Basic Linear Algebra

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Lecture - 21
Determinants and their Properties - III

See till now we said that let us start with a function D on n * n matrices with those 3 properties, what are the properties? First property was, if 2 rows are identical then = 0.

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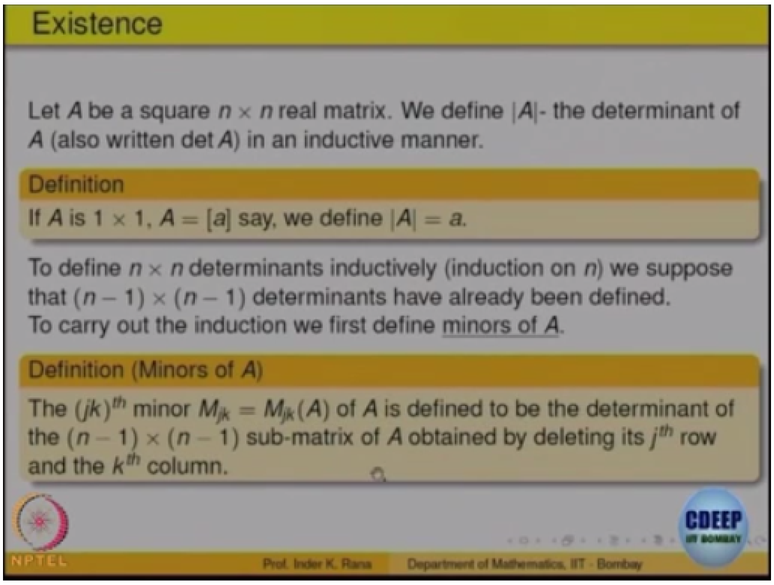

Second property linearity in each of the row vectors, third determinant of identity =1 and we deduce all the consequences, but we did not show that there is a function with that properties those 3 properties, does there exist a function with those 3 properties? We assume there exist one and deduced what are it is further properties. We did not say there is one right. Now the stage has come to say that they actually exist and there is only one such function.

So what we have done is we assumed existence of a function called the determinant function with some properties, basic properties and deduced some other properties, we did not say that such a function exist right. So existence we have to show in mathematics we not only have to say that something exist as these are properties, we have to show these are the properties and this either exist one function and only one.

Then only we can say that all function will have those properties right. That means for every matrix A there is a unique concept of associating with it a determinant, such that those 3 properties hold right. So that infect is what you have been doing all along expanding a matrix by rows or by columns. See when you are given n * n matrix you have used that fact that we can expand it by any row or by any column, that means what?

See when you try to expand, let us say 3 *3. Let us say how do you expand 3 * 3 so that you understand what you are going to do.

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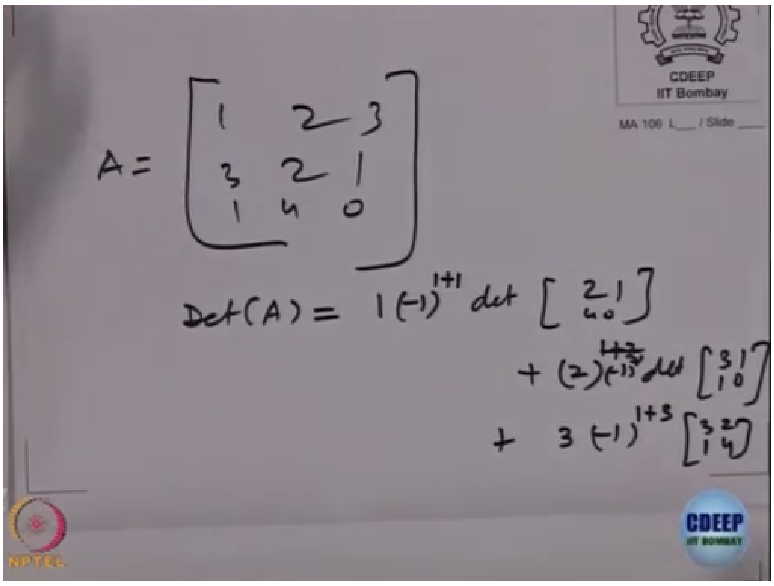

Here is a matrix let us say 1 2 3, 3 2 1, 1 4 0. So these are matrix A. Expanding determinant of A what we normally do is let us say by first row, so you look at the coefficient that is 1 * -1 raise to the power 1 + 1 forget this row and forget this column, so you write determinant of 2 1 4 0 right. Next one 2 so + 2 raise to power, so in the first row second so 1+ 2 * determinant of, sorry this is -1 raise to power that thing, okay into determinant of that row is gone, that column is gone it is 3 1 1 0.

Plus, the next coefficient is 3 - 1 raise to the power first row third column. So first row third column multiplied by that row is gone, that column is gone, it is 3 2 1 4 right that is how. So what you are doing is computation of determinant expanding by first row you are reducing the problem to determinant of 2 * 2, we started with 3 * 3 computing you have reduced the problem to 2 * 2.

How do you compute for 2 * 2 by the same method? So it is the inductive process of computing same is the definition of determinant. So let us formulate the definition of determinant. If the matrix is 1/1 induction, we want to define the notion of determinant of a n* n matrix so how do you define? Define it for n = 1. So for n=1 what is the definition, it is only 1 scalar right. So it is that scalar itself as the determinant only 1 row 1 column so all properties will be satisfied.

Assume you have defined notion of determinant for n-1 * n-1 matrices, that means for every n-1 * n -1 matrix you have the notion of determinant with those 3 properties. I want to define what is the notion of determinant for a n * n matrix so that is defined by looking at what is called, what we wrote as removing that row, that column that is called a minor right in your thing.

So define what is the notion of a minor. So jk'th minor is right determinant of the sub-matrix deleting j'th row and the k'th column right j'th row k'th column delete you get a smaller matrix of the lower order, take determinant of that that is called a minor okay.

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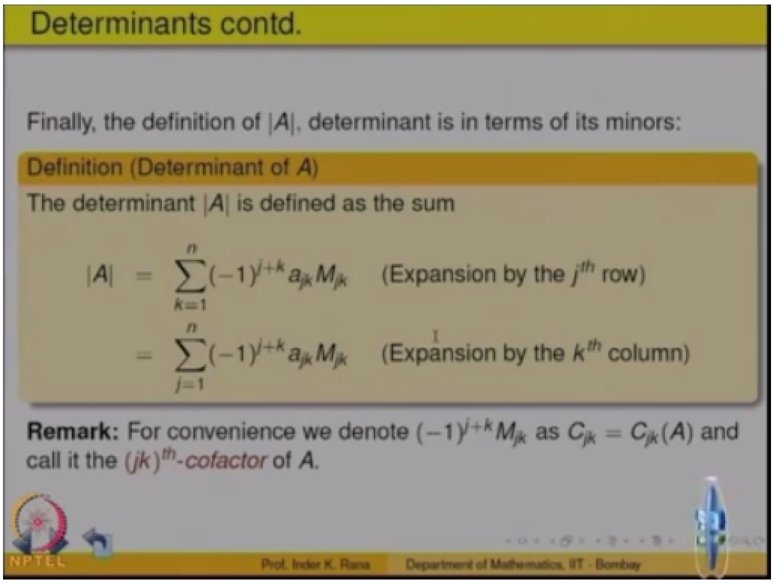

So in terms of minors this is what how did you find determinant. Determinant of n * n matrix is -1 to the power. So this is expansion by the j'th row, you are expanding it by the j'th row. So -1 to the power j+ the k'th column entry will be, coefficient will be Ajk into the corresponding determinant of the minor removing those row and column, is it okay. So this is the definition by expanding by j'th row.

So there could be k definitions if there are n rows you can expand by any one of the rows. You can also expand it by any one of the columns by the similar things, by deleting appropriately right. So this does not say there is only one definition. How many definitions are possible for determinant? There are k square definitions possible right and one has to prove a theorem that all of them have the same property and all are equal.

Basically that normalizing factor says that all have to be same okay. So we will not prove that so this gives you existence and one shows that the fine this way there is only one concept of determinant meaning what that this definition of determinant which ever row or column you take as those 3 properties. So it is rightly deep theorem. It has all those 3 properties and each one of them will give you the same value right, they are all equivalent to each other.

So you can use any one of them as per your convenience right. So this is more useful, this definition is more useful, this definition is useful in computing that determinant. It is the inductive process. So you can feed it to a machine also to do the job, but this is also existence and uniqueness, while proving if you use this definition and try to prove that for example that determinant of A inverse is 1/determinant of A.

You will have hell of a time, first of all how do you compute using this definition of determining what is A inverse, at present we do have no method of computing A inverse and then proving for every matrix determinant of A inverse if it is invertible it will be one over that, this is of no use. There the theoretical definition that these are the abstract properties they were helpful.

We were able to prove without much of problem right. Okay, so expansion by rows and columns I think example you already know what is the example.

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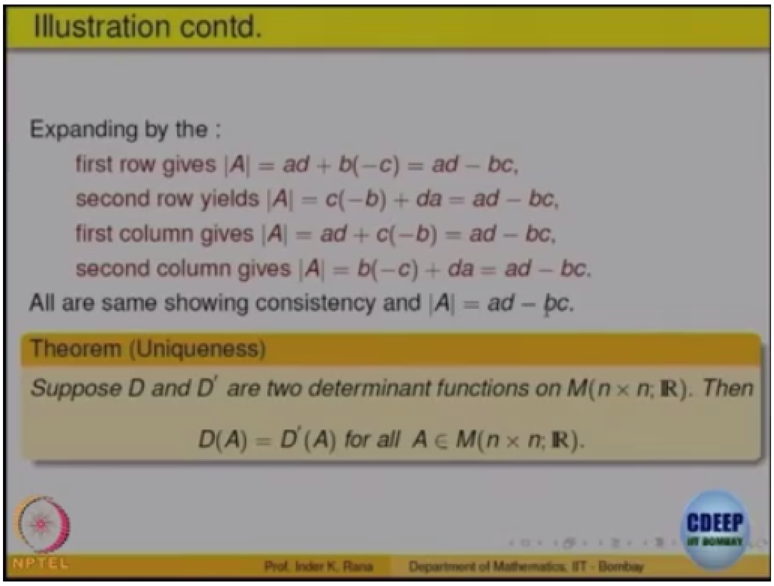

And you can verify that by 2 * 2 whether you expand by any row by any column you will always get the same number, that essentially is not a magic it is happening because of the property that expansion by rows or columns is a unique definition. It gives the same property. So that is uniqueness. We will not prove this we will assume that. So definition of determinants by expanding by rows is column is a definition which gives a unique way of expressing determinant as a function on a matrix with those 3 properties.

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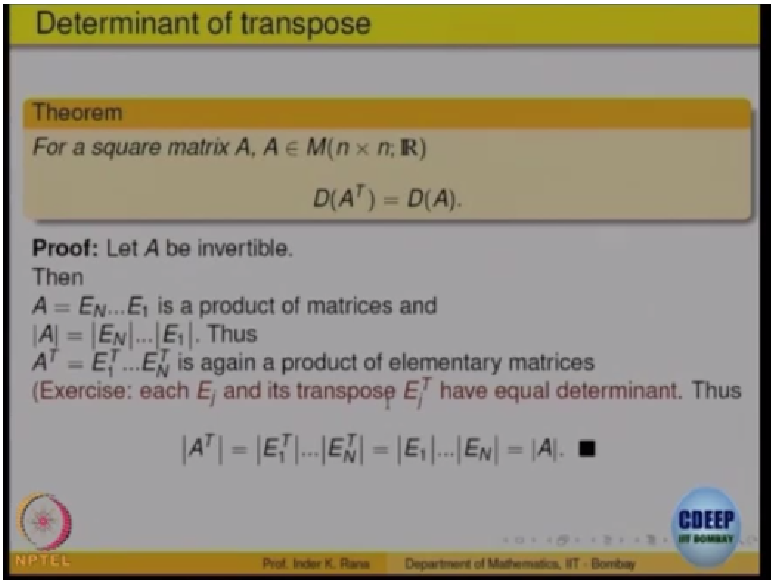

How do you prove determinant of A = A transpose? You can say okay let us prove we have expanding by rows and columns and same we try to expand by columns and rows and do something, but there is very simple way of doing it, again by using our abstract definition, let us say A is invertible, A invertible, not invertible 2 possibilities right. If A is not invertible then determinant of A is 0.

What is determinant of A transpose? The one of the columns or rows right. If the row is 0 in A then column will be 0 in the reduced row echelon form, so similarly. So if A is invertible it is the product of elementary matrices. So determinant of A will be product of those elementary matrices. A is invertible means it is the product of elementary matrices right is it okay. Now only thing is to show if elementary matrix is there and take it transpose they have the same determinant.

For elementary matrix you take elementary matrix and take it is transpose the 2 will have the same determinant that is simple because lot of zeros are coming right. So that we leaving as exercise note it down that if Ej is any elementary matrix of anyone of those types and you take it is transpose then they have the same determinant. One set is (()) (11:25), what is A transpose?

If A is this, A is En * En what is A transpose? Transpose like inverts the thing right. So if A is = this then what is A transpose it is E1 transpose * E2 transpose * En transpose. So what are determinant, they are all again elementary matrices. So determinant is the product, is product of the elementary and that is equal to this (()) (12:00). So the problem is reduced to elementary matrices again.

Again and again we are using that the matrix if it is invertible it is the product of elementary matrices and determinant for elementary matrices have the product property. Determinant of product of elementary = product of the corresponding elementary and here we are using extra fact which is the exercise that if E is elementary look at it is transpose it is determinant is same as right, it is same as determinant of the matrix.

If you like you can also use row expansion by row and column because we have assumed now uniqueness. Now all those are useful, these are available now right, expansion by row or by column okay, but again important thing to notice to prove such a thing if you try to expand such a matrix by rows or by columns and then try to do that computations, compute the inverse expand that by rows and columns and show determinant of A = transpose you will have a hell of a job of verifying that right.

As a theorem abstract theorem it is quite useful and need to prove.

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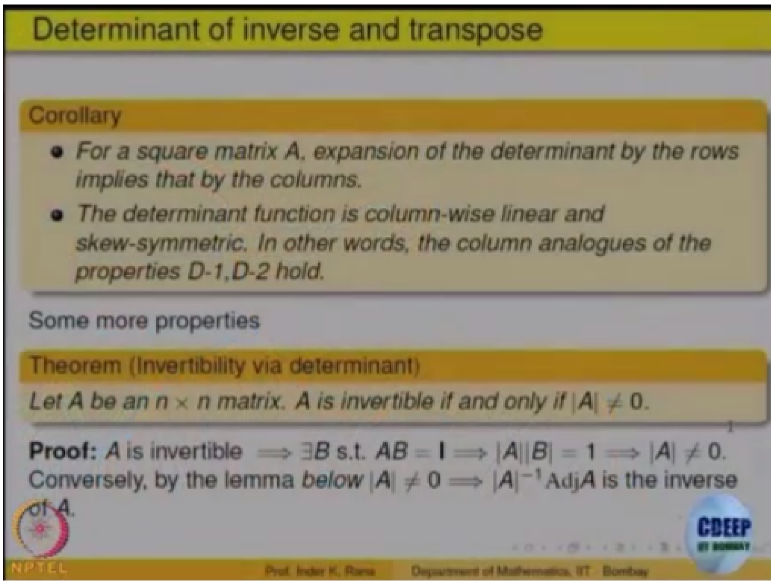

So as a consequence of that now you can say that like you have been applying row operations you can also apply now column operations to compute the determinants, they would not change because determinant of A is same as determinant of A transpose now. So you can do that. So invertibility we have already checked if only if determinant A is not = 0.

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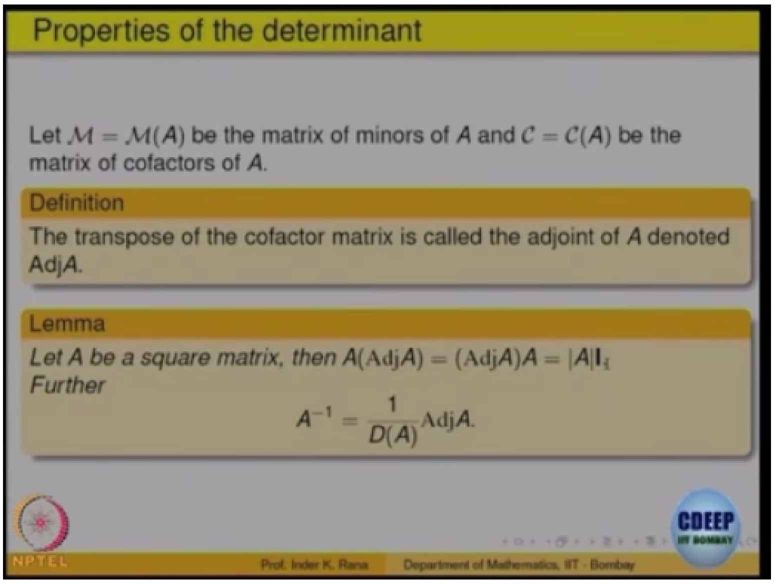

Now here is something which will not prove everything here because they are messy to write. See look at the transpose of the cofactors of a matrix that is definition of what is called adjoint of a matrix right. So what is the adjoint of a matrix? First of all, we looked at minors and then we looked at the cofactors right. So look at the transpose of the cofactor, look at the cofactors that give you a matrix.

Look at the transpose of that that is called the adjoint of the matrix A and one proves a theorem namely A * adjoint is same as adjoint of A * with A and that is = okay, actually one should write not this is determinant of A that is scalar times identity. The product of the 2 matrices namely if you look at the adjoint of the matrix, adjoint of the matrix multiplied using A right and the left or on the right you get a scaler times identity matrix and the scalar is noting but the determinant of the matrix.

This theorem is not really straightforward to prove, workout this lemma, you have written as lemma, one has to do a bit of computation to see this, so we will assume this fact then this gives you what is A inverse, if this is true then what is A inverse determinant right. You can divide by that. So A inverse is 1/determinant of A * adjoint of A that you have been using earlier.

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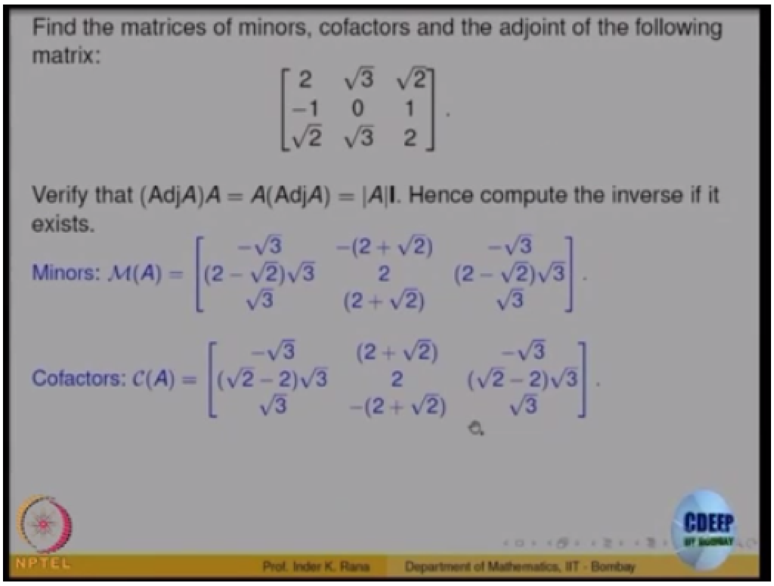

So let us just these computations we have been doing so we will not go into all aspects. Let us look at one example you are given this matrix okay. So how do you compute adjoint of this matrix? Look at the minors right. So this row, this column okay and this row, this column expands by any, these are minor, these are so let us not bother too much about the computations verification of that. Basically you compute the minors matrix, you compute the cofactors matrix.

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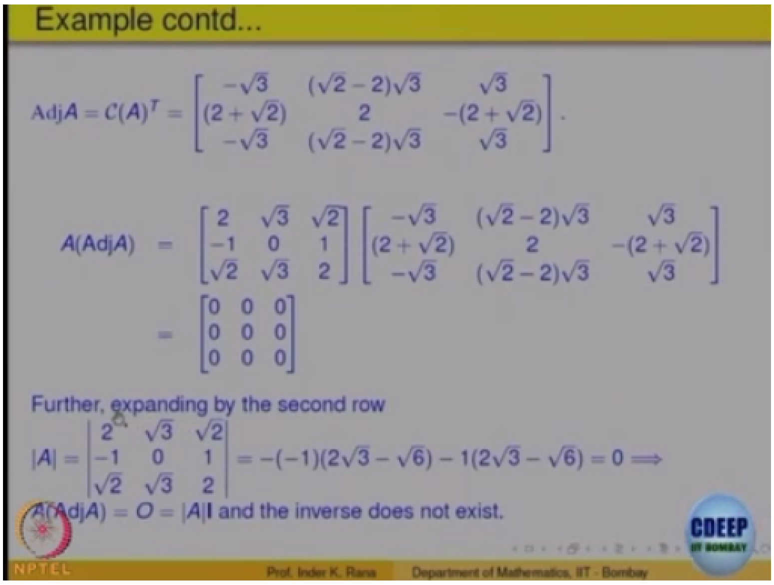

And then you compute what is adjoint, the transpose of that gives you this right and you multiply and see that it may come out equal to determinant of A identity or it may come out to be something else. Here it comes out to be equal to 0 that means the matrix is not invertible right you cannot do anything. So earlier that was only for invertible matrices because you can divide only when determinant, that identity is okay.

A * adjoint of A = adjoint of A = determinant of A times I that is okay, but you can divide only when A is invertible that means determinant should not be = 0 right. It could come out to be = 0 determinant actually if A is not invertible this product will come out = 0 because in that case determinant of A is, what is determinant of A if A is invertible? it is 0, so A * adjoint of A will be = adjoint of A * A = 0 times identity.

That means it should come out to be a matrix which is identically 0 for, so this is one way of what is hell of a tedious way of checking some matrix is not invertible we have a simpler method of checking, reduce it to row echelon form and check whether it is invertible or not right and that also gives you a inverse. How to compute the inverse also quite easily by tracking what are the row operations you are doing?

This is the complicated way of finding the inverse. So anyway this example illustrates the fact that for a noninvertible matrix this could have come out to be = 0 because we can check the determinant of that = 0.

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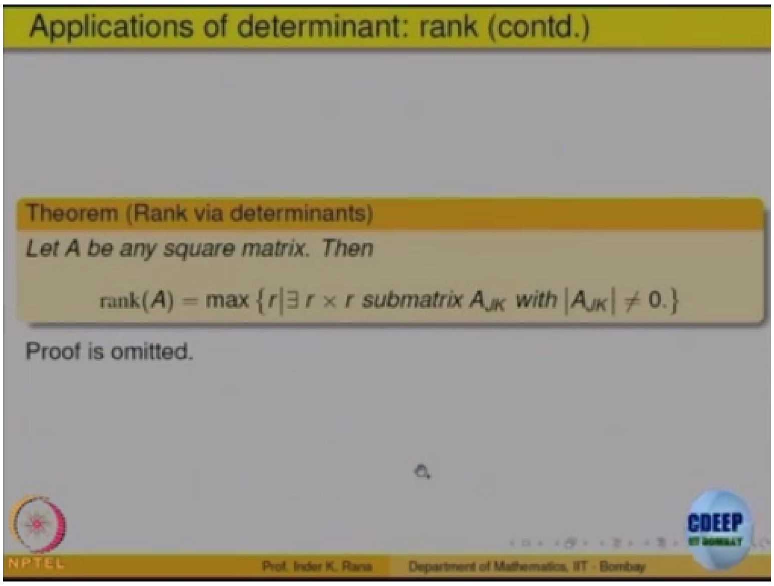

So here is another theorem which will not prove, namely it says that the rank is related to also determinant. You are given n * n matrix right you can form sub-matrices of lower order right. Either the determinant of the whole matrix is not = 0 what does that say, determinant not = 0 the matrix is invertible, rank is full or the determinant is 0, so matrix is not invertible now look at the sub-matrices of order m-1 * n-1 there are many possibilities right sub-matrices.

If one of them has full rank n-1 cross right, then the rank of the matrix will be n-1. So go on looking at smaller sub-matrices till you find a matrix of some order which is having rank full right and higher order have rank not full understand what I am saying. For example, let us look at a 3*3 what are the possibilities.

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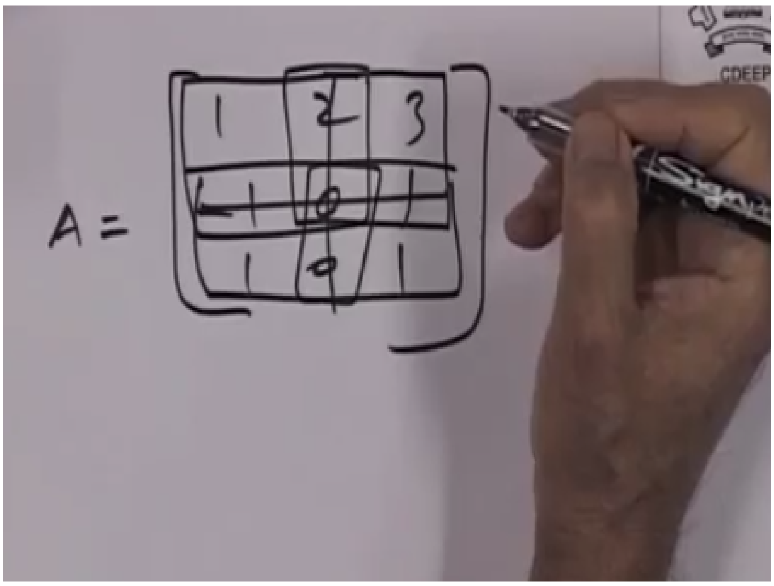

So let us 1 2 3 – 1 0 1 1 0 1, so this is the matrix of order 3 * 3, I do not know invertible or not let us not bother what I want to indicate is how many sub-matrices are possible. You can have a sub matrix 2 * 2, you can have this 2 * 2, you can have this 2 *2, you can have this 2 *2 or you can have forget about this you have another 2 * 2 lot of 2 * 2 sub-matrices are possible. Check that determinant of each one of them.

If one of them is not 0 right that means rank should be = 2 because 3 * 3 if it is 0, if the original matrix has determinant 0, it ranks cannot be full, but I have what a sum matrix of order 2 * 2 one of them at least which has determinant not = 0 that means rank of that = 2. So rank of the full matrix will be = 2. So find out, so it says look at the R, look at the maximum value of R, the largest value of R, so that the matrix of that order R * R determinant is not = 0.

Or higher are 0 but that is not 0 at least 1. Then the rank is that number. That is another complicated way of finding the rank but it is a useful theorem improving theoretical results right. So this is one of the ways of relating the determinant with the rank of the matrix right. So I think we do not have much to say about determinants example is there right.

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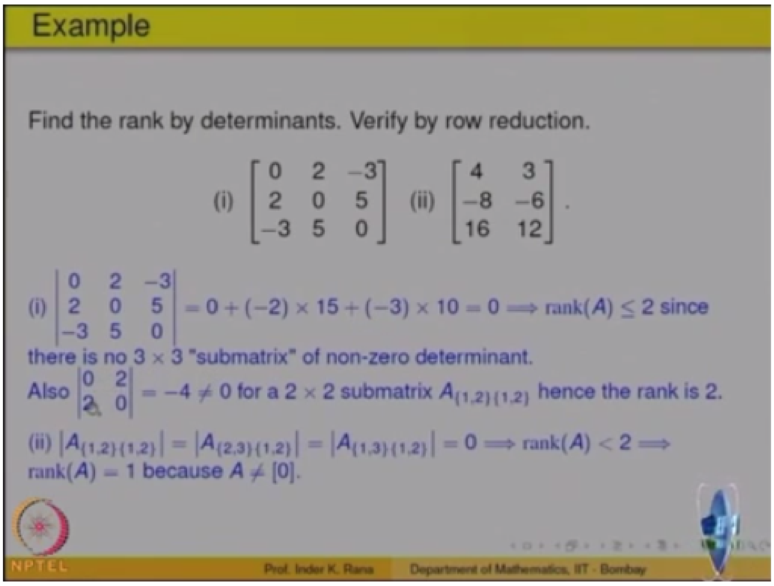

So here for example for this matrix right if determinant comes out 0 but the rank is 2 since there is a matrix 0 2 2 0, so 0 2 and 2 0, this matrix right 2 * 2 has got determinant not = 0, determinant is -4, so rank of this matrix = 2, so that is another way of saying the rank.

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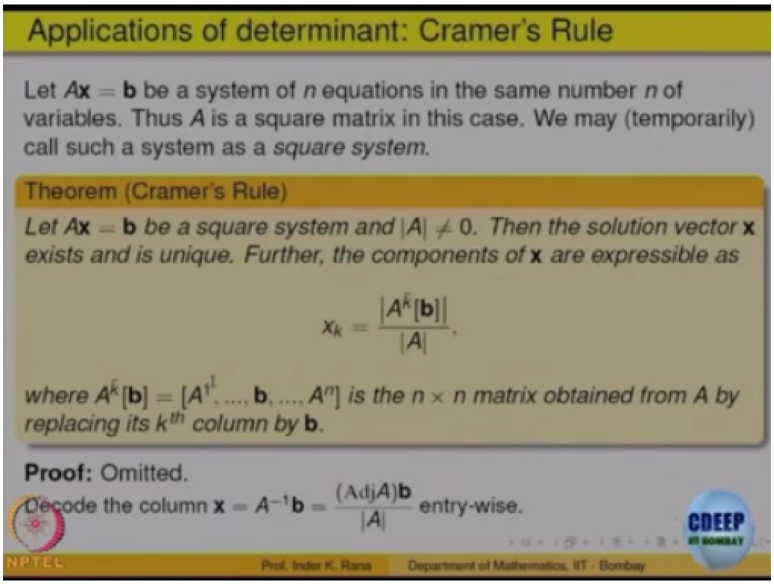

I think I would not say much about this but just recall there is something called Cramer's method of solving a system of equations that is a very specialized method that works only when the matrix is as first of all n * n matrix, n system, n equations and A should be invertible then how to find that unique solution right. Historically it was of importance and useful. So what you can do is you can find xk by that formula.

What is this Ak had, that is replaced right, you want to find the k'th entry in the solution so in the given matrix remove that k'th column by b, remove that k'th by b, see Ax = b, you want to find a solution of that you know already A is invertible, you want to find x, that means you want to find X1 X2 Xk right. So what you do is find out determinant of A which is not going to be 0 because we have assumed that okay.

Look at the matrix A in that the k'th, we are looking at the k'th component of the solution. Look at the k'th column replace that column by the vector B, which is on the righthand so you get another matrix right. So that matrix is the numerator determinant of that this divided by this gives you the k'th entry. So in the given matrix if you want to find the k'th component of the solution replace the k'th column by the B find determinant.

So that is the Cramer's rule, we do not normally use it for computational purposes, because it is very time consuming. You have to find determinants of all of them and computation of determinant is a very time consuming job on a computer because for n * n how many sub determinants you have to find out. For n you have to go to n-1, for n-1 you have to go to n-2 and so many right.

So finding determinant by a machine is not time efficient way of doing it, but this method has used in some other places so it is kept okay. So with that we finish this lecture on determinants. We want to start now something new. We will start looking at a dynamics of linear algebra. The dynamics means when things move right. So we look at what is called linear transformations in linear algebra.

Linear transformations are map it takes one vector and throws it somewhere else. So we will do it next time okay. Let us stop here. Thank you.