Basic Linear Algebra
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Lecture- 28
Isometries, Eigenvalues and Eigen Vectors-I

Okay so let us begin with today's lecture. We will start recalling what we had done last time. We looked at what is called the Orthonormal basis of a vector space.

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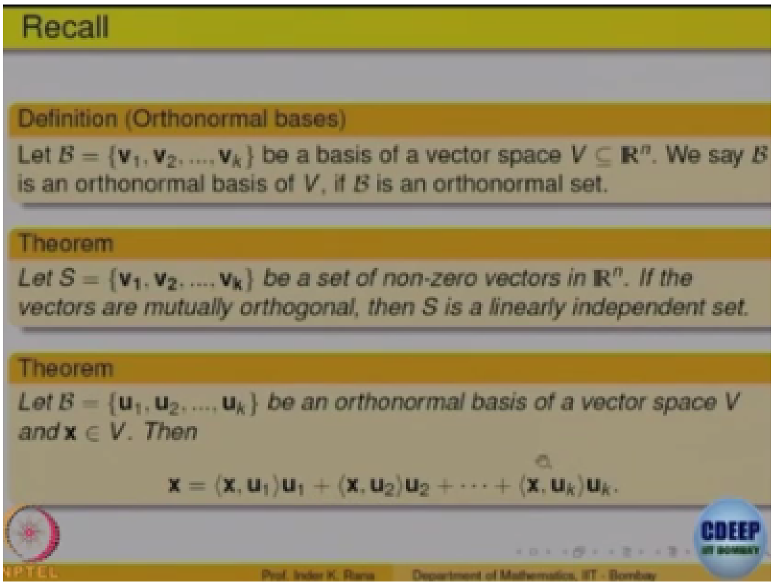

And then we looked at orthonormal basis was defined as a basis such that it is a non-zero vector such that mutually orthogonal and form a basis. So orthonormal basis is a set of vectors which is a basis and any 2 are mutually orthogonal. And then we prove the result that if a set of vector is orthogonal and of course none of them is 0 then that is linearly independent.

The advantage of orthonormal basis was that given any vector X in the vector space you can immediately write down those scalars alpha 1, alpha 2, alpha n such that X is a linear combination of bases elements. So those scalars come out to be the dot product of the inner product of x with u1, x with u2 and so on. So coordinate of a vector are immediately known once you have orthonormal basis. So that was the advantage.

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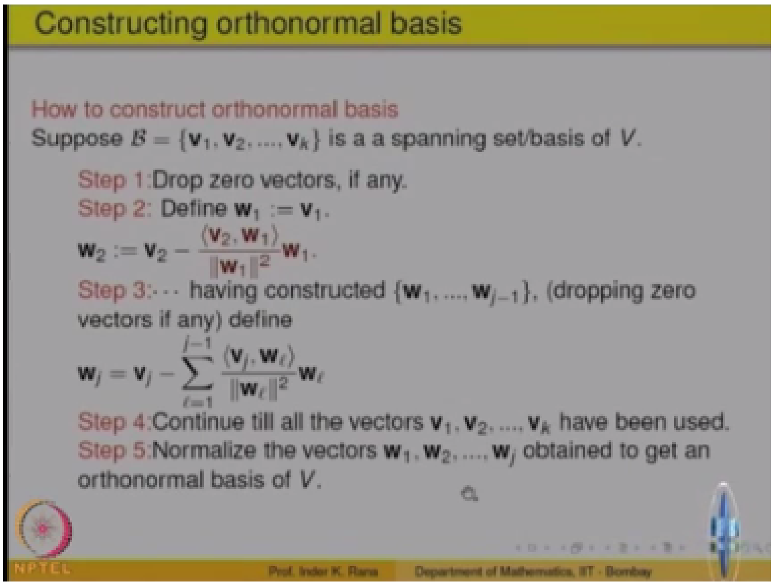

And then we saw the process of constructing orthonormal basis. So given set v1, v2, vk if his basis well and good even if not if it is a spanning set that is good enough we can construct a set of vectors which is orthonormal and will give the same span as that of v1, v2 and vk. So the first step is if there are any zero vectors in this drop them because anyway they are not going to contribute anything in spanning.

And next we define by inductively w1 to be v1 and define w2 to be the next vector v2- the projection of v2 on w1. So that projection is removed so that the difference is orthogonal to w1 so that is a next stage v2. At every stage we go on removing the projection on to the previous ones which I have defined. So having defined w1, w2, wj-1 and if there are any zero we drop them. So assuming that none of them is zero once we have reached the stage.

Then look at the next vector which is vj from the given set and look at this projections of vj on each one of the previous ones all this are removed will get a vector wj which is perpendicular which is orthogonal to all previous ones. So continue this process still we have finished all the vectors v1, v2 vk. Once we have finished we will get a set of vectors some w1, w2, wj which will be orthogonal right.

None of them will be zero and which will span the same space at that of this. Once we have got on that you normalize them to form a orthonormal basis. So that is a process of constructing a orthonormal basis from a given point and we have looked at examples of that. So we will continue with the further ideas.

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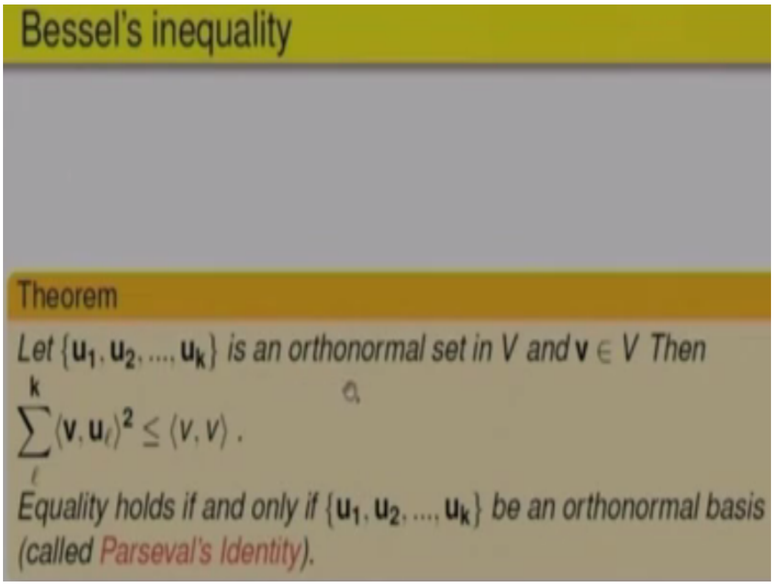

We also proved what is called Bessel's Inequality which said that if you are given a orthonormal set not necessarily a basis then the coefficient v ul square (()) (04:27) is always < or = norm of u square and equality holds there if and only if this form a orthonormal basis. So that is what is called Parseval's identity. So Bessel's Inequality becomes equality if and only if the given set of orthonormal vectors is a basis also.

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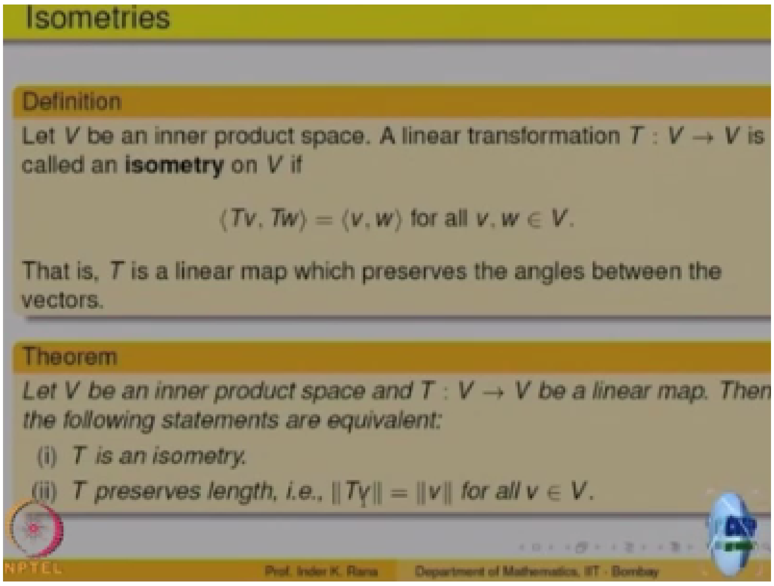

Now let us look at if we recall we looked at linear transformations as the map from one vector space to another R2 to R2 or R2 to other things such that they preserve co linearity they takes lines-to-lines. Now on a vector space we also have a notion of inner products because there all vectors spaces are subsets of some Rn. So there is a notion of dot products so we would like to specialize those linear transformations which not only preserve co linearity also preserve angles they do not change the angles.

So such things are called Isometry. So a linear transformation from v to v is called an Isometry if the inner product Tv and Tw is same as the inner product of the original. So it preserves the inner product and we have seen that inner product is relative to the notion of angle and distance. So it is not surprising that we have a theorem namely T is an Isometry if and only if it preserves the length also. So here we said it preserves the inner product so inner product is related basically to angles.

So it says T is an Isometry if and only if it preserves the notion of distance also the magnitude of vector is kept intact. So an Isometry takes lines-to-lines right preserve angles as well as preserves the magnitudes so they are sort of the perfect transformation in the planes. Example when you physically do something physically you move an object the shape of the object does not change right.

That means all the straight lines remains straight lines angles remain same right and distance remain same. So those are basically called the rigid motion of (()) (07:03). So this is coming from that translation is not a linear transformation, but still it is a rigid motion okay. So this is how they arise algebraically Isometry or is a map from the vector space v to v such that it preserves of course the angle that is given.

And we are saying that it is equivalent to saying that it preserves also the magnitude and this is not very difficult to show basically how is the magnitude related to the inner product that is square root of the v dot v right that is a magnitude. So using that one can easily prove.

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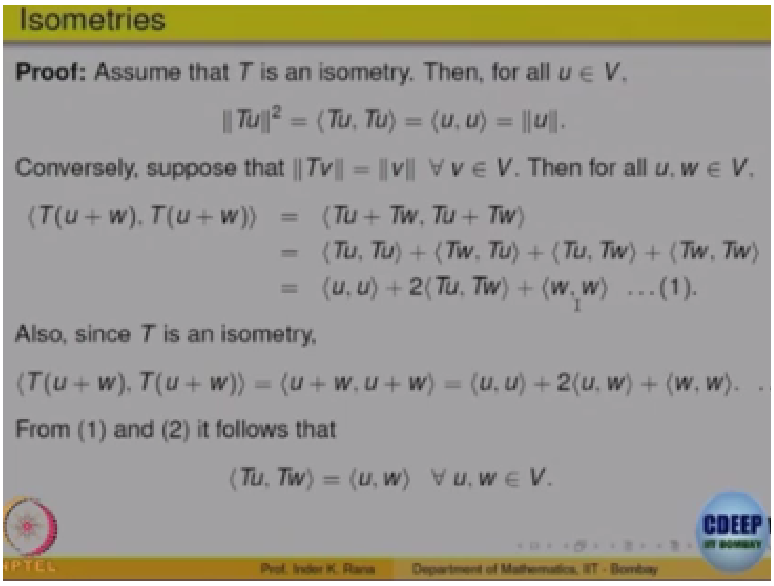

So let us say first T is an Isometry that means it preserves inner product then look at the norm of Tu square by definition it is Tu, Tu inner product, but T preserves inner product so that is same as u, u inner product of u, u. So if T is an Isometry if it preserves inner product then it also preserves this should be square here u, u that should be Tu square so it preserves the distance magnitude also.

So that is one way conversely let us suppose T has the property that it preserves the magnitudes. We want to show it also preserves the angles it also preserves inner product. So for that let us look at T of u + take any 2 vectors u and w look at the sum and look at the image of Tu+ w and it is inner product with itself. Now we will T is linear so this will be Tu + Tw this also will be Tu + Tw use the property of the dot product with linear in both expand we will get these 4 terms right.

So 2 of them are common so it gives you u, u inner product + 2 times Tu, Tw and ww inner product, but T is an Isometry right. Since okay now T is an Isometry one should actually, but what is this = Tu, Tw right so that is= expand that. So from these 2 what you get see norm is preserved right so what you get. So from 1 and 2 this is see what we are assuming is Tv preserves the notion of distance right.

So norm of Tv square is same as so this is what is a norm of Tv square right so that =this so this is norm of u square this is norm of w square. So once you expand that you will get Tu this is anyways it is simple I think let me I think there is some mistake in this proof what I have written, try to prove it yourself okay because I think there is some what we have given is suppose this is true okay.

Then what do we want to prove that Tu, Tv right this is= this is what we want to prove right for any given. So if I take this then this is 4 terms that is okay, but this is norm of u square and this is norm of w square. So this is norm of so this side gives you norm of Tu T of u+ w square= norm of u square + norm of Tu Tv, Tu and norm of (()) (11:13) so what do we get. So let us just see if we look at T of u+ w, T of u+ w.

So that comes out to be= uu + 2 times Tu, Tw + ww right.

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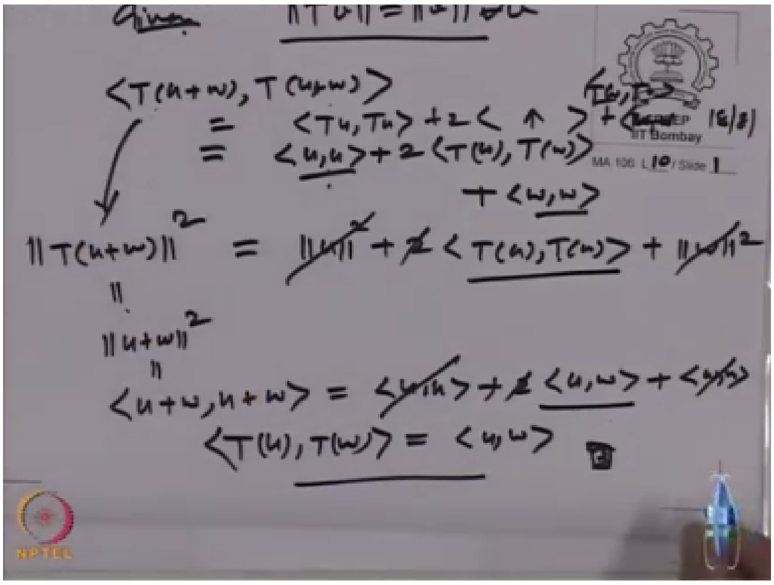

So what we are given is that norm of Tv= norm of v for every v so that is given to that. So what is this, this is norm of u square + 2 Tu, Tw+ norm of w square. And what is this, this is norm of Tu+ right this one is T of w same vector with itself so that is norm square right. So this is= this, this is= this. So what is this= by this given property that is norm of u+ w square right again by the given property.

So what is that= I can write as u+ w right is it okay norm square. So what is that= that is uu+ one will give you u, wu and other will be that is 2 times uw+ ww right 4 terms. Now this is same as this so what happens this cancels with this, this cancels with this, this 2 cancels right. So what you get is Tu, Tw= uw right. So that is what we wanted to prove is it clear. So assuming that the magnitude is preserved we just look at dot product of T of u+w. u+w T of that right and expand and you get the required thing okay.

So that is what is it clear to prove now. See the first step okay so that I have just not written the earlier thing that was= Tu Tw right. So when you expand this will be Tu, Tu+ 2 times this thing right + Tww right T of w Tw, but that this is= this okay and that Tww= this by the given norm of that that is basically norm of Tw square so that is= same as this okay. On the other hand, if you expand use that property given property and expand you get these 2 equations and that gives you this is= this.

Expanding 2 different ways that is all nothing more than that okay. So that shows that if T is an Isometry that we preserve inner products then it also preserves the magnitude of the vectors okay.

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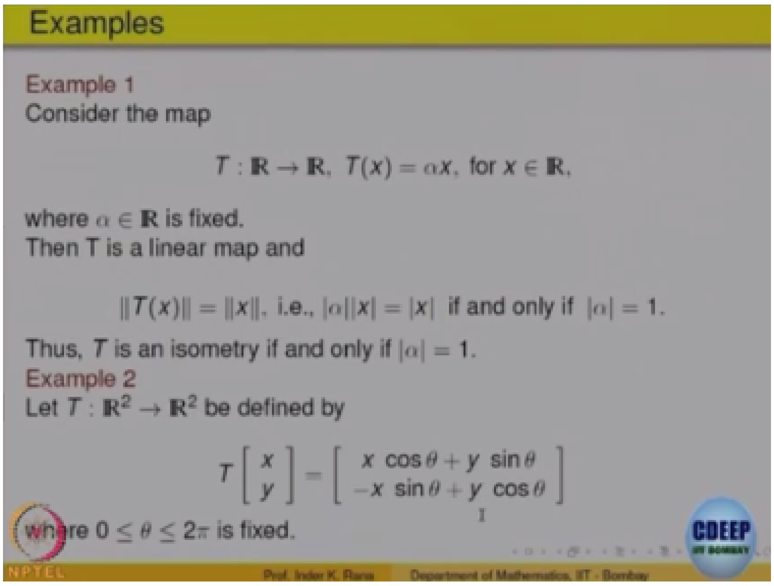

So let us look at some examples which are quite obvious which we normally use actually in rigid motion what is called. So consider the map from R to R, R is the vector space over itself right additions, scalar, multiplications. So look at alpha times that is a magnification in R. So the claim is this is a linear map obviously right if T of x+y is alpha times x+ y so T is a linear map okay. Claim is that it is a we are going to find out for what alpha it is an Isometry.

So if it is an Isometry what should happen norm of Tx must be= norm of x, but what is Tx that is alpha x so you get a mod alpha times right absolute value of x must be= absolute value of x so that we want for all x right. So that will happen if and only if mod alpha =1. So the simple example at scalar multiplication by alpha is Isometry if and only mod alpha is it has to preserve distance right so it cannot be anything else.

Let us look at another one in plane R2 to R2 so T of xy is x cos theta+ y sin theta- x sin theta + do you recognize this we had looked at this example when we looked at matrix multiplication as a linear transformation right. It is matrix multiplication by a matrix. What is that matrix cos theta sin theta right- sin theta cos theta. So if I look at where theta is fixed we want to check whether it is an Isometry or not.

So let us look at T of xy= this thing right. So let us compute what is the norm of this magnitude of this vector. So what is the magnitude of this vector x cos theta+ y sin theta square + -x sin theta+ y cos theta whole thing square that is a norm square.

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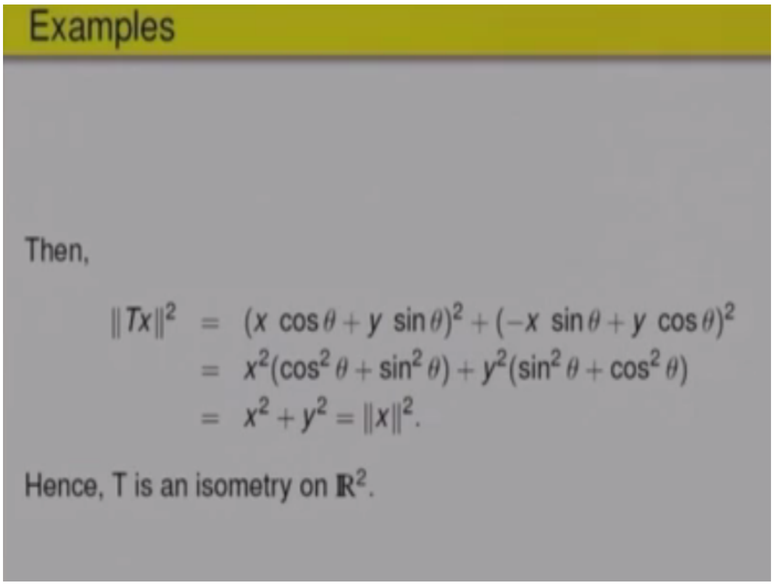

So let us compute that and simplify. The usual properties of sin theta cos theta gives you that norm of Tx square= norm of x square that means T is an Isometry. So that condition that preserving inner product= preserving distance or magnitude you can use either of them to prove something in Isometry or not. So here computing the distance is easy norm is easy so we use that property and that as I said was a rotation right.

So rotation does not change the magnitude of something right. So as expected so that is an Isometry. So rotation in R2, R3 also is actually is Isometry right. What about translations or not linear so we cannot call them Isometry, but they are called rigid motions. What about reflection in R2. We take a line and reflect as if that is a mirror every point is reflecting not necessarily x axis or y axis but any line.

So physically it should not change right angle or distance right so you can try that. What should be called as a reflection, how do you define a reflection in R2 to R2 what should be the formula for that right, what would be the formula for reflection against a line. It will depend on what is that line the line is at angle theta. So the formula will come out in terms of theta again okay. So see in R2 xy goes to its reflection against a line of slope theta angle theta.

What should be the coordinate of the image point? You can take it as a good exercise and then show that it is an Isometry okay. Physically it looks okay it should be it does not change angle, it does not change distance magnitudes right so try that.

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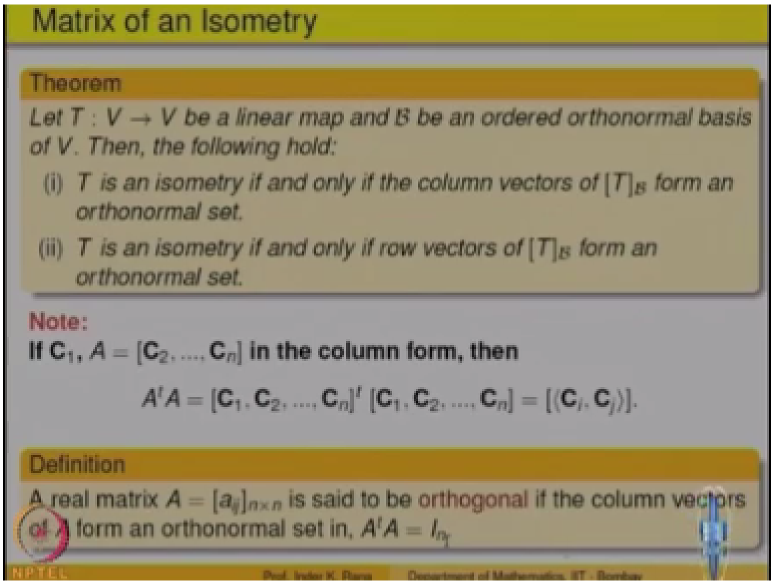

So here is a once we have said that we have got a inner product space means what is a inner product space vector space on which inner product is there so all subspace of Rn once we take the inner product also it minds that becomes a inner product space right. So T is linear and B is an ordered orthonormal basis. Earlier we looked at ordered basis and looked at matrix of that right.

Now there is angle also available you can have a basis which is orthonormal. So T is a linear transformation from V to V and we have fixed an ordered orthonormal basis on V not only a basis that also is orthonormal any 2 vectors are perpendicular to each other and norm is 1. Let us compute the matrix of that and the important thing is if T is an Isometry right if and only if the column vector of that matrix of T forms a orthonormal set.

So given any linear transformation given an ordered basis you will get a matrix so you will get the column vectors right. Here if the starting basis is orthonormal right and if T is Isometry one can show that the column vectors are actually orthonormal. They are perpendicular to each other and norm=1 so that is a special property of linear transformation when you get their matrix with respect to ordered orthonormal basis right. So we would not write the proof of that, but it is a very nice property.

And another one is the row vector also form y only right column vector or row vectors also form okay an orthonormal. So it says that if T is a linear transformation from one vector space to another you take a ordered orthonormal basis of V look at the matrix corresponding to that then T is an Isometry if and only if the column vector are orthonormal and equivalently row vectors are also orthonormal right.

So let us observe we will assume this theorem, but let us seek what are the consequences. So let us write the matrix say its column this is some extra thing has come. A being column vectors are C1, C2 oh C1 should have been here actually okay typo. So let us A is a matrix whose column vectors are C1, C2, Cn right and matrix can be written as the column only. Then what is A transpose A?

Column transpose will give you the proof right. So what is this? This is just C, Cij, Ci transpose, Cj column multiplied by the row multiplied by column right multiplication. So that means A transpose A is just Ci, Cj inner product. Remember what was the inner product in terms of vector rotation A dot B inner product was A transpose right B as if you multiply as vectors right multiplications as vectors.

So here if you multiply so that is a dot product and if these are orthogonal what will happen (()) (23:47) 0 or 1 depending on i=j or not. So that says that if T is an Isometry right and if you write its matrix with respect to ordered orthonormal basis then it has a property that A transpose A= 0 if I is not= that means what it is identity matrix. What is this matrix then if these are orthogonal it is 0 when I is not= j n=1 if i=j that is identity matrix right.

So for an Isometry its matrix representation with respect to ordered orthonormal basis has the property that A transpose A=identity right. So let us give that as a name matrices which have that property A transpose A is identity let us call them a name. We say that a matrix is called orthogonal if the column vectors form A right from a orthonormal set in (()) (24:59) that is same as saying A transpose A= identity right.

So matrices which have that property square matrices which has the property A transpose A is identity we will call them as orthogonal matrices. So the matrix of an Isometry is a orthogonal matrix right. So you can write it as this theorem that T is an Isometry if and only it is matrices is a orthogonal matrix right. But saying the row vectors that is same as saying not only A transpose A is identity A transpose also is identity right both are same.

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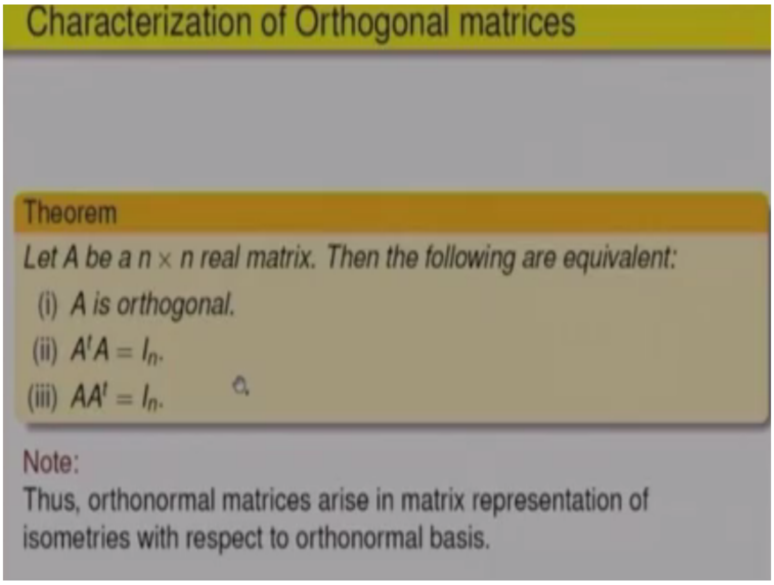

So this is an equivalent way of saying that a matrix A is orthogonal if and only if this is a definition right, but you can take the transpose of this what is a transpose of this it will be precisely A transpose, transpose right that is precise with this. So saying that a matrix is orthogonal is equivalent as A transpose A=identity or AA transpose that is same as saying A transpose is the inverse.

The transpose of the matrix is itself is an inverse of the matrix. So if a matrix is orthogonal it is invertible as a consequence obviously right and it inverse is the transpose. So they become very nice to compute the inverse you do not have to go to determinant or adjoin or anything row echelon form nothing just take the transpose you get the inverse of the matrix we have a very special matrices right.

And they arise as matrix representation of Isometry right which preserves conversion of angle and distance. So that is what we are saying that orthonormal matrices arise in matrix representation of Isometry with respect to orthonormal basis right.