, right. So that A-lambda i for lambda=1. So what, this itself is in reduced row echelon form. We do not have to do anything, right. So what is the rank of this matrix. We want to solve, homogenous system A-lambda i applied to X=0. We have to find solutions, to find eigenvectors for that, right. So for that, we have to know what is the rank of this. So what is the rank of this matrix?

That is 1, right. So rank is 1, so what is null t, that is equal to 1, rank+null t=the dimension, right. So null t means what, for the homogenous system, that is the dimension of the solution space. So dimension is 1. That means there is only 1 linearly independent eigenvector for the eigenvalue lambda=1, right. We can find that. So how do you find that? So this applied X1, X2=0, right. So what does it mean? You can give X to any value, right 1, what is X1.

You can give it 0. So you get 1 0 as solution for this space, right. So this is 1 dimensional, so this solution space of A-lambda i, lambda V1 is of dimension 1 and you get a vector for this, which spans this one dimensional. So 1 0, right that is one of the vectors. So there is only 1 eigenvalue and that is repeated, but there is only 1 eigenvector. So that means what, I cannot find a basis of R2 consisting of eigenvectors. So what does the previous theorem tell me.

This matrix is not diagonalizable, right, because there is only 1 eigenvalue and for that eigenvalue, lambda=1, I can find only 1 linearly independent eigenvector because the solution space has got dimension 1, right. So it depends on for a root, if it is repeated, can I find that many linearly independent eigenvectors for it or not, right. Essentially is boils down to that. So let us keep some names to these things.

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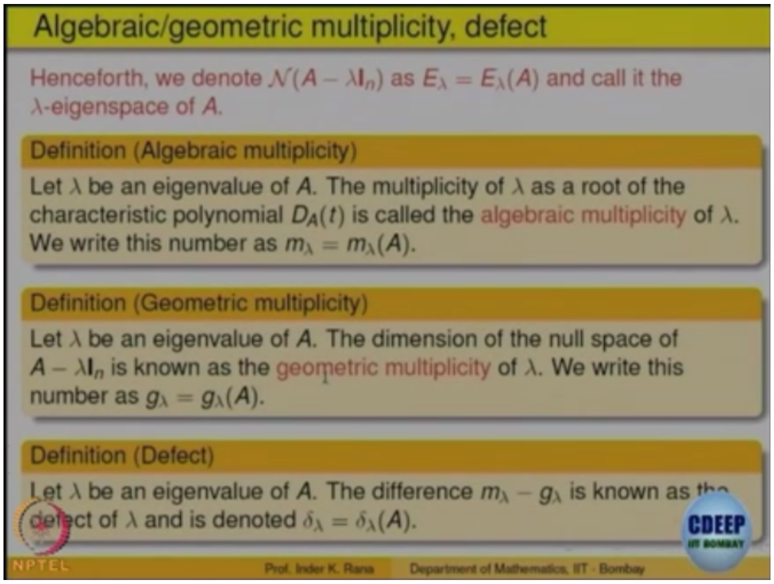

So for the null space of A-lambda i that is solution space, we will start denoting it as E lambda, that is Eigen-subspace for the eigenvalue lambda, right. All the vectors, which form a solution for A-lambda i=; applied X=0, right. So that is null space of that matrix. We have different names, okay. So what is, so we will start calling if lambda is an eigenvalue, that means it is a root of the characteristic polynomial and root may be repeated.

So the number of times it is repeated, that is called the algebraic multiplicity of that eigenvalue. So as a root of the characteristic polynomial, it will be at least appearing once, because it is a root. It may appear twice, thrice or so many times, so number of times it appears, that is called algebraic multiplicity of the eigenvalue, right and we denote it by M lambda. So that is M lambda or M lambda of A depending on if you want to stress, that is the matrix A.

So that is the algebraic multiplicity. Now on the other hand, for this eigenvalue, we are interested in eigenvectors, right. So how many eigenvectors exist and how many exists? That depends on the dimension of the null space. So that we call it as geometric multiplicity of the eigenvalue. So what is geometric multiplicity? That is the dimension of the null space of A-lambda In. So that is same as the nullity of this matrix. Nullity gives the dimension, so that is called the geometric multiplicity and geometric multiplicity is written g lower lambda or g lambda A.

So g for geometric multiplicity, M lambda for algebraic multiplicity, right. So when do you think a matrix will be diagonalizable? There may be roots, which are repeated, right, but if you count with multiplicity, there are N roots for a matrix of order nxn. Some of the roots may be repeated, right, but for the roots, which are repeated, there are number of times it is repeated is algebraic multiplicity. Dimension of the null space of A-lambda A is geometric multiplicity.

If as many independent you can find as algebraic multiplicity, then you are through, right. then you will have total number of eigenvectors will be linearly independent and they will be n, right. So if it is less as happened in the previous example, lambda=1 was an eigenvalue of multiplicity 2, algebraic, but geometric it was only 1. So that was lot of the defect in diagonalizing the matrix, right. So sometimes this number is called the defect for the matrix to be diagonalizable.

If both are equal, then it will become diagonalizable, right. Because for each eigenvalue, we will have as many eigenvector as is the algebraic multiplicity. So put them together, you will have a linearly independent basis, linearly independent set of eigenvectors forming a basis. So we can diagonalize that, okay. So that is what, so this is what normally called the defect, but that is not really important, okay, what you call that difference, okay.

Algebraic multiplicity will always be bigger than geometric multiplicity. Is that clear? Yes. The number of times root is repeated algebraically that many eigenvectors, you may not be able to find. You will be able to find only less of them, because they are independent.

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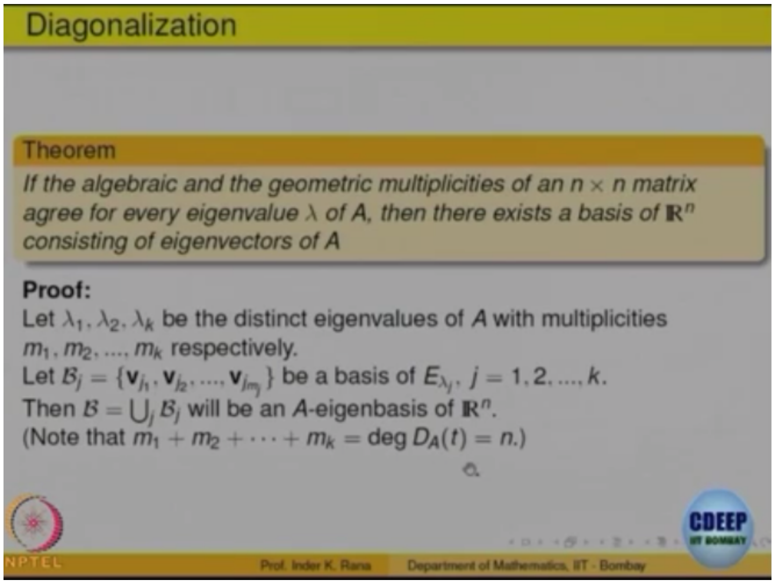

So theorem says if the algebraic and the geometric multiplicity of a matrix agree, for every value eigenvalue lambda, then the matrix is; then there is a basis consisting of eigenvectors and hence the matrix is diagonalizable, right. So it is quite clear what is the proof. Basically, for each if they are distinct eigenvalues with multiplicity M1, M2, and Mk, right for each we can find a basis right of as many as the algebraic multiplicity. So find the basis and put all the basis together.

For each eigenvalue, there is eigenspace, there is a basis, which has as many eigenvectors as is the algebraic multiplicity, if they are equal. Put them together, you will get a basis consisting of eigenvectors for the underlying space Rn, okay. So their sum is equal to the degree of that is equal to R, right. Algebraic multiplicity equal to geometric multiplicity, total=n, so for each one of them, there is a basis, put them together. So there is nothing complicated in this theorem.

Basically, that is how you will analyse whether a matrix is diagonalizable or not. So the main problem is given a matrix to show it is diagonalizable or not, what we have to do? We have to find, for the given matrix, what are the eigenvalues. Once you have the eigenvalues, find out corresponding eigenvectors, right. Find out the dimension of the corresponding eigenspace that is same as the null space of A-lambda i, right.

If each is equal to the algebraic multiplicity dimensionality of A-lambda i=the algebraic multiplicity, then we are through, then we can find a basis consisting of eigenvectors, right.

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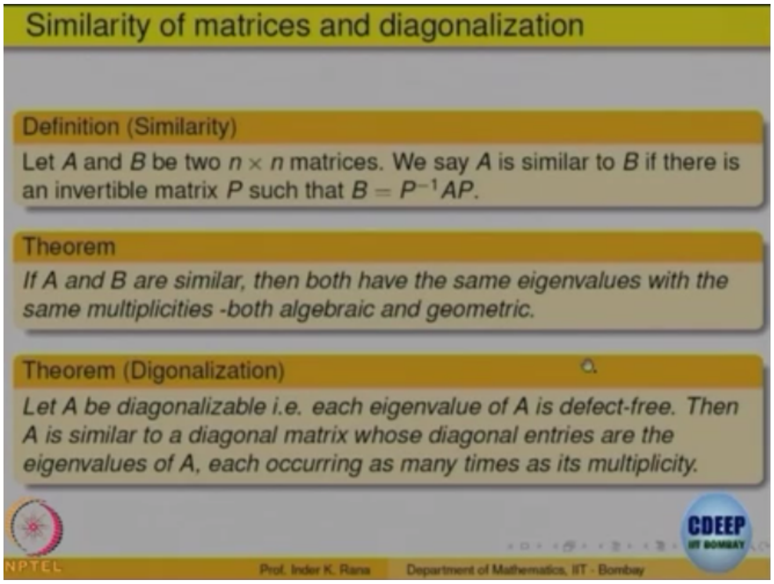

These are just rewording the things. We say 2 matrices A and B are similar if A can be obtained from B or B can be obtained from A by this process. B=P inverse AP, where P is invertible, right. So in terms of this the diagonalizability can be said as a matrix, any matrix A is similar to a diagonal matrix, if and only if, the algebraic multiplicities are equal to the geometric matrix and we can find that. So that is also the diagonalization process.

So we are rewriting in terms of the terminology of 1 or 2 matrices said to be similar because that useful later on in the other context also. So for us, it says that a matrix A is diagonalizable or similar to a diagonal matrix if and only if you can find an eigen basis. That means find a basis consisting of eigenvectors and that is same as, for each eigenvalue, the algebraic multiplicity is same as the geometric multiplicity.