Basic Linear Algebra
Prof.Inder K.Rana
Department of Mathematics
Indian Institute of Technology- Bombay

Lecture - 23
Linear Transformations-II

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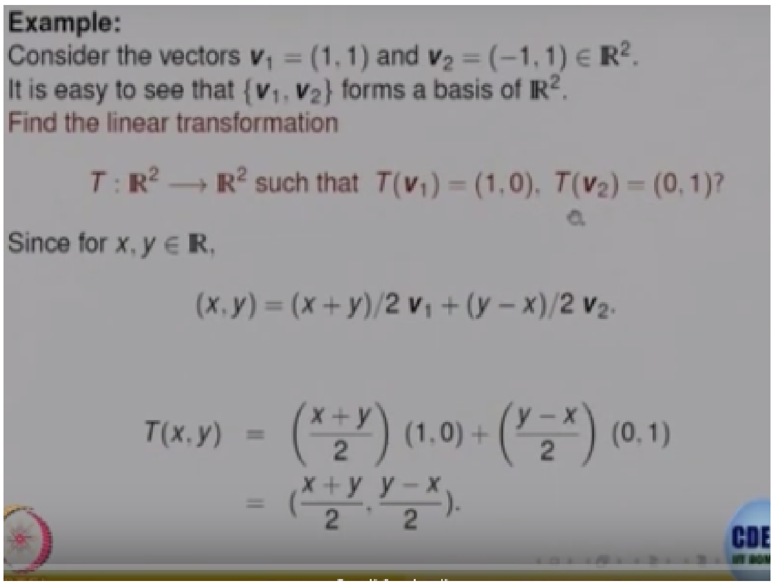

So let us look at is this v1 v2 form a basis let us look at R2 right v as R2 look at the vector v v1 is 11 v2 is -1 -1 and R2 right if there are two linearly independent vectors then they will form a basis right dimension of R 2 is 2 1001 if I am able to find any two linearly independent vectors right? they will form a basis. Are these two linearly independent I can easily check yes what is one way of doing it.

Simplest way is find the determinant right look at the determinant whether rows or columns whichever way you like right? for a square matrix when are the rows and columns independent if they are independent than it is invertible? So it is 2/2 and the invertible is the determinant is not equal to 0. So for 2/2 3/2 3/3 it is not very difficult to check via determinant the advantage of determinant checking directly whether something is independent or not?.

For simple things computation so these are independent because determinant is not 0. So, let us fix T of v1 is 10 so we want to find we want to check whether what we have done we have to fix

T of v1 as 10so v1 v2 are basis elements T of v1 is 10 T of v2 is 01. Okay I want to know what is T of V general vector so how do we compute t of v general vector for any v I should write first a linear combination of the basis elements.

V should be written as sigma alpha i vi an then T applied to it T of vn you know so let us do that so if you look at xy okay that is x+y/2 times v1 so that is a linear combination of v1 and v2 you can check any vector okay xy it is a linear combination . Okay so what is T of this it will eb x+y/2 T of v1 +y-x/2 T of v2 by linearity that is what they are saying . if we do that what is T of v1 is 10 T of v2 is 01 what is this scalar multiply by the vector 10+y-x/2 multiplied by.

So, what is this x+y2 y-x/2 so that is the linear transformation so it says once I fix values of T on the basis vector every other vector because its going to be representative uniquely as a linear combination of the basis. T of that will be linear combination of the T of the image vectors of the basis. So that values we know right so this is given images on the basis elements basis elements we know.

How to find the general formula for the transformation right? so its okay so this we have to do the linear equation right? Xy=alpha v1+beta v2 so you will get two equations and two variable you have to solve them. So, basically that previous knowledge will be coming handy here now you want to write a vector as a linear combination of known vectors. So, xy= alpha times 11+beta times so you will get two equations alpha and beta in terms of x and y.

So, x and y are fixed for you given to you so find the value of alpha and beta in terms of x and y you have to so that is you check whether a given vector is a linear combination of some other vectors or not right? so this is obtained by solving two equations with two variables alpha beta.

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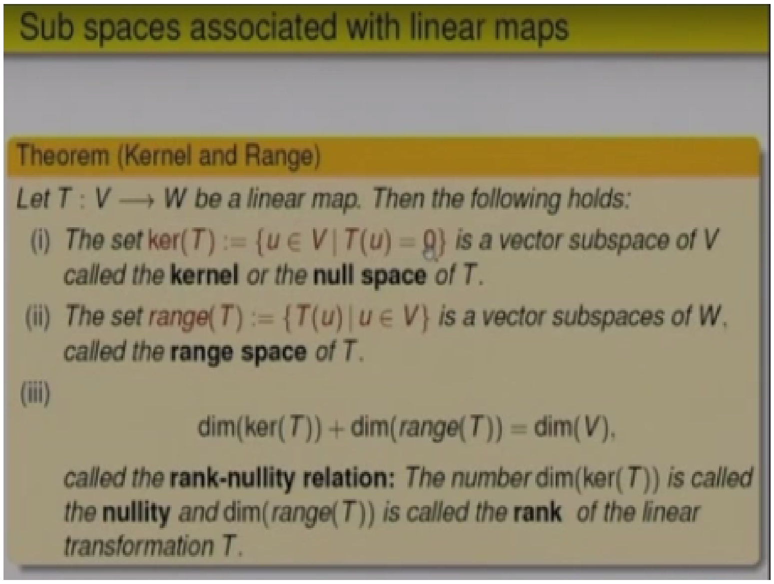

Okay so here is the important theorem which will prove it says where associating with the linear transformation from sub spaces V is a linear transformation from vector space V2 vector space W if we look at all the vectors in the domain which go to 0 look at all the vectors u in the domain u belonging to v say that u of u=0 all the vectors in the domain which go to the 0 vector will let them together that set is called kernel of the linear transformation.

So, what is the kernel? all the vectors which are killed by the linear transformation which goes to 0. So, in the domain collate together all vectors whose images are 0. So, that is a subset at present but the claim is actually it is a subspace. What is the difference between a sub set and a sub space? In a subset obviously all u will go to 0. But if u1 and u2 go to 0 and alpha times u should also go to 0 it I a vector space.

Right so we will prove that so this kernel is all the elements of the vectors space v will go to 0 it is called the kernel the range like range of a map right? The image side what is the range rage is T of u belonging to the domain that is a range of a function as it is right? so claim is if T is linear then the range is also is a vector space okay second and third now kernel T and range T both are vector spaces.

So they allow dimensions so it says dimension of kernel of T +dimension of the range of T =dimension of the domain space V this formula holds this is as same as the dimension of kernel it is called the nullity actually and dimension if range is called the rank of the linear transformation it is same formula that we had for rank nullity theorem for matrices. We will see how it is related to that also later on

So it is the same formula okay same theorem for linear transformation that the rank right dimension of kernel is nullity dimension of range is called the rank . So, rank+nullity= dimension of the domain space .So, let us prove this theorem okay proofs are quite simple straightforward so let us prove them so i will write the proofs so that it is easier to understand slowly.

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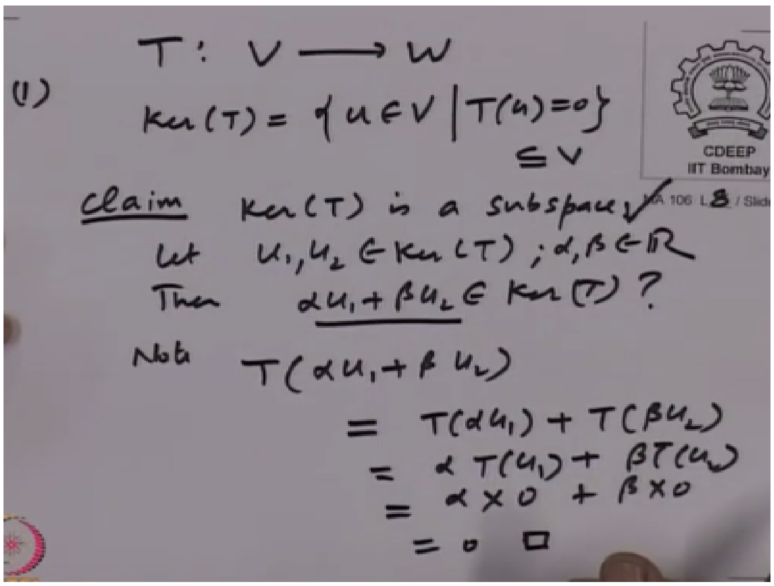

So, the first one is T is from v to w right so what is kernel of T that is =all u belonging to v say that T of u=0. So, as such this is a subset of v right? so claim kernel of T is a subspace . So, to prove this so let us take let u1 u2 belong to v kernel of two elements and scalars nullifying beta belonging to scalars. Take a linear combination then alpha u1+beta u2 should belong to kernel of T that is what we have to show right to show it is a vector space.

The linear combination of elements should be inside. So, let us to show this what is to be shown that this element also has a property T of this element is 0. So, let us note T of alpha u1+ beta u2 what is that=linearity plays a part so it is T of alpha u1+T of beta u2. But again by linearity alpha comes out that is T of u1+beta T of u2 right I am using linearly. So, this is =alpha times this is 0+beta times 0 T of u1u1 belongs to kernel u2 belongs to kernel.

So, T u1 is 0so this is =0 right so that mean the kernel is a subspace whereas if we cut the range our range is a subspace of what . let us put the range so what is the range of T what is tis this is all W belonging to W.

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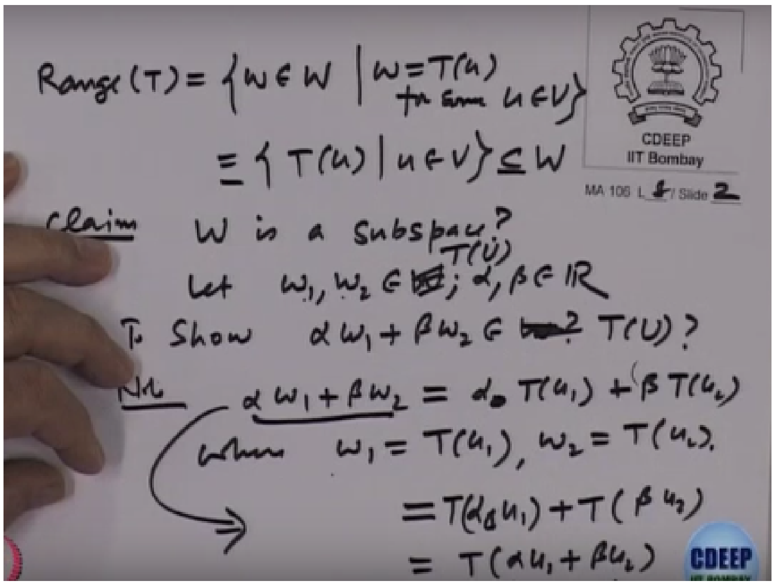

In the image say that W=T of u for some u belonging to v is that okay right all the points and the co domain which are images of something right? another way of saying this is another way of saying T of u belong to v both are same okay they are equivalent of both alpha, So this is a subset of W range is a subset of W claim W is a subspace right so what do i have to check? so let W1 W2 belong to W alpha and beta will be scalars. .

So, to show alpha W1+beta W2 belongs to W clear how do you show that I should represent this as T of something as image of something? right? so note so this is what we saw alpha W1+beta W2 what does that equal to we know W1 is in the W1 sorry I wrote W right I should have written of T of u W1 W2 in the range I should show that is unexplained if I am saying W (()) (12:03) anyway to show of the range is a.

So, here is the mistake W1W2 belonging to the range right should imply the linear combination inside the range right? That is to be shown I will start with tis one W1 belongs to a range that means what alpah1 what is W1 some of T of u1+beta what is W2 that is T of u2. So, where W1= T of u1 and W2= T of u2 is that okay because if they are in the range there must be the images so what does this give you?.

So, this is equal to implies that this right hand side is alpha of T1 so we can write as T of alpha1 u1 +T of why did I write alpha 1 anyway there is no alpha 1 where is alpha 1 here right +beta times u2 this alpha can be taken in because of linearity beta can be taken inside with our own linearity this is same as all linear alpha u1+beta u2. But this new element in u right so that means W alpha W1+beta W2 is an image of some element right using linearity.

Nothing more than that were not doing anything just using what is given us linearity and bring it inside. So, it is a subspace right.

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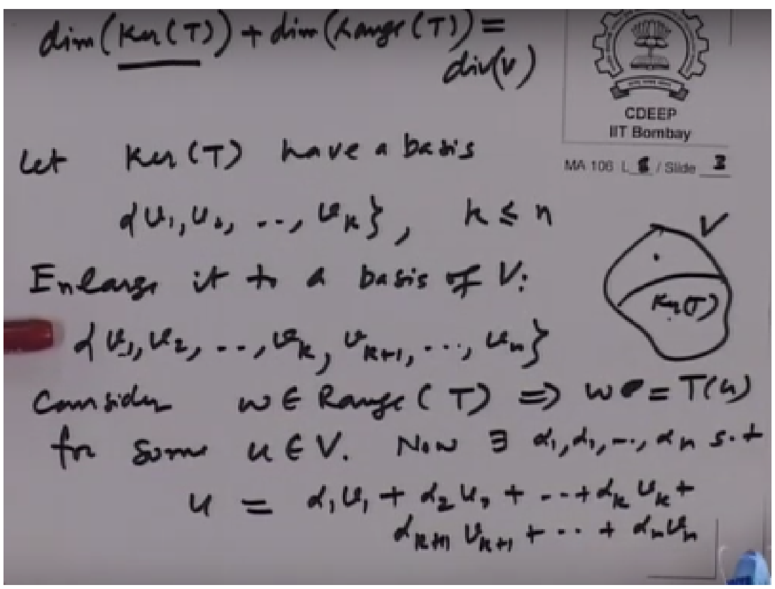

So, let us look at a third property the third thing we want to show we want to show that the dimension of kernel of T +dimension of range of T =dimension of v . So, T is formed v to w keep that in mind right so as I say this is dimension m and this is dimension n right v is a vector space it will have some dimension n right W will have some dimension n so let us look at that we do not have to write everything.

Okay so now we want to compute what is the dimension of kernel? how does what is the definition of dimension or the dimension of a vector space is a number of elements in any basis of the vector space. If the dimension is n right we will have some n vectors as there is a basis for it we do not know what it is we can find let us start so let us let kernel of Have basis kernel of T is a subspace right? so does that make the space in itself? so it has a basis right?.

So, let us write it has a basis so let us call this as v1 v2 vk k <=n is that okay? this kernel uses a subspace of vn right so it will have some basis let us say k elements are the basis dimension is k right? and this k will be <=dimension of n because it is a subset of vn , so k<=n so I will just a picture imagine this is v right? it does not look like a Venn diagram I am trying to show and here is a kernel of t this is a part of it right.

So, kernel were taken as a basis for the whole space also has a basis right? so what we are trying to do in our ways and large this given basis of kernel were fully basis of v. I can go on adding more and more a nonzero elements right? Till I get n elements which are linearly see because if kernel is not whole of v there must be a vector outside right? there must be a vector which is not a linear combination of elements of the Kernel.

So, add that to that basis of the kernel you will get 1 more k+1 probably that itself finish everything. If not take one more and add till you get n elements which are linearly independent right? So what one says is enlarge it to a basis of v enlarge it that means you are given k vector it form a basis of the kernel add how many more do you will be adding n-k so that you have n vectors which form a basis of v.

So, enlarges the basis of v1 v2 vk up to where this is kernel the new one which are added is v k+1 up to vm. We get a basis of v is that okay right so this is to form a basis of v so the whole thing has got a basis . So, let us consider any w belonging to range of t that implies w must be= in the range there must be mage of something for some u belonging to v is that okay if w is in the range it must be image of something.

Now I have got the basis of v and u belongs to v right. So, now this u can be written as I have got a basis for v u is element of v so what should I have that means now there exists some alpha 1 alpha 2 alpha m such that u must be a linear combination of the basis elements m is the basis element right for this basis. So, it will be alpha 1 v1+alpha 2 v2+alpha kvk+ alpha k+1 vk+1+alpha m vm right it is a linear combination.

So, u must be written that way but what i am interested is I am not interested in u I am interested in w because I want to get a basis for the rage so I should go to the image. So, what does this imply what is T of u .

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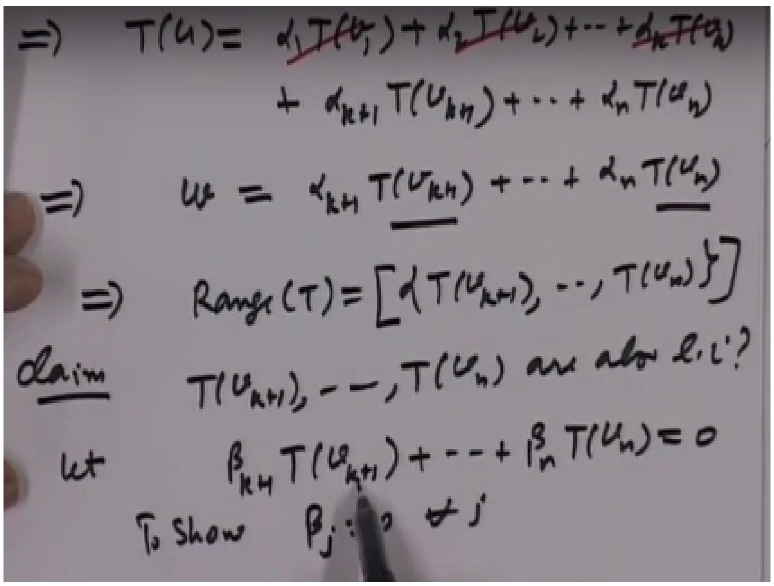

So, imply what is T of u linearity that is alpha 1 T of v1+alpha 2 T of v2+alpha k T of vk+ I am using linearity again and again+1 okay + alpha N T of Vn using linearity and using u is the linear combination T of u must be linear combination of the images right? so that implies what is T of u that is W right so W= what is T of v1 where is v1 these are form a basis of the kernel so what is T of v1 0 T of v2 is 0 up to T of vk is 0.

So, let us use that fact this is 0 so this goes this goes off this is not required this is not required this is 0 so what is left alpha k+1 T of v k+1+alpha n T of vn. So, what does it mean any vector so this implies any vector w which is of range is a linear combination of these ones right? that means the range of T is span of the vectors T with k+1 T of vn right these vectors span only thing you have to check we have to show the basis once it is generated.

We want to show our basis what we should do prove it is linearly independent right? so claim the T of vk+1 T of vn are also linearly independent so let us check that so how do we check their linearly independent then a linear combination=0 should imply all is equal. So, let us say beta k+1 T of vk+1 I am choosing purposefully the betas now right? +they are arbitrary beta n T of vn=0 what is to be shown to show beta of j=0 for every j .

All these betas are 0 right but now what do we know? we only know that this vk are independent right? we know something what you recall because we had the basis the basis of v. So, these ones are independent we have to now for now go back to the domain space and use this fact there so how do i go back? so the idea is this one okay? so let us look at the left hand side of this=okay is equal to okay i add to it T of u1 T of u2 and T of uk all of them.

Whether all are 0 right so it says sigma we can write beta T of ui I=1 to n=0 okay for any beta 1 to beta k . Take first any beta 1 beta k but T of u1 is known T of u2 is known and T of k is known as first take n=0 whatever beta1 beta 2 and beta k right remaining comparison is given to us right we are going to make it =0 is it okay . So, it implies linearity now take linearity that means T of sigma beta i vi i=1 to n=0 by linearity and I am going back now.

So, that means what T of something is 0 that must be in the kernel right?

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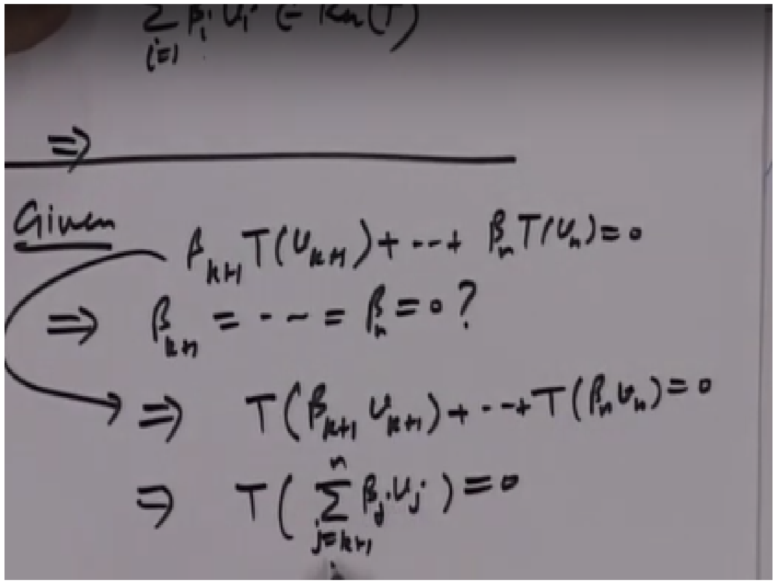

So, it implies sigma beta i vi i=1 to n belongs to kernel of T right but that was for any beta I so what does this imply what is the property of vi to vn they were a basis right and I have got a linear combination now right. So, this belongs to kernel of T that is a linear combination which T=0 this is =0 so this is okay sorry i should not be hurrying through let me just from here we have to rewrite what we have to show it says beta here okay.

I will write it again in a simpler way I have done it given beta k+1 T of uk++beta n T of vn=0 or beta implies all of these are 0 beta k +1=beta n=0 that is what I want to show right okay let us show this okay. So, this one implies T of beta k+1 v of k+1 + T of beta n vn=0 that by linearity. So, let us bring her again linearity that may sigma of we will write sigma beta j vj j=(27:03) v j=k+1 to n=0 right.

Now that means this element belongs to kernel right t of something is 0 so that element makes clear. So, implies sigma j=k+1 to n beta j vj belongs to kernel of T right from k+1 onwards.. But I know that kernel got a basis already v1 v2 vk. So, this element must be a linear combination of v1 v2 vk . So, implies sigma j=k+1 to n beta j vj must be =sigma alpha I vi I =1 to k right . Now bring everything on one side so what does that imply? so bring everything.

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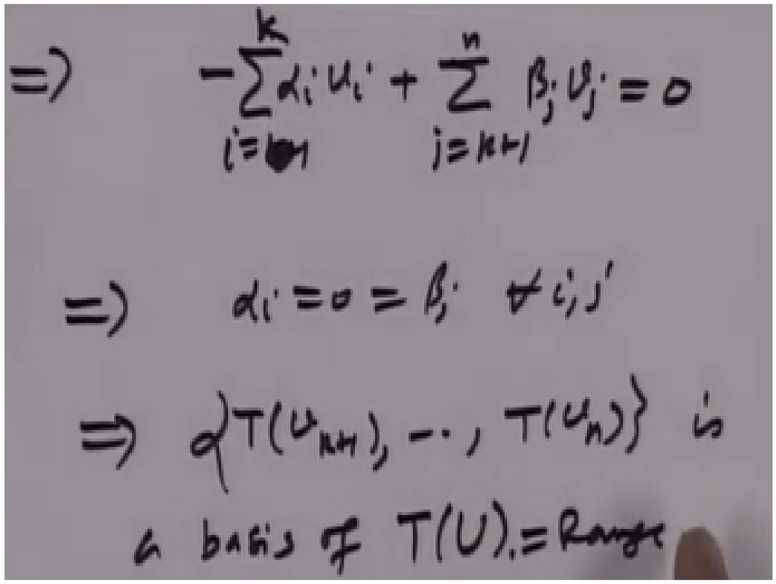

So, implies that – sigma alpha i vi I =k+1 to n+ **“Professor - student conversation starts** pardon i=1 oh i=1 sorry “**Professor - student conversation ends”** 1 to k+j=k+1 to n beta j vj =0 everything on one side. Now what is the left hand side is a linear combination of v1 v2 vk up to vn all of them but i know that it is linearly independent all of them are linearly independent. Because they formed the basis of the whole space v.

So, implies what all the alpha=0 beta j for every I and every j all of them must be 0. v1 v2 vk and n is linearly independent. So, implies T of vk+1 T of vn is a basis of the range T of U right that is =range . So, basically the idea is we start with a basis of the kernel which is in v enlarge it to a basis of whole of v whatever you edit the first is the kernel the image will be 0. They do not contribute anything idea is that the remaining n-k will contribute to the range of t.

They will be independent and everything will be generated by them. So, how many are there? N-k right? so dimension of the range space is n-k dimension k. So, total=n so that is rank related theorem. Basically doing what? looking at the bases of the kernel enlarging and saying that images remaining ones that you added form a basis for the range. That is all okay