Basic Linear Algebra
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Lecture - 25
Orthonormal Basis and Geometry in the Euclidean Plane- I

Let us begin todays lecture by recalling.

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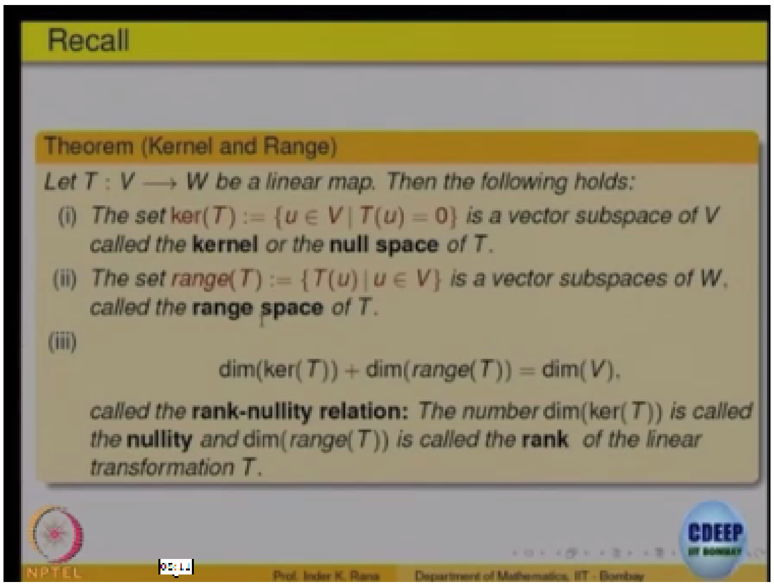

What we have done last time so given a linear transformation over vector space free T to W we define what is called the kernel of that linear transformation so that is all vectors in the domain which go to 0 vector whose image is 0 that is called the kernel or the null space of the linear transformation then we defined what is called the range so the range is the image type so T U .U belonging to domain V so that is called the range space.

And we showed that both of them are sub spaces not only that they form a relation called that rank nullity relation. The dimension of the kernel that is the dimension of the null space so the kernel is sub space of the domain + the dimension of the range of T which is range of T is the sub space of the co-domain so if you add this 2 together you get the dimension of V the domain space so that is the rank nullity theorem.

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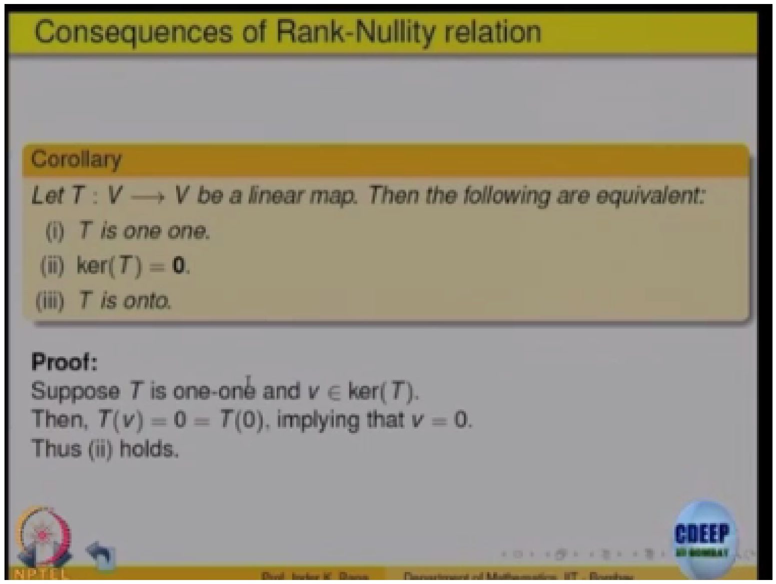

And as the consequence of that we stated this theorem namely Ft is the linear map from V to V from the same space to itself then the following statements are equivalent namely T is one- one. It takes different vectors to different vectors the kernel is 0 space no other element in the kernel other than 0 and T is onto. So, as rank nullity theorem what we are saying is a linear transformation of T is one-one if and only T is on to.

One of the property is good enough to say rather for the functions it is not true in general for linear transformation V to V that is true and the proof is quite easy let us say T is one-one that means different elements go to different elements so if element V belongs to kernel of T then we know that T of V is 0 so that is same as T of V = 0 so that implies by one-one property that V should be 0 so 1 implies 2.

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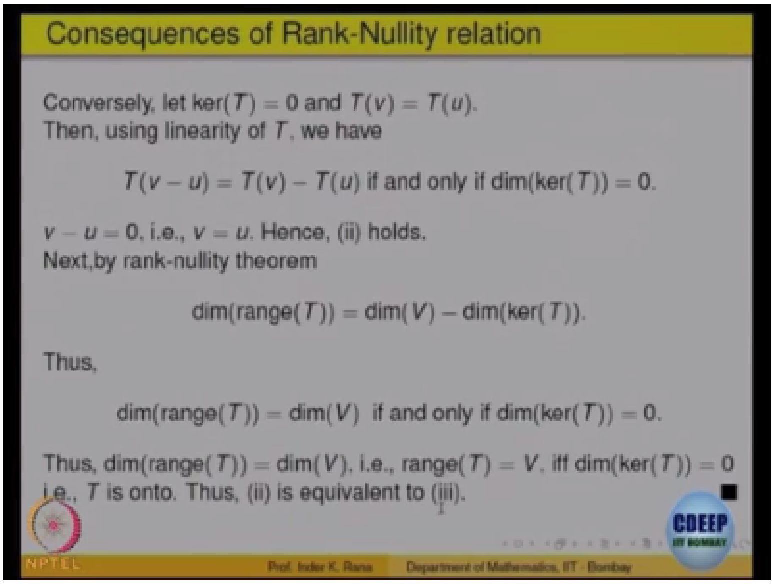

Conversely you can show 2 implies 1 namely look at the kernel of T and if it is 0 and we want to show it is one-one and so expose T of U = T of V then you can bring it on this side and use it as linearity property so that says T of UV is = 0 that means that belongs to V- U belongs to the kernel that means V = U because kernel is only 0 so V-U must be 0 so V=U so 1= 2 2 is = to 3 is a direct consequence of the rank nullity theorem.

By rank nullity theorem dimension of the range is = dimension of V – dimension of the kernel so if this is 0 that will happen if and only when dimension of the range = dimension of V so dimension of kernel of T will be 0 if and only if these 2 are equal but this is a sub space so that means range = the full space so that is what it says so dimension of range = dimension of V if and only if kernel is 0 that says 2 is = 3.

So, saying a linear transformation is one-one is equivalent to saying it is also onto 2 and vice versa that is very useful property.

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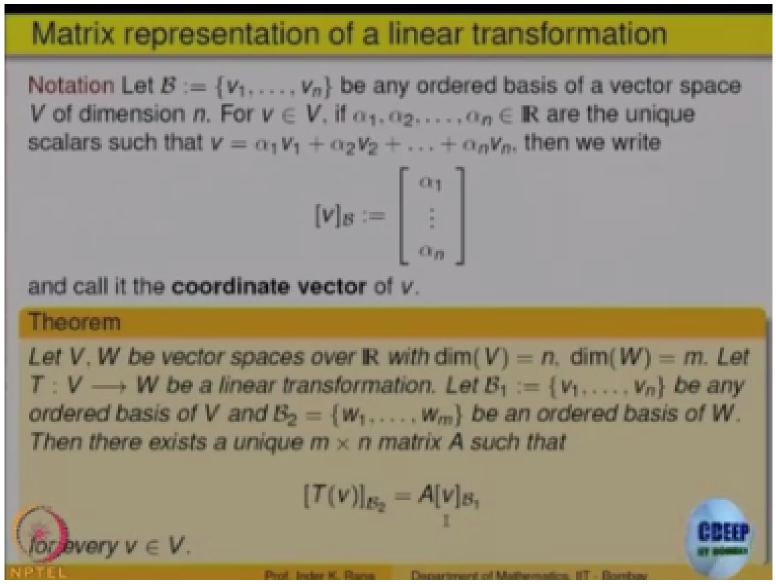

Later on we then looked at what is called the matrix of the linear transformation so we started looking at given a basis V1 to Vn to fix the order then this is called an order basis of the vector space V for any vector V in V we know because this is a basis so there must be a unique vector alpha1 alpha2 scalar alpha1 alpha2 alpha n say that V is a linear combination so these unique scalars are alpha 1 to alpha n.

If you put them as a column vector then this is called a co-ordinate vector of the vector V either called as co-ordinates and how much you have to go in the direction of V1 we have to go alpha 1 in the direction of V2 we have to go to alpha 2 and so on and so these are called as co-ordinates of the vector V so the theorem says the following suppose T is the linear transformation from V to W and we fix an order basis for V and fix a order basis for W.

So, B1 is an order basis for the domain and B2 is an order basis for co-domain that is W so given this ordered basis for the domain and the co-domain fix and keep in mind V is dimension n and W is dimension of m so their claim is there exists a unique matrix order m cross n m is the dimension of W and n is the dimension of V so there is a unique matrix A of order m cross n says that if you take a vector V take its image.

So, that will be n W look at its co-ordinate vector it is same as take the co-ordinate vector of the original vector V so that is the co-ordinate vector V and multiplied by the matrix so that is essentially saying that T of V can be obtained as matrix multiplication. The beginning viewers lecture you have seen every matrix multiplication will give you linear transformation this is other way around this gives you association between linear transformations and matrices studying 1 = studying other 1.

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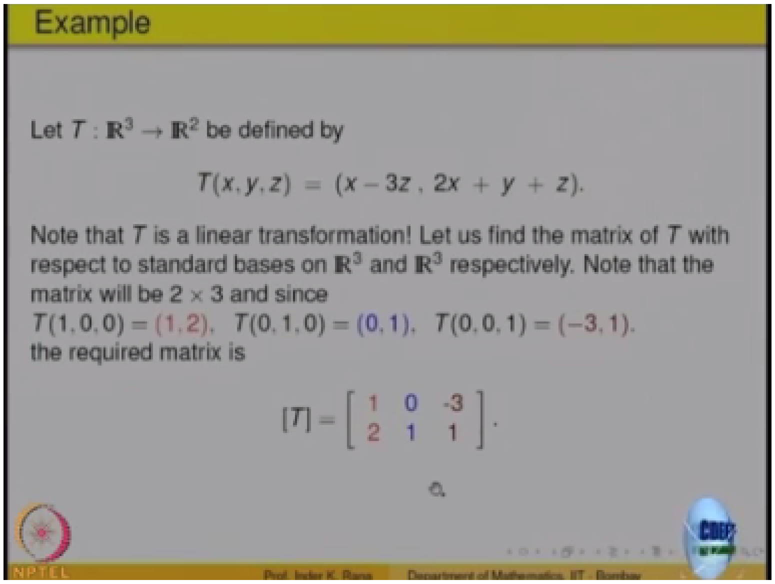

So, we looked at example last time so look at linear transformation T from R3 to R2 so this is given by T of x y z and the element in R3 the image is x-3z 2z+y+z it is easy to check that this is the linear transformation let us fix the basis on R3 and basis on R2 and try to find out what is matrix of this linear transformation so let us fix a standard basis on the domain as well as on the right so on R3 what is the standard basis that is 100 010 1001 that ijk.

Similarly, i and j on R2 so let us take the first vector in domain R3 the basis vector is 100 and find out its image according to the formula that is the definition so x =1 y = 0 z=0 in this that gives you T of 100 is 12 similarly T of 010 by this definition is 01 and T of 001 is -3 1 so what is the matrix so for the V1 for first vector whatever you get that is the first column this is the second column and that is the third column.

So, that gives you the matrix 1 2 so this comes as the first column and this comes as the second column and this comes as the third column so that is the matrix of the linear transformation.

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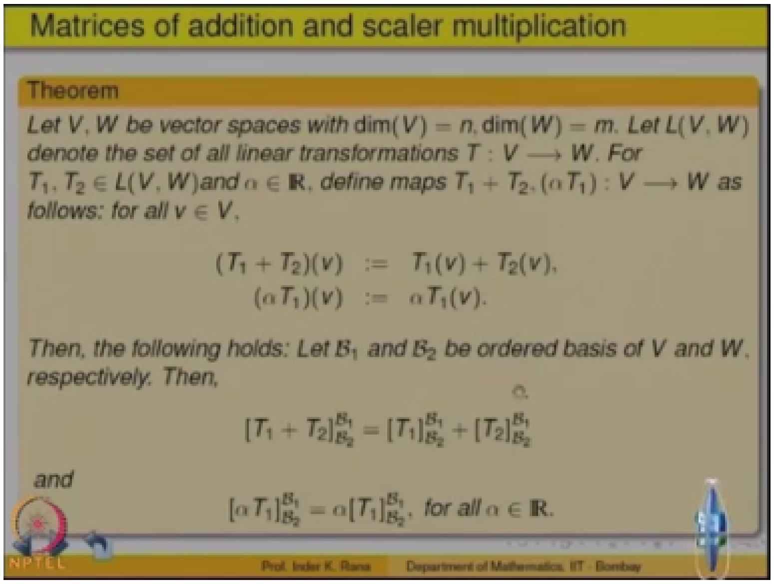

There is some simple properties which will not prove but we can use them so the given the basis on the domain and co-domain you can write on a matrix the question is if I take two linear transformations and add them what is the matrix of that transformation and what is the matrix of scalar multiple of a linear transformation so this gives the relation it says supposing T1 and T2 two linear transformation L W VW means set up all linear transformation from V to W.

That is a set so if T1 and T2 are linear transformations then the claim is T1 +T2 which is defined as T1+ T2 of vector B is T1 of V + T2 of V like functions real value functions alpha of T1 is defined as alpha times T1 of V so given T1 and T2 you have to find the addition of linear transformation this is scalar multiple of linear transformation the claim is both of them are again the linear transformation.

And if we fix a basis on V and W then look at the sum of the linear transformation T1+T2 find out its matrix with respect to B1 on domain B2 on co-domain it is same as taking the matrix and finding the matrices this of the individual one and adding them so just says addition its matrix is same as finding of each individual and adding them up the same for scalar multiple one can write and proof which is very easy but will not go into the proof of this. Let us try to see some illustration of this

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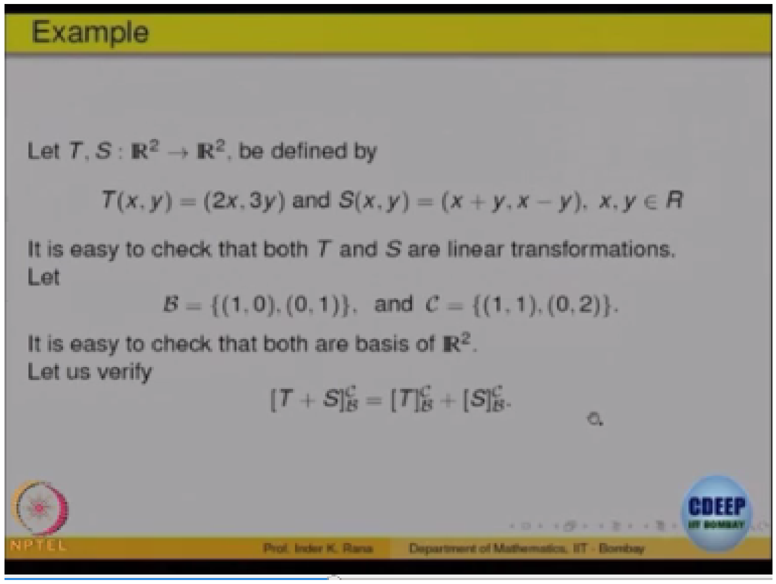

Okay let us look at this two transformations T is given by and T is from R2 to R2 okay T of x y =2x ,3y another linear transformation is x y is x+y x-y and they are fixing the basis on 1 0 on the domain and fixing the basis C on the co-domain fixing a different basis and want to see want to verify all those relations so what are the relations we want to verify that if I take the sum find out the matrix of T +S with respect to C on the domain and B on the co-domain.

Then this addition of this let us compute and verify try to see whether it is okay or not right and this will also gives you an idea of how to compute things.

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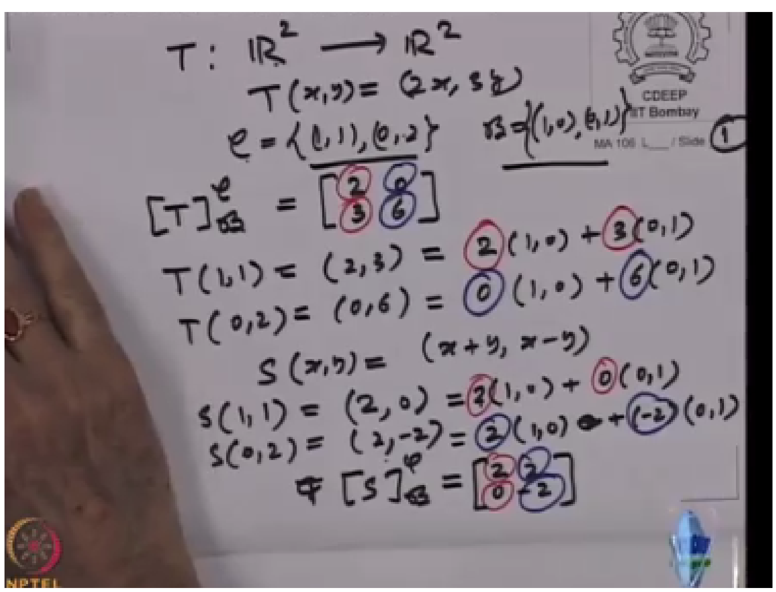

So, let us look at T is from R2 to R2 T of xy= 2x 3y and what is the basis there in domain that is given as C the basis is 1 1 0 2 and the basis on the co-domain is given by B which is the standard basis 1 0 0 1 so this C is the basis on R2 which is the domain and B is the basis on the co-domain so we want to find what is the matrix of T with respect to C from C to B what do I do I look at the basis of the domain this is V1 V2.

So, let us find what is T of 1 st that is 11 and similarly T of other one 0 2 so what is of T 11 that is gives me 2 3 and T of 0 2 so this is 0 and this is 2 and so that gives you 0 6 now this is to be written as linear combination elements of T of something goes in R2 in the co-domain where the basis is this B so to write it as a linear transformation of elements of B so what is the linear combination of 2 3 it is 2 times 1 0 + 3 times 0 1.

Similarly this one is 0 times 1 0 + 6 times 0 1. Is that okay? So, what is the matrix so this matrix comes out to be what is the 1 st row 2 3 and the 1 st column is 2 3 2 nd column is 0 6 so that is the matrix so let us find the matrix of linear transformations is what was S defined as x of y x x + y x – y same process so what is S of 1 1 and what is S of 0 2 this two are the elements in the domain basis elements of the domain.

So, 1 st vector 2 nd vector what is S of 1 1 x +y so that is 2 x-y that is 0 what is S of 0 2 so 2 -2 so what is this = so this 2 0 so this is 2 times 1 0 + 0 times the basis there is 1 0 0 1 so I should write it as a linear combination so that is 0 1 and this is = 2 times 1 0 - let us write + -2 times 0 1 clear so take a element in the domain basis element in the domain and its image according to the expression that is given.

Write it as a linear combination of the elements of the basis in the co domain so what are the vectors so that means our matrix S with respect to C and B = 1 st row or 1 st column 2 0 write 2 nd one is 2 - 2 okay is it clear to everybody what I am saying? So this is 2 and this is 3 so that terms as 2 and 3 here. Here the 1 st one is 2 0 and this comes out 2 0 and if you want to write 2 nd one that is 0 and 6 and so that is 0 and 6 here it is 2 and -2 so 2 and -2 .

So, that is how you write the matrix properly in the linear transformation given a basis order basis in the domain an order basis in the co domain so i have computed for T. I have computed for S you as an exercise I will leave you to try and write down what is T + S so what is T + S.

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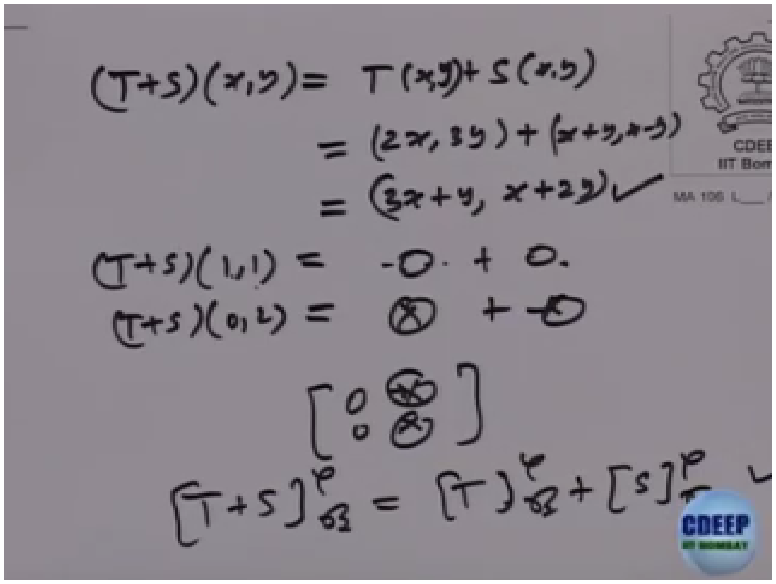

T+S of x y so that is T of x and T of xy + S of x y so what is T of x y that we know is = 2x 3y so it is 2x 3y + and S of x y is x+y x-y so that means it is = so it is 3x+y and x+2y once we have got that we have to find T +S of the basis element so 1 1 for C what is that? What is T +S of second element is 0 2 find out this as a linear combination of the basis so find out that so you will get some vector here and some vector here +.

Some vector here + some vector here so what will be the 1 st row and 2 nd will be I am not doing that computation I want to do it yourself so according to this formula compute what is T of S write this as a linear combination of elements in the co domain that gives the rows and then verify that was T + S C B is same as T of C B those two we have already computed as S of C and B so verify that so that is the verification of the theorem it is not proving the theorem.

We are just verifying but in the process it is quite clear how to find the basis of given an order basis in the domain and the order basis in co-domain how to find a matrix of that linear transformation and then you can easily verify this also so that is the process required to verify let us go a step further not only for addition and scalar multiplication it holds and this also holds for composition say in functions it can compose 1 function with another one.

If the range of 1 is in the domain of the other and same is possible here.

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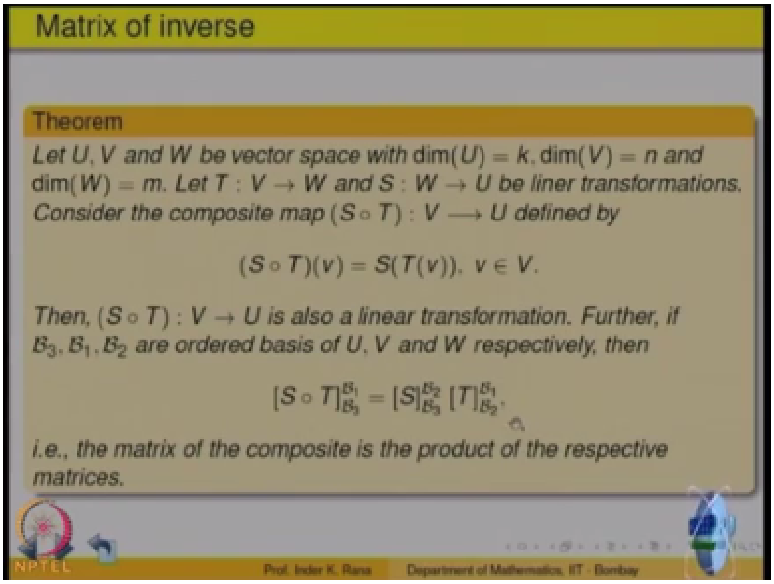

For example let us take 3 vectors basically U V and W the dimension are dimension of U is k and dimension of V is n and dimension of W is m but different dimension different vector spaces possible now I have got linear transformation from V to W and another transformation from W to U so what I can do is i can take a vector V by T it will go to vector TV in W and under S that vector will go to S of T V which is in U.

So, that is the composition of 2 linear transformation like composing maps so let us define what is called the composition as composite T from V to U so this is T here T 1 st applies to V so that what you will get T of V that will be a element in W compute the image of that under S that is S of T of V that is the element in U so this is called the composite of the linear transformation and so this is the map from V to U right from first to the last one.

The claim is that is again the linear transformation composite of linear transformation is again a linear transformation now what we want to do is find out a relation supposing on V I fix a basis and W I fix the basis and on U i fix the basis then I can find out what is the matrix of T I can find out the matrix of S I can also find out the matrix of composite the theorem says the matrix of the composite as composite T whether it is B1 and B3.

B1 is the first one and B3 on the last one is same as find the matrix of S with respect to B2 and B3 and multiple with matrix of T with B1 and B2 so the matrix multiplication that we defined in the funny way in the beginning to take this row and take this column and add and you will get the i th entry that actually comes because of this reason. For composite that is how it should be actually.

So, we will not again prove this theorem but note 2 of computing things for the composite B1 B3. B1 is on V B3 is on U and when you compute the matrix of that here S is from W to U B2 B3 B2 is the basis here and B3 is the basis there so it is B2 B3. T is the matrix is with respect to B1 and B2 in the composition what is happening is these two are assign something like B2 is getting moved.

So for this is S composite T it is B1 and B3 okay again we will not prove this theorem but you can compute examples you can take the previous example S and you can take as T and you can compose and find out the matrix of the dimension and see what is the product of the matrix but it gives a very nice supposing this S and T are related they are inverse of each other then what do you get this will be the identity map.

And this will be saying that identity matrix = product of these 2 that means that gives you the inverse of the matrix of the linear transformation the star way is T is one- one or onto and you know it is matrix then you can conclude the matrix of the inverse by just inverting the matrix we do not have to do the computation again.

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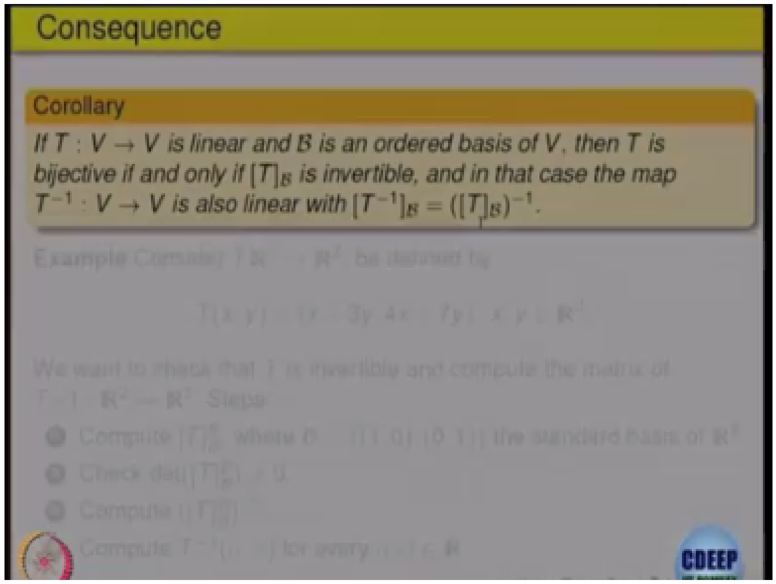

So let us write it as the consequence of the theorem namely if T is the linear transformation V2 to V and B is the order basis and T is bijective that is same as one-one and onto if and only if its matrix is invertible and the matrix of the inverse transformation is same as inverse of the matrix proof of interchange with each other.