Basic Linear Algebra
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Lecture - 24
Linear Transformations-III

Right. So this is what is called the Rank nullity theorem. I once of all write dash and it looks like minus. It is Rank + nullity=N right rank nullity theorem.

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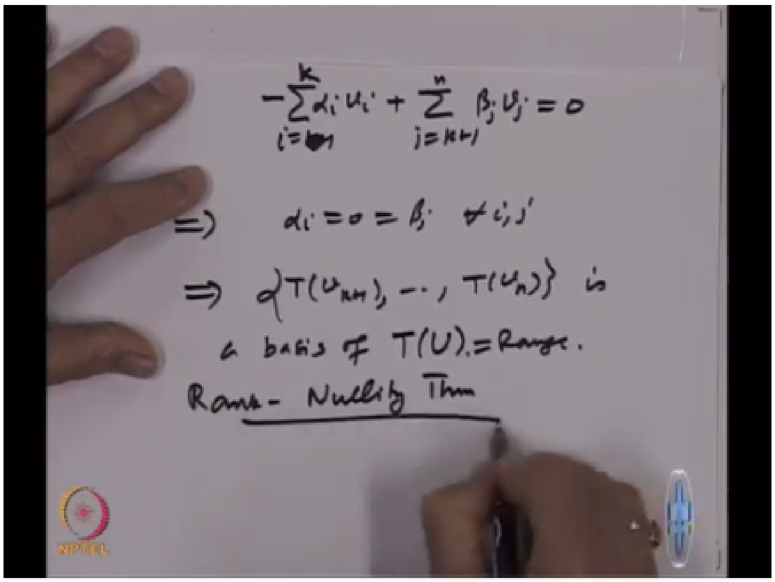

So this is the last one.

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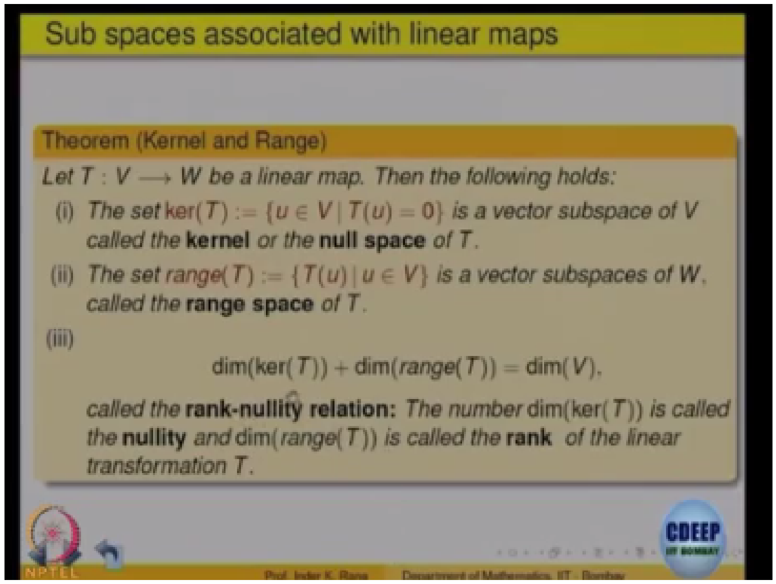

Dimension of the kernel + dimension of the range=V so that is what is called the rank nullity theorem. We will see that this is same as diagonality theorem of matrices a bit later . Okay good let us look at a bit further so important thing what you have done is what a linear transformations kernel is the sub space of the domain range is the sub space of the co- domain and the dimensions added up will give you the dimension the domain. Right important consequence of this very important consequence.

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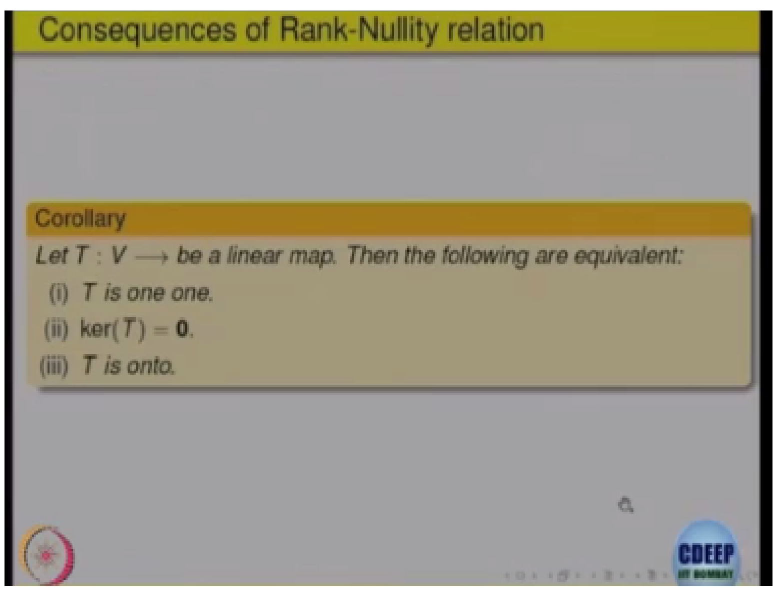

There is a P V L linear map and then these statements are equivalent see if you have got function one set to another in general set theory a function and this function need not be one-one right. This function need not be onto and there is no relation between function being one-one and function being onto. You can have function which are one-one but not onto and functions which are not on to but are one-one and both also.

But for linear transformation it is very important and very nice that a linear transformation if it is one-one and that is same as saying kernel=0 and that is same as saying T=T is onto so a linear transformation if it is one-one it must also be onto right and it cannot have one of the properties. Let us see a proof of that so T is the linear transformation

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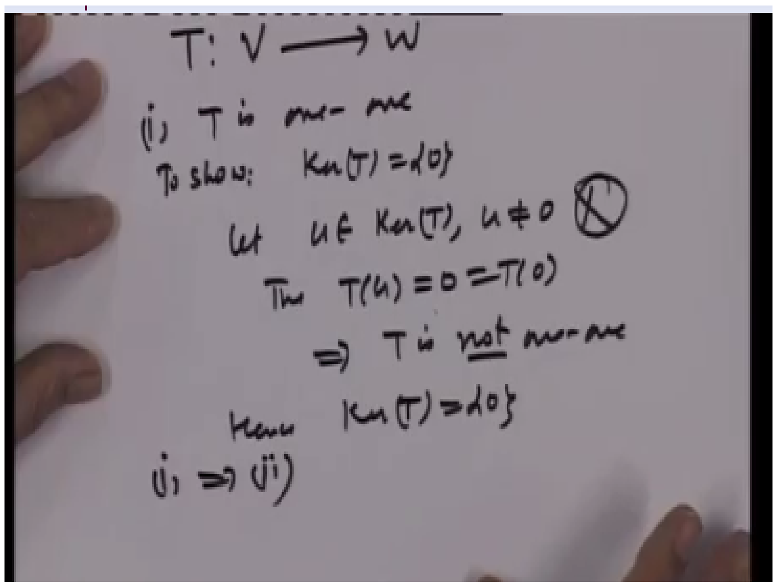

V to W so let us assume (i) T is one-one to show T is on what is to be shown T is to show that the kernel of T is only 0 that means in linear transformation 0 always goes to 0 so kernel will always be having 0 inside it. Claim is that it is=0 let us show that let us take U belongs to kernel so let us say U belongs to kernel of T and U not=0. Let us say something else also possible right then what is T of U that is 0 that is also equal to T of 0.

So, i have got two element U not equal to 0 its image is 0 which is also=T of 0 so what does it implies T is not one-one right I have got two different elements one-one 0 and other 0 the image is 0 so that means what this is not possible. Hence Kernel of T is only 0 so let us so this proves to so (i) implies (ii) if T is one-one then kernel of that is=0. Let us say (ii) implies (iii) so let kernel of T=0.

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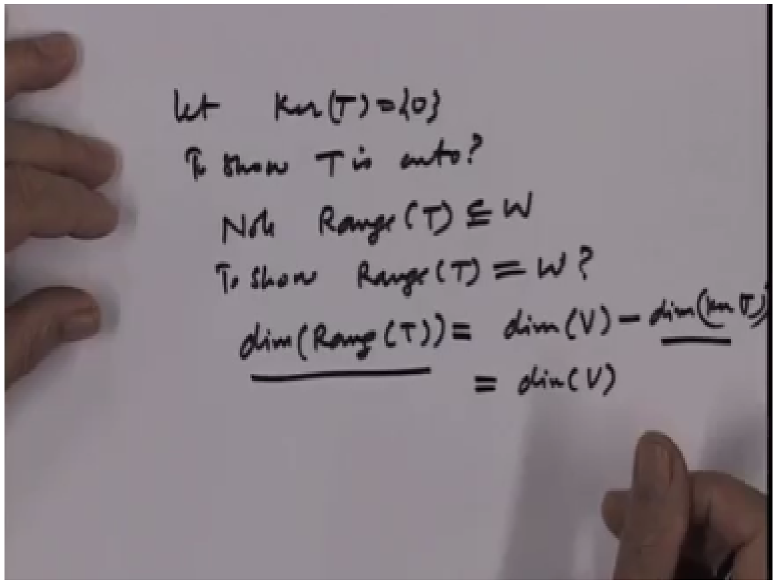

To show T is onto that is what it should be shown so note range of T is a subset of W to show it is onto to show the range of T=so to show actually range of T is=W that is what should be shown to show it is onto if the range is full co-domain then it is onto. Now let us compare the dimension of range of T what is the dimension of range of T from rank nullity theorem says it is a dimension of range of T should be=rank nullity theorem so what is the rank?

Rank is the dimension of the range so that is=dimension of V - dimension of kernel of T but what is that dimension kernel is 0 so what is the dimension? Dimension is 0 so it is dimension of V so dimension is full that means what it has to be=so it has to be so that it has to be 3 rd it is onto 3 rd follows because of consequence of . Is it clear to everybody? That these 3 are=3 rd is the consequence of rank nullity theorem. Now comes the crucial thing namely.

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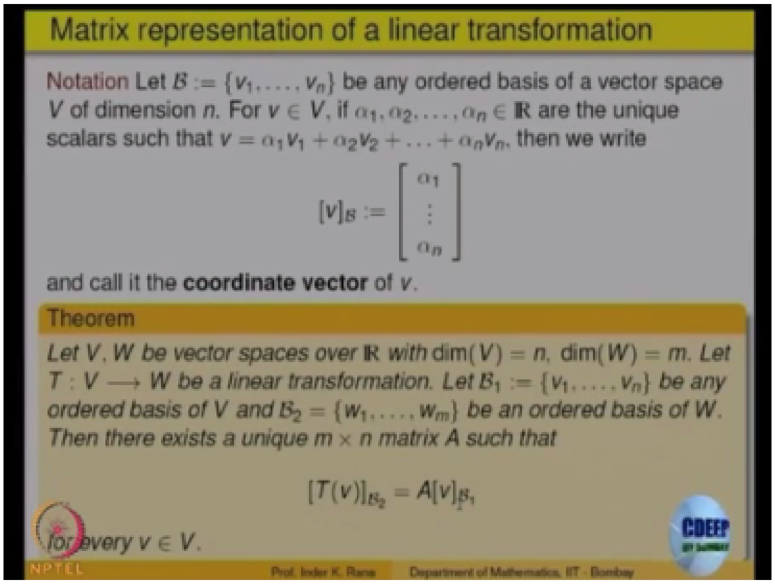

How are these linear and we said that given a matrix you can generate a linear transformation out of it now conversely we want to show that given any linear transformation it arises that way only it arises through a matrix multiplication that is only way of getting a linear transformation so for that let us have this notation if V is a vector space and you have got a basis V1 V2 Vn ordered basis.

Once again the order is fixed so what does it mean it is a basis every vector V you have got unique constants alpha 1 alpha 2 and alpha n unique scalers such that this vector is a linear combination that is the basis that means you have got unique scalers alpha1 alpha2 and alpha n and their order is fixed alpha1 is for V1 alpha2 is for V2 so let us form and write this as a column vector alpha1 alpha2 and this unique scalar which coming as their representation.

So, you get a vector in rn this is called the co-ordinate vector for the vector V. V is the element in the vector space these are co-ordinates which is written as a matrix form unique linear combination and so this is we called as the co-ordinate so what is the theorem the theorem says the following that if T is the linear transformation from V to W on V whereas the dimension is n dimension of W=m then you have got a basis dimension is n.

So, there must be a basis fix the basis V1 to Vn order basis for W there is a dimension m so fix on order basis W1 W2 Wm then it says that if you fix these things then here is a m cross n matrix

T is from where V to W what is the dimension of W m and dimension of V is n so you get a matrix of order m cross n it says that there is exist a matrix says that if you look at the image vector V of T look at its co-ordinator vector with respect to the basis in W.

It is same as take the matrix A multiple with the co-ordinate vector of V in the domain so it says T V can be obtained by matrix multiplication so every matrix is being associated with every linear transformation is being associated with the matrix of order range cross domain m cross n such that this linear transformation can be realised as matrix multiplication. How is this matrix obtained what is the question? How do you get that matrix for a given linear transformation?

So, let us look at a proof of that I will write that proof first and then show you so it will be a revision.

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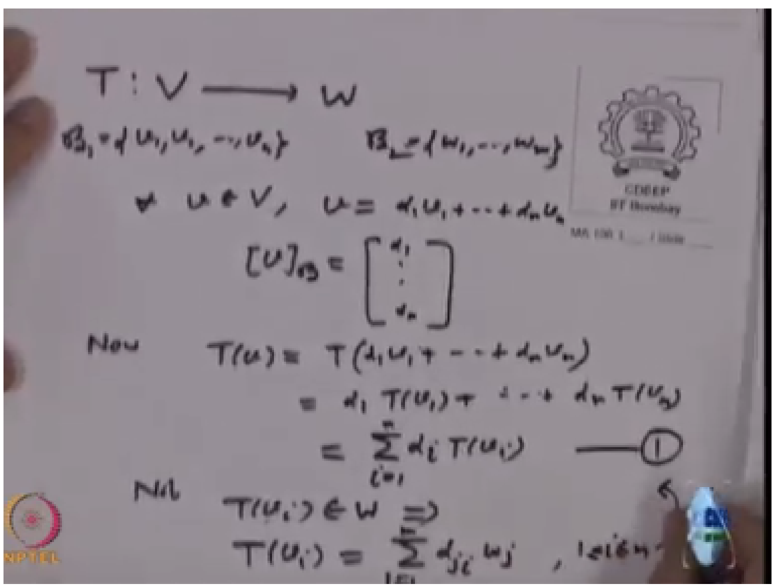

So T is from V to W here you have got a basis B1 which is V1V2 Vn here you have got a basis B2 which is W1Wm T is from V to W so let us write for every vector V belonging to V. V is the linear combination of V1 V2 and Vn that is equal to alpha1 V1+ alpha n Vn and these are unique because of basis so what is the co-ordinate vector that is alpha1 alpha n that is what the theorem said.

Now consider what is T of V T of V is V is this so what is T of V alpha1 V1+alpha n Vn this is linearity alpha1 T of V1+ alpha n T of Vn so let us just shorten it instead of writing dot dot dot sigma alpha i T of Vi so let us keep this it as it is that means if i know T of Vi that is I know T of V that is I said earlier also what is T of Vi so note T of Vi belongs to W is the image and for W we have got a basis W1 W2 Wm.

So, implies that T of Vi must be=sigma scalar multiplies of Wj so let us call them as something say alpha ji of Wj j=1 to m. Is that okay V1 V2 Vm is the basis of W for wj and j=1 to m is the linear combination and these are the scalar which are appearing i between 1 and for every i you will get a scalar right T of Vi so these scalars which appear in the representation will depend on i they will be j and for each i they will be j of them.

So, that T of Vi is the linear combination multiplied by Wj 1 to m is that okay? What I am going to do is from here I am going to put the value in T of Vi. I am going to put back the value so what will happen so let us see and put back the value.

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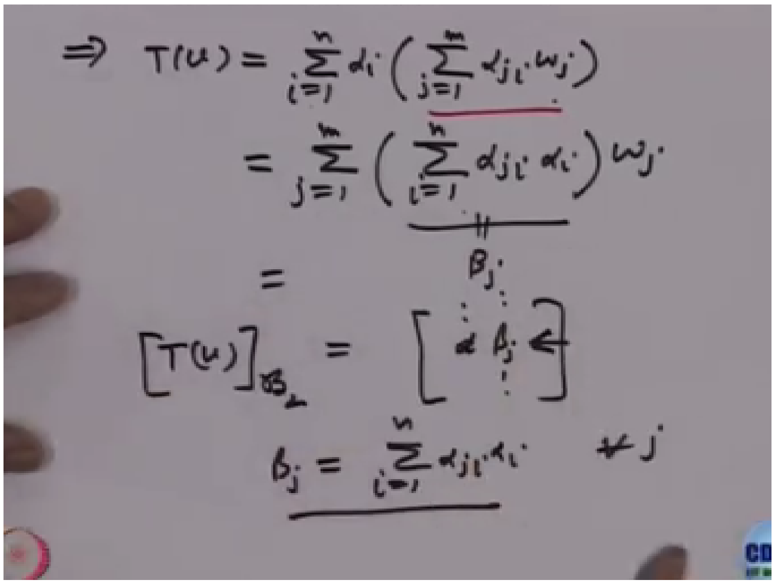

So, implies T of V=we had sigma i= 1 to n alpha i T of Vi that is sigma j=1 to m alpha ji Wj. Is it okay? I have taken this value T of Vi which is this I have put this value here in this expression 1 so alpha i i=1 to n and T of Vi that value is put here so this value T of Vi from here so this is my

T of Vi. Is it okay? Check T of V was equal to sigma alpha i T of Vi and T of Vi being a element of W must be a linear combination of W j.

So, that is the expression for that so this value I am putting back in that equation T of V so it says T of V must be=sigma alpha T of Vi so this I put and so this is the expression I get but where is T of V that is the range. I should express it as linear combination of Wj but they are inside so I interchange the order of summation they are finite number of sums only so I interchanged what I get so j=1 to m i=1 to n alpha ji alpha i of W j.

I is interchanged i was outside and I have got it inside j was inside and I brought it outside so alpha ji alpha i summation over i so this thing is independent of i now because i have been summed up so scaler it will come so let us write this as beta j and this depends on j so what is T of V is sigma beta j Wj that means what? That means this T of V which is in W its base and its co-ordinate vector with respect to W must be=alpha this summation.

Let us write beta j beta 1 beta j beta n so what is beta j definition is i=1to n alpha ji alpha i for every j. Do you recognise what is this what is beta j if you take a matrix with alpha ji and raise and how do you get the corresponding entry here multiply by alpha 1 multiply this by that one that is precisely=this is matrix multiplication so implies that this vector so let me just write here.

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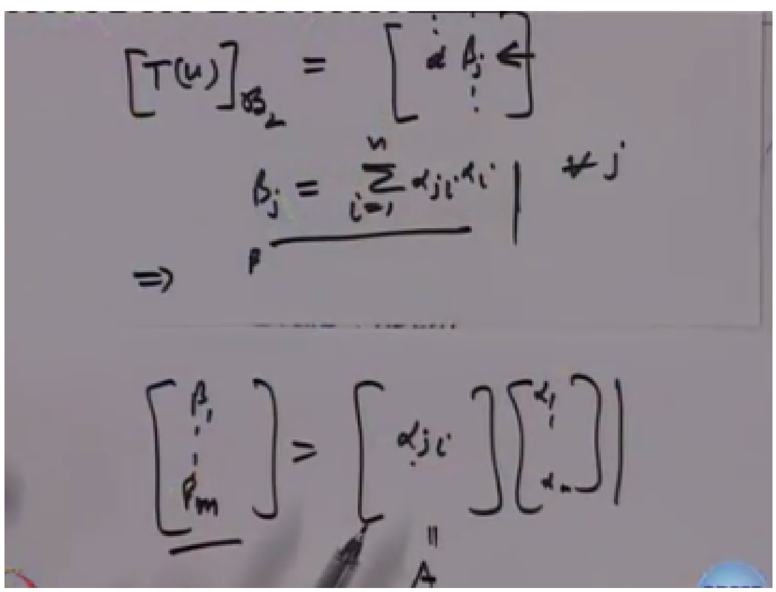

That beta 1 beta n beta m=the matrix alpha ji and alpha1 up to alpha n that is precisely this equation and this precisely saying this see alpha ji what is j? First one indicates the row and the second one indicates the column so in that row look at and multiply by alpha 1 will get this vector so this is the matrix A which corresponds to the linear transformation so I think we will spend some more time because he indicating his time is running out. But let me just show what it says the following.

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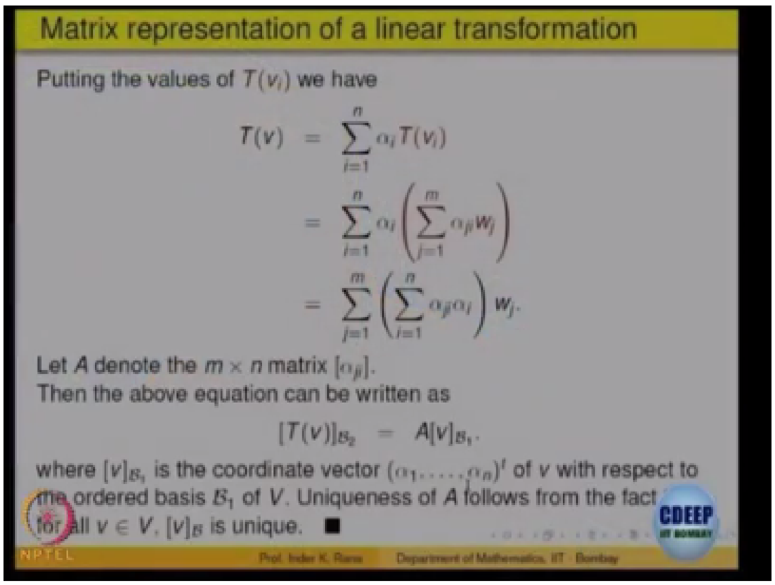

So, the proof so the important thing is T of V is the linear combination that we get here. This is what we get coordinates of T V is A times where A is m cross n matrix alpha ji. The important thing you should notice I purposefully wrote something which is when I write T of Vi as a linear combination I wrote j first and i later purposefully I wrote because then only it gives me a corresponding multiplication.

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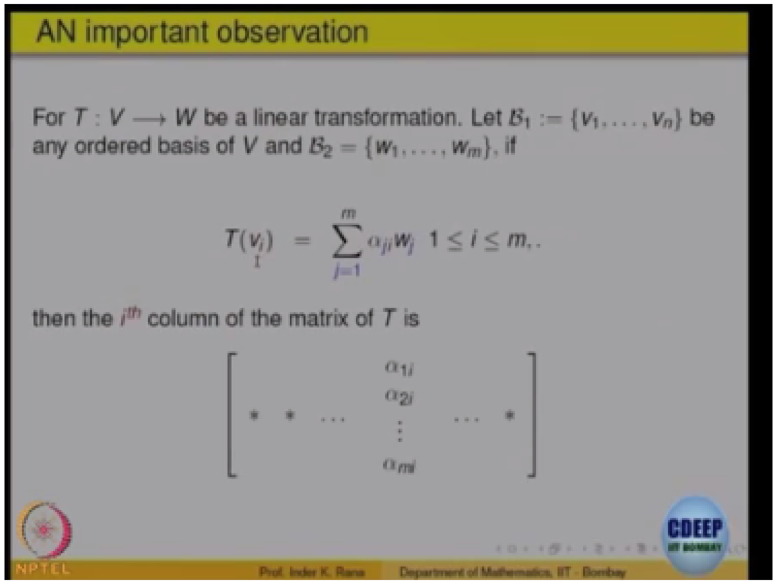

That means what from our point of view it says the following that if T of Vi is linear combination ith element in the domain basis if you write as a linear combination you will get scalar those scalars are nothing but the ith column of the matrix that is the crust in the domain you have got V1 V2 and Vn in the co- domain you have got W1 W2 to Wn look at the ith element of the domain basis T of Vi.

That is the ith element of the domain that must be a linear combination of w s those scalars which are coming here and how many are there m that form the ith column so this i gives you the column and W1 W2 these scalars will give you that ith elements of that column so to find a matrix of a linear transformation fix a basis of the domain for each vector look at what is the linear expression write down the column.

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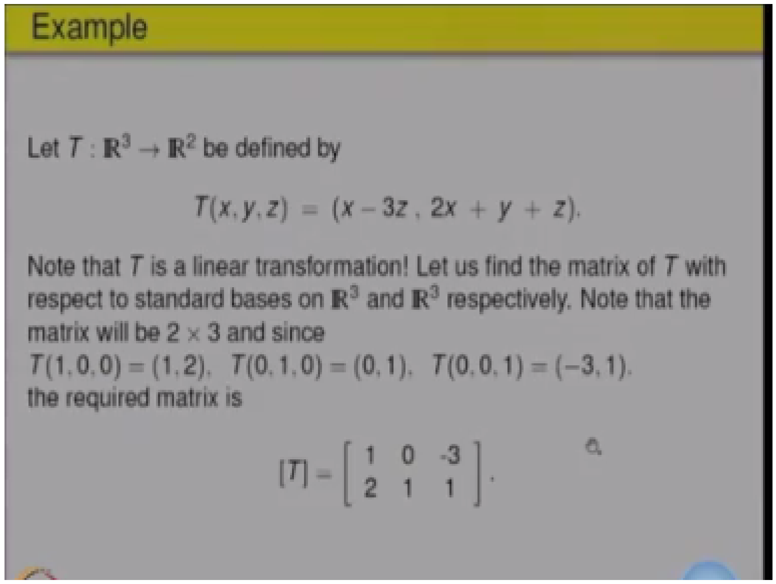

Just look at one example so let us look at this and finish off T of and this is the linear transformation okay from R3 to R2 x y z goes to comma this expression check it is linear and it is easy to check so what will be the matrix of this order will be 2 cross 3. I want to write the columns of that matrix so in the domain R3 let us fix the 100 010 standard basis in the domain 3 elements are there.

For the first one T 1 0 0=this formula compute write according to this formula compute what is T of 0 1 0 so I got this 3 what are these 3 things 1 2 is the first row because it is the image of the f=1 st element of the basis and this is the image of the 2 nd element of the basis in the domain . This is the 3 rd element in the domain its image is – 1 3 so that is the matrix. In the domain V1 V2 and Vn look at the ith entry write this as the linear combination of their side.

And that gives you the ith column that is how you get the matrix. We will do some more examples in the next time. Thank you.