Basic Linear Algebra
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Lecture - 22
Linear Transformations -I

Okay, so we will start with our today's lecture. Till now we have seen various concepts of linear algebra, more of theoretical, they are motivated by geometrical concepts. Today we are starting to look at a concept, which is more dynamic in nature and so to motivate that concept let us look at a small video clip rather.

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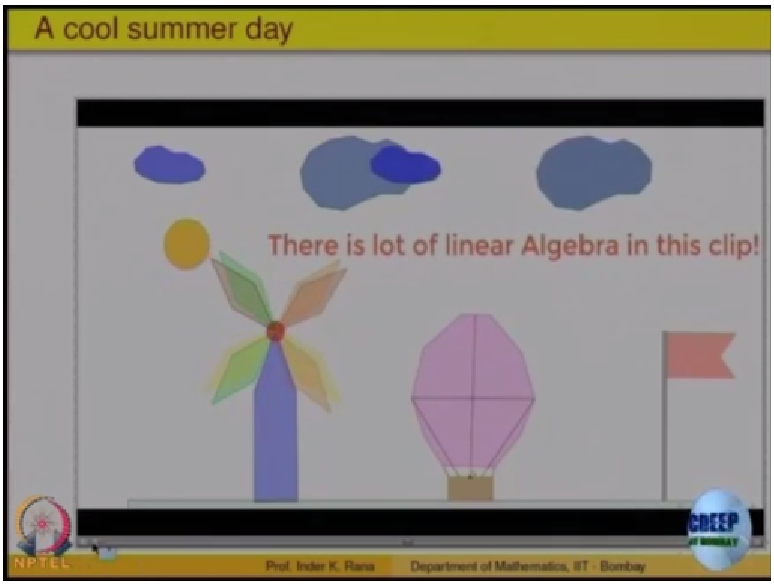

Observe the movements of the objects in this clip. So what are the kind of movements that you saw in this clip. These clouds were moving, this wings of the windmill they were moving, the sun was moving and this balloon was changing and this flag was changing, let us see it once again some part of it and observe the various movements.

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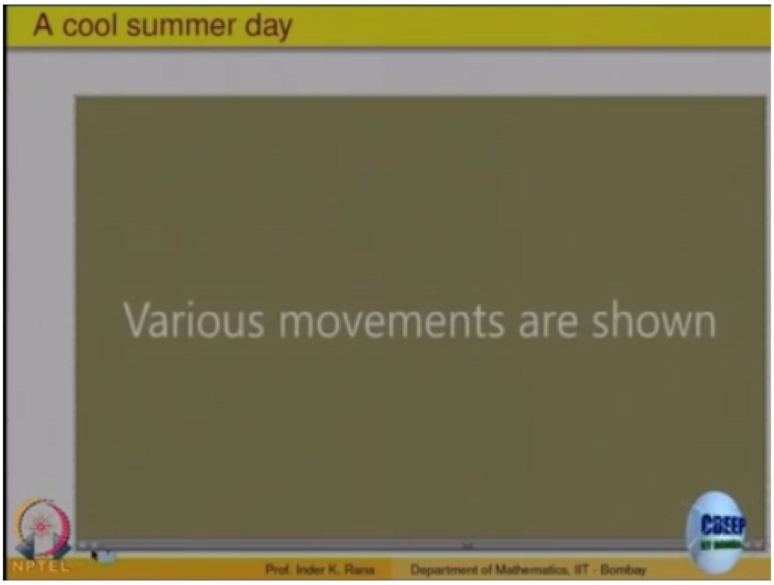

So what are the various movements which are happening? Clouds were removing in certain direction. The sun was moving along a particular path, the flag was shifting it is position and the balloon was becoming bigger or smaller. So these are various things which we use in life which we observe in our day-to-day life and which have been used in this particular clip. So let us see what are those things. So movements were shown and some of them were reflection.

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The flag changing okay. Then scaling.

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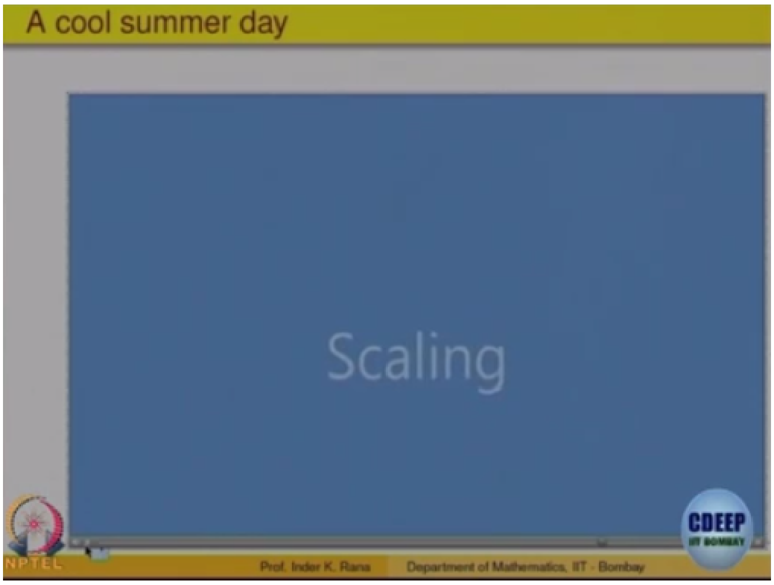

Balloon becoming bigger and bigger retaining the shape that is called scaling.

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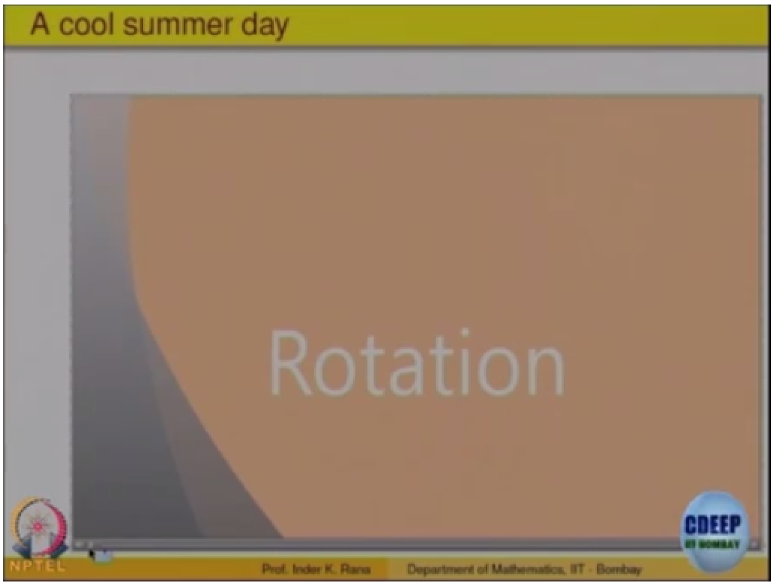

Then there was rotation. The sun was moving along a circular path.

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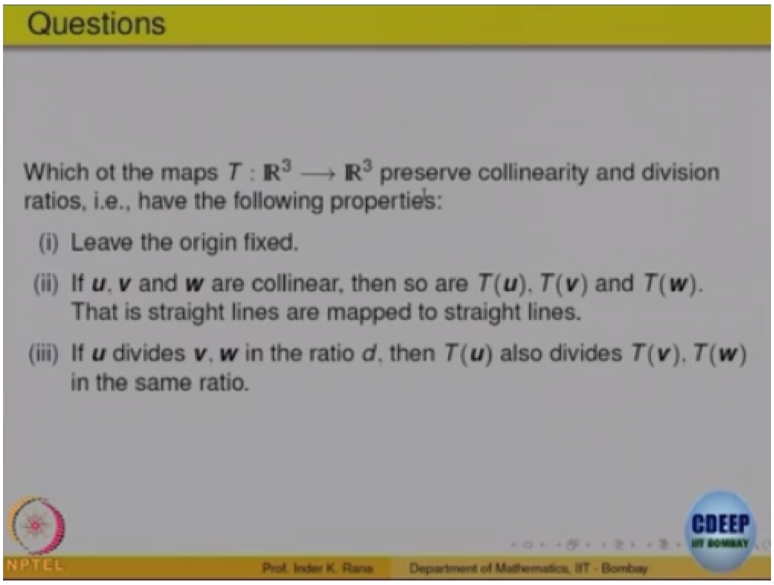

How are these movements described mathematically? So basically the idea is to look for maps, let us call it as a map T, from R3 that is our word okay, to R3 which preserve collinearity. What does collinearity mean? They take lines to straight lines to straight lines they do not change the straightness of 3 points. If 3 points are collinear to start with then their images also remain collinear.

Right, so lines go to lines, so origin remains fixed in some of this right, in translation origin does not remain fix, but in rotation, reflection, scaling all these thing origin remains fixed and third the proportionality is maintained. The proportionally division ratios are mainlined if u v and w are in a particular ratio these points on a straight line were in a particular ratio then the images also maintain the same ratio.

So question mathematically comes what are the maps from R3 to R3 which have these 3 properties, which we observe in day to day life and which are useful.

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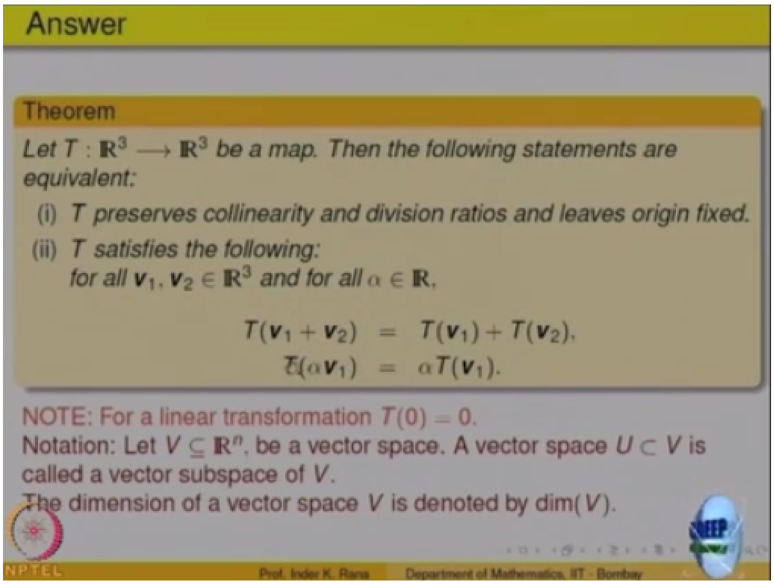

So one can prove a theorem will not prove that theorem it is not very difficult but we will assume it that if T is the map from R3 to R3 then these 2 statements are equivalent one T preserves collinearity, division ratios and leaves origin fixed. Those are the transformations we want to look at. It says it is equivalent to saying the algebraic condition comes now T of you take 2 vectors V1 and V2 in R3 take their sum and take their image that is same as the sum of the respective images okay.

And if you scalar multiply a vector, magnify a vector, then the image gets magnified by the same scale. So these are the 2 properties which are equivalent to saying if a map T from R3 to R3 will have these 2 properties. So these are algebraic version of geometry right. We started with geometry namely what are the maps which preserve collinearity, division ratios and leaves origin fixed, this is algebraic way of saying the same thing and equivalent way of saying.

So this motivates the study of maps from vector spaces to vector spaces which have these 2 properties. So we will note one thing important that because T of alpha V1 = alpha TV1 for all alpha if I take alpha = 0 what does that give you? T of 0 = 0 so such maps will leave origin fixed right. So that is the consequence of this property. Origin is fixed, T of 0 is 0 right and note another thing.

Here it is from R3 to R3, but when define we will for vector spaces we will see that they need not be from same vector space to same vector space right domain and range need not be same. We will use this notation and if V is contained in Rn is the vector space we define the notion of a vector space and u is inside v which is also a vector space then we will say u is the subspace of v right.

A subset also a vector space will call it as a subspace and this dimension of a vector space will write as dim (v) that will denote the dimension of the vector space, there is notation so that we do not have to write everything again and again.

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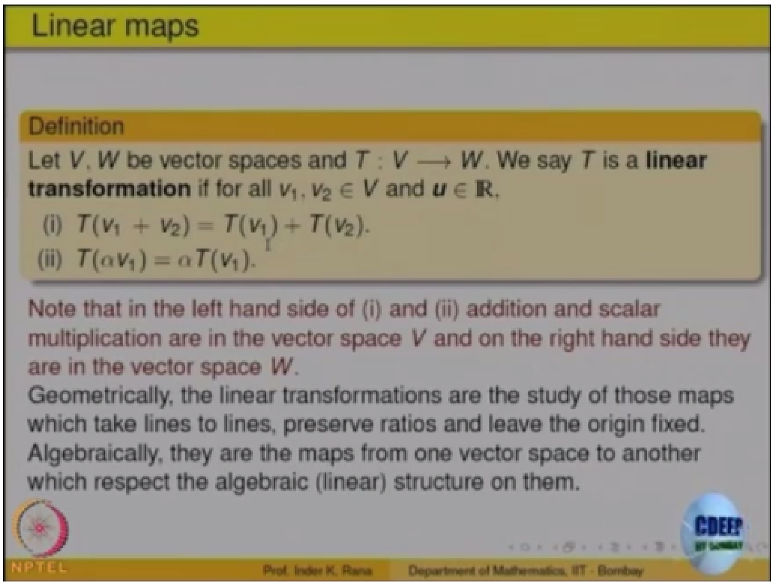

So here is the formal definition of a linear transformation, v and w are vector spaces, T is the map from V to W. we say it is a linear transformation if T of the sum, image of the sum of vectors is = sum of the images and scalar multiple of image of the scalar multiple is same scalar multiple of the image right. If you add vectors, then the images can be added and that is same okay.

And if you scale a vector, magnify a vector then the image gets magnified by the same scale. So these 2 properties okay, one thing we should note here you see this when you add V1 and V2 where are V1 and V2? They are in the domain, they are in the vector space V. So you are adding V1 and V2 here, but TV1 and TV2 are in the image space W. So there this addition is of vectors in W right.

So here this addition is in the domain, here is in the codomain okay keep that in mind because here it could be R2, T will be R2 to R3 for example okay. So that is similarly here alpha V1 is scalar multiplication of the vectors in the domain and that is in the image space. So these are one observation, you should note. So here addition is in the domain, here is in the codomain, scalar multiplication in domain and here is in the codomain.

So geometrically a linear transformation as we said preserves right, it takes lines to lines, preserves ratios and leaves origin fixed algebraically. See algebraically what it is essentially says is that there is a linear structure in the domain and this is the linear structure in the codomain. So it gives due respect to the corresponding structures right, in the domain as well as that is the algebraic meaning of that.

And this phenomenon comes in lot of branches in mathematics whether you study group study here we are studying vector spaces. So whenever there is a set and there is structure and there is one study maps between those 2 structure and if they preserve those are the important aspects that is what whole of mathematics is about. You take a set, look some structure on it okay. You get an object and look maps between these objects.

For example, in your calculus course you had the real line, right. On the real line what is the structure there is notion of distance on the real line. So notion have limits make sense on the domain. Closeness makes sense. You can take a range to be R or R2 or R3. There also is the notion of distance. So closeness is there. So you can study if something is close air, whether the images are close or not, continuity and such things.

So that happens in most of the branches in mathematics so okay. Let us go ahead with our concept of linear transformations. So it is clear what is the linear transformation, linear transformation is the map from one vector space to another which gives due respect to linearity that means T of the sum, image of the sum = sum of the images and image of the scalar multiple = scalar multiple of the image.

So these 2 because on a vector space there is addition and there is scalar multiplication. So these 2 are given due respect that is the linear transformation.

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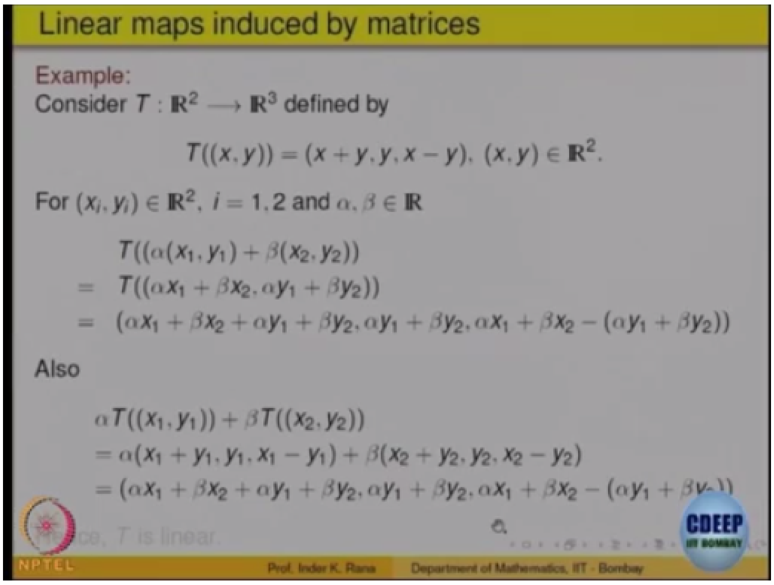

Let us look at this map, R2 to R3, so the map is T of xy right. So points in R2 R3 will whenever we are writing as points will write as ordered pairs or ordered triplets when we treat them as in terms of matrices will go as columns vectors right keep that notation in mind. So here we are treating R2 as ordered pairs, R3 as ordered triplets, so for a point xy in R2, so xy in R2 what is the image, images 3 components.

First component is x + y, second is y and third is x-y right. So this is the map from R2 to R3. We want to know whether this is linear or not. Is this map linear or not. So what we should look at, we should take 2 point x1 y1, x2 y2 take scalars alpha and beta, take a linear combination of them alpha x1, y1 + beta y1 y2, x1 x2, look at the image.

Look at the images of the original point C of x1 y1, T of x2, y2 and take the linear combination that too should be equal then it will be a linear map right. So to verify this let us look at T of alpha x1 y1 + beta of x2 y2, these 2 vectors are added right with scalar multiples, T of that we want to compute. So first what we do is, we add these 2 vectors we add anyway right. So if we add them we will get alpha x1 + beta x2, the first component and second component is alpha y1 plus.

Now T of that so what is T of a ordered pair, add x and y components, y component remains as it is, x – y. So this first component alpha x1 + beta x2 + alpha y1 + beta y2 that will be the first component. Second remaining as it is. So alpha y1, beta y2 that remains as it is. Third difference of the two, so that is the image of this right. On the other hand, if you want to calculate you can calculate alpha times T of x1 y1 + beta times T of x2 y2 so what will be that.

So this will be alpha times x1 y1, y1 x1 – y1 right that is the image of the first one. Beta times image of the second so x2 y2, y2 x2 – y2 and simplify both will come out to be same. So that is what it means by checking whether something is linear or not right. So take 2 vectors v1 and v2 in the domain, take a linear combination take the image, take the image of the original vectors, take their linear combination by the same scalars and 2 come out to be same. So that is linearity. Okay so T is linear.

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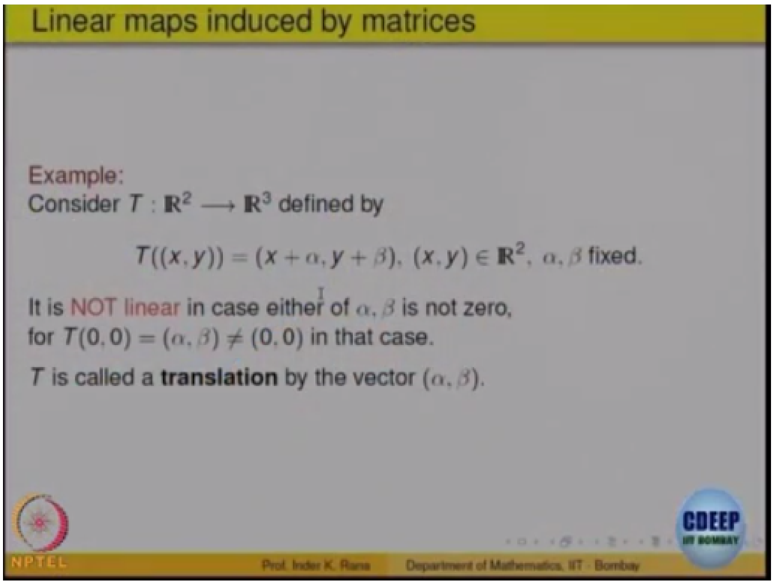

So let us look at this, so T of xy it is from R2 to, this is a typo here this is R2 to R2 right, xy goes to x + alpha, y + beta. So x component you add alpha, y component you add beta. Question is is it linear or not right. Obviously it is not linear because in the linear, if T is linear T of 0 should be 0. If I put x =0 y =0 then it is alpha, beta. So alpha may be 0, beta maybe 0, both may not be 0 one of them maybe 0.

So if both are 0 then it is a linear transformation but in that case what you are doing nothing it is the identity map, xy goes to xy. So if either of alpha or beta is not 0 then this is not linear right. And what is this basically? This is what is the translation, you are taking a vector and just translating the vector by a fixed vector alpha, beta. So translations in plane or even in R3 or in any Rn are not linear transformations because it would not leave 0 fixed unless the translation by the trivial vector.

So this is called translation. So keep in mind translations are not linear. Whenever you want to check something is linear or not the first thing you should check is whether T of 0 is 0 or not right, that is a necessary condition. So T linear implies T of 0 is 0, so that is a necessary condition. So always we want to check something is linear or not first verify the necessary condition alright that is how necessary conditions are used.

So it is not linear okay and this it causes some problem in computer graphics see, we will show later on that these motions translation, rotation, deflection, such things are used in computer animation right. So when you want to do computer animation you have to tell the machine that at this point this line is to be rotated like a robo is there standing and arm force right.

So that means this is my center wave I want to rotate and by how much angle I want to rotate but how do I give command to the computer, it does not understand to rotate. So all this is done via matrices okay. We will see how this command can be given by matrix and then to a computer and then computer does the job right. So that is possible, but the problem comes this translation is not linear.

That causes a problem because to give a command to a computer we need to convert a linear transformation to a matrix and translation is not linear so you cannot convert directly into a matrix. So what is done is one introduces a dummy variable right and then coverts this into from R3 one goes to R4 actually and then does it. So you will learn these things if you are doing computer animation, computer graphics.

You will learn their order called projective coordinates. So we will not go into that because idea is to introduce things okay.

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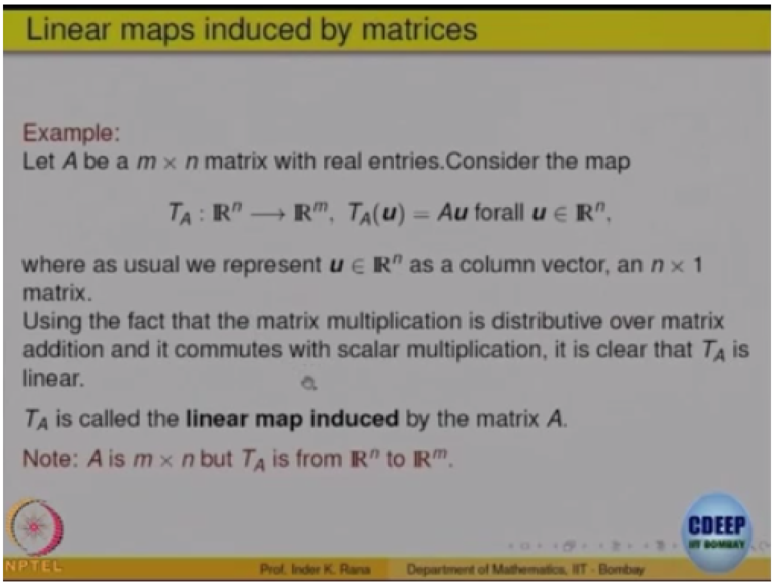

So let us look at some particular ways of introducing linear transformation examples they comes from matrices, so till now we have been treating matrices as collection of rows and columns of numbers which help us to reduce right express linear equation as system of linear equations and do various job. Here is a dynamic version of it. So take a matrix of order m * n okay.

Consider the map TA this A indicates this map depends upon the matrix A, it is from Rn to Rm. So keep in mind here matrix is of order m*n. the map we are defining is from Rn to Rm, so there is switch and how it is defined for a vector u in Rn, if u is a vector in Rn, A is the matrix of order m*n, so you can multiply A with u, you can multiply A with u. So m*n, this vector is n*1 u as a column vector we have treated as column.

So what will be the result m*1, so that will be element in Rm clear. So given a matrix of order m*n you take a vector u in Rn and multiply as matrix multiplication you will get a vector in Rm. So take a matrix, take a vector in Rn multiply you get a vector in Rm. So this transformation we are writing it as T lower A, T index by A. It depends on the matrix A, question is is this a linear map.

Right obviously it is liner because we have observed the matrix multiplication distributes over addition. If you take vectors u1 and u2 and add them and multiply with the sum with the matrix A is same as multiplying A with u1 adding it to A multiply with u2. Similarly, scalar types right. So that property of matrix multiplications says that this is a linear. So using the fact that matrix multiplication is distributive over matrix addition.

And commutes with scalar multiplication right. It follows the T of A is A linear map is it okay everybody, T of A is linear because A of A multiplied with u1 + u2 is same as matrix multiplication it is Au1 A multiplied with u1 added to A multiplied with u2 and similarly scalar comes out. A alpha u is alpha times Au right okay. So this is linear. So this is called the linear map induced by the matrix A.

So we will give it a name. So every matrix m * n gives you a linear map from not from Rm to Rn but from Rn to Rm okay. So this keep that in mind. So now A is m * n but TA is from Rn to Rm, want to look at some particular cases of this. We will take a special matrix and see what is the effect happening. So let us look at that.

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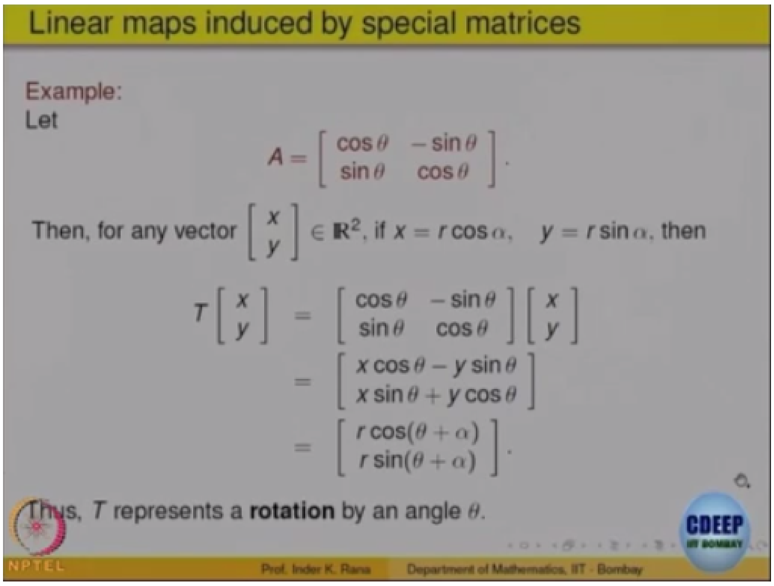

So look at this matrix cos theta – sin theta, sin theta, cos theta right. Theta is some angle between 0 to 2pi, let us take a vector xy in R2 right and let us say that x is the point in R2 xy so polar coordinates. So what are the polar coordinates x = R cos alpha, y = R sin alpha right. So let us multiply this matrix by this vector and see what is the outcome. So when you multiply T of xy right, T of the image the matrix multiplied by the vector.

This vector is right what was x, x was R cos alpha, y is R sin alpha, so when you put these values R cos alpha, cos theta - R sin alpha, sin theta input and used a trigonometric identity what is this R of cos theta + alpha and this is R of sin theta + alpha. So what has happened earlier the point had polar coordinates R cos alpha R sin alpha, now R remains the same, the distance of that image now origin remains the same but he angle has changed to theta + alpha right.

So if R theta were the polar coordinates of the starting point. The image point has got polar coordinates R theta + alpha right. So that means what geometrically it is the rotation by an angle theta right. So this matrix multiplication gives you rotation by an angle theta depending on theta positive or negatives clockwise or anticlockwise okay right. So let us look at one more. So this rotation by an angle theta.

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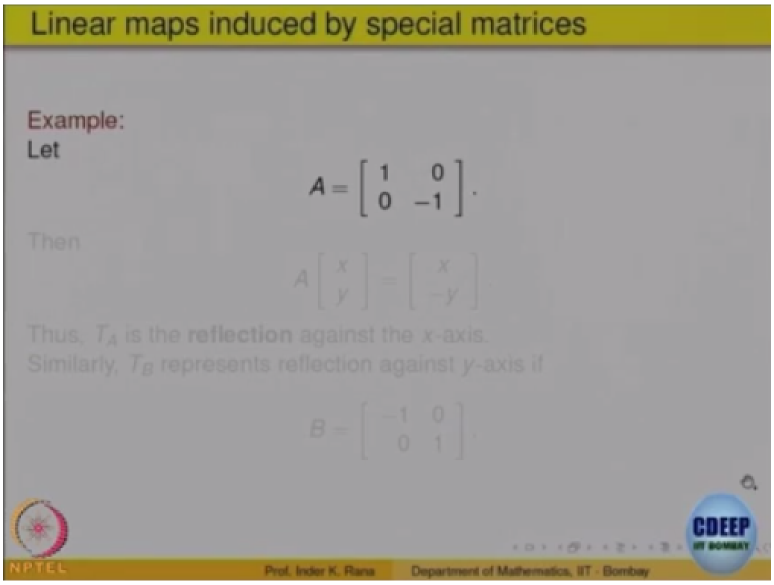

Let us look at this, 1 0, 0 -1, so if i take a vector xy and multiply by this what will happen, 1 0, 0 -1 multiplied by xy so what will be the image? x-y right x will remain as x, but y changes to –y. So a point xy is changed to x – y so what does geometrically looks like? x coordinates remain the same, the y which was there it has become –y. So this is what is refection against the x axis. Y goes to –y, so it is defection against x axis right.

And similarly if you look at -1 0 0 1 that gives you reflection against the y axis so reflections they are obtained by this simple matrix transformation in the plane right.

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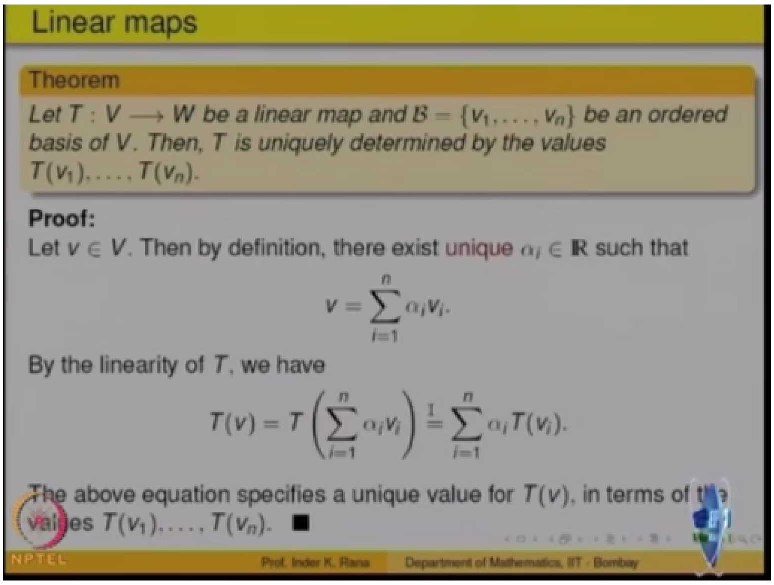

So we will come across many more slowly. I just wanted to introduce 2 of them. So rotation has been done. Reflection has been done okay. We will look at examples more a bit later. So now here is, so linear transformations become important from these 2 examples anyway right. They are the transformations which help us to represent rotations and reflections at least. Now here is an important theorem which says if T is the linear transformation from V to W.

And let us fix a basis of the domain, that is V. So V1 V2 Vn is the basis and not only it is the basis we have fixed at the order of the vector which are up hearing in that as a set that also is fixed. We fixed the order also right. So we will remain as the first one always we will not shift the position of V1 V2 or anything. So such things are called ordered basis right. You take a basis and fix the order in which the vectors in the basis are going to be written.

Okay that is called an ordered basis so what does basis of V mean, what is the meaning of saying that something is the basis of vector space V? Every vector in V is the linear combination of the basis elements not only it is linear combination it is the unique representation. If a vector V is written as sigma alpha iVi then those alpha is are unique you cannot have more than one representation because of independence if you recall.

That is what the importance of independence was generates and independent. Everything should be a linear combination and unique representation should be there. So we are fixing a ordered basis V1 V2 Vn, then the claim is T is uniquely determined by the values on this n vectors V1 V2 Vn that means T is the liner transformations on a vectors that is V. If we fix it is values on basis elements, then it is value for every vector is fixed.

That means to determine a linear transformation completely on a vector space of dimension say N there is an N elements in the basis. You need to know only the values on the basis elements. Where does this basis vector go that is all, nothing more is required. So let us see why there is so. So let us look at any vector V by definition we have got a basis V1 V2 Vn so every vector must be a linear combination and unique one.

So V is the unique linear combination of V1 V2 Vn, that means V is a linear combination that means what V = alpha 1 V1 + alpha 2 V2 + … alpha and Vn and this scalars alpha 1, alpha 2, alpha n depend on V, but they are unique for V, you cannot have 2 choices for any one of them we have only one choice for representing V as a linear combination that is uniqueness. But if T is this so what is T of V, T of V is T of the linear combination.

But T is linear, so what is T of the linear combination, it is the linear combination of the individual ones. So T of V is T of the sum, sigma linear combination which is same as sigma alpha iT of Vi/linearity of T right. Now if this Tis are fixed T of Vi are fixed right. We have said and only choice is alpha I right. If you know the values of TVi's alpha i's are fixed, then this value is fixed now right.

Because given a vector V alpha i's are fixed right. This is weak choice and if we know T of Vis we know T of V right. So once you fix the values of a linear transformation on the basis vectors it is value for every vector is fixed there is only unique way you can do it right and that is here that if v is sigma alpha iVi then T of V is sigma alpha i of linear combination of T Vis so if T Vi are fixed alpha i's are unique for the vector V so everything is fixed right.

So this equation specifies that unique value for T of V in terms of the values TV1 TV2 once you fix them so these are very important which says that to know value of T on any vector in the vector space you have to know only the values on the basis elements right that is what is important right.