Basic Linear Algebra
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Lecture - 20
Determinants and their Properties - II

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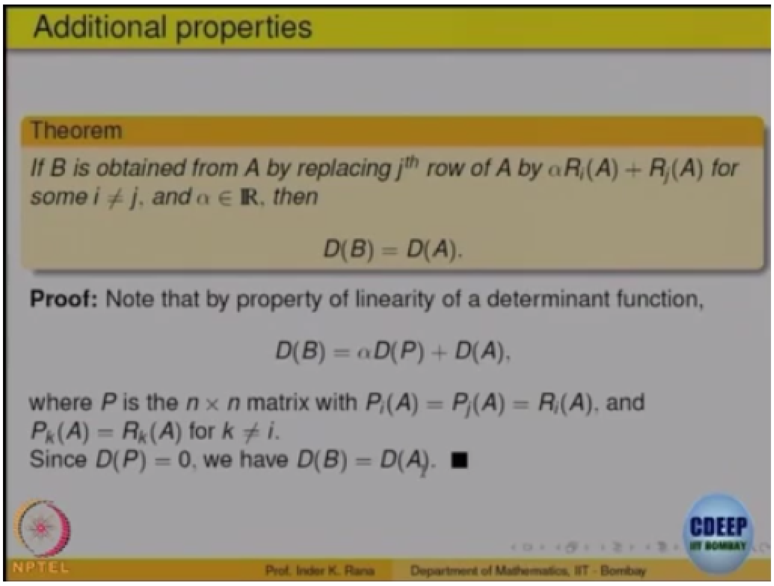

If B is obtained from A by replacing j'th row by alpha Ri + Rj I am slowly I am adding towards that elementary row operations remember one of the elementary row operation was multiplying by a (()) (00:41) nonzero that we already taken care by the definite property, alpha, lambda comes out right and this if you are taking a i'th row, multiplying by alpha and added to j'th one then what happens to determinant, I am trying to look for that and it says then determinant does not change.

Determinant remains the same. Why what will happen because in this row right this is only this row is right changed j'th one so in the right hand side when we apply linearity what will happen one matrix will be R1 R2 and j'th place and j'th place same row will be repeated right. This is i'th one so this will be i'th and i'th will be coming and here i and j will be coming that will be A itself.

Are you getting this is the j'th place this is the j'th entry for the new matrix, this is the j'th row for the new matrix. So what will be the i'th row if I take only the first one for linearity, it will be alpha Ri, Ri is already there this is j'th place. So alpha comes out so i'th and the j'th row in the new matrix will be identical. So this part is not going to contribute in the determinant for the second part it is the A itself, i and j so that does not change right.

So determinant of the new matrix will be same as determinant of the original matrix right. So let us write that, the new matrix is alpha times D of P + D of A what is P? P is the matrix where i'th and j'th rows are same = Ri right i'th and j are Ri, so that will be = 0 right and the DB = D of A other one is stable. Again linearity thus expanding those rows right. Using linearity and writing in the new form, is it clear?

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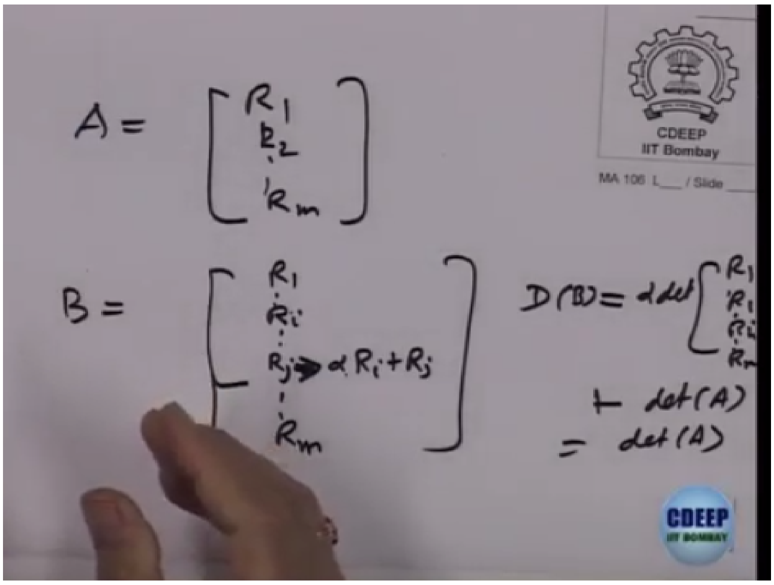

See let us say A = R1 R2 and Rm right. Now what is B, I have interchanged i'th, so I have to replace, so R1, up to Ri is the same. In Rj what has happened this Rj = alpha Ri + Rj. So this I should not write equal this has gone to this row right and remaining is Rn. This is a new matrix B. So what will be determinant of B R1 Ri and this will give you alpha Ri Rm. So this will be alpha times determinant of R1 Ri and again Ri at the j'th place and Rm + R1 Ri Rj and Rm. So that is determinant of A is that okay by linearity right.

R1 Ri alpha Ri alpha comes out Ri Ri i'th and j'th place other is just A. So this is = this is 0 so it is = determinant of A that is all nothing more than that right. Again using linearity property and the property if 2 rows are identical then it becomes = 0.

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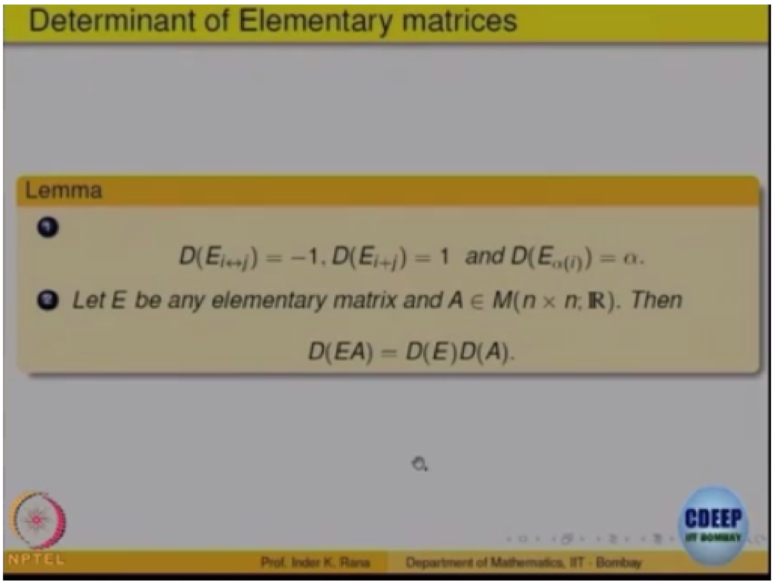

Next we want to compute the determinant for the elementary matrices. So what is the first elementary matrix that was interchanging i with j right, i'th row with the j'th one. Take identity matrix in that interchange i'th with the j'th that is same as interchanging i'th with the j'th. So what is determinant of that -1, just now we showed that in any matrix if we interchange i'th and j'th row then it gives a negative sign right.

So in particular for elementary also, second one was i + j, if you take j'th row and add it to i'th row what you should get, just now we saw if alpha Ri + j right it is the same as the original one, here what is alpha 1, it is i + j, so one times i + j that is put in the j'th place. So that is same as determinant of the matrix ei and what is that? that is 1. What about alpha times i, that is linearity? When you multiply a particular component by alpha rho, alpha comes out, linearity property.

So that is alpha time, so determinant of the elementary matrix which is multiplication i'th row by alpha is just alpha. So these are the 3 elementary matrices and these determinant we have computed. First one interchange i with j that gives you a negative sign right. So that elementary matrix determinant is -1. You take the elementary matrix with i'th row add it to the j'th row, determinant remains 1.

Determinant original was 1, it remains 1, it does not change and third one multiplying a particular row by alpha by linearity that is = alpha times original one that is = 1 right, is it okay. Is that clear? alpha times that is 1, that is alpha, identity matrix, what is that determinant of identity matrix is 1, that is the definition, normalizing factor, third defining property, first one was if 2 rows are identical that is 0, second was linearity, third was normalization.

Determinant of identity is 1 right, so because of that this is alpha, okay. Now let us apply this to matrices, take a matrix A and multiply by an elementary matrix, you will get a new matrix right. So if this A is interchanged what you get here that will be negative or determinant of A -1, but the -1 is the determinant of E, right, so I can write that property as determinant of E times A is same as determinant of E times determinant of A when E is interchange of 2 rows.

Same with other 2, i + j that is same as 1, so that is same as i + j and the third one same, right. You can easily see that these 3 properties. So what I am doing is I am interpreting these 3 property in terms of production of, so it says determinant of an elementary matrix any of the 3 types multiplied by a matrix A right. So this above equation can be written as determinant of E times determinant of A, is it clear to everybody, yes.

Because here determinant of interchange was -1, so what will this give you interchange right. So basically now you can see that I am heading towards trying to prove a property that determinant of A times B = determinant of A * determinant of B. This is the first step towards that. So what you are saying is determinant of a matrix multiplied by pre-multiplied by a elementary matrix is same as the product of the determinants right and that follows from the simple identity, okay. So let us go a step further.

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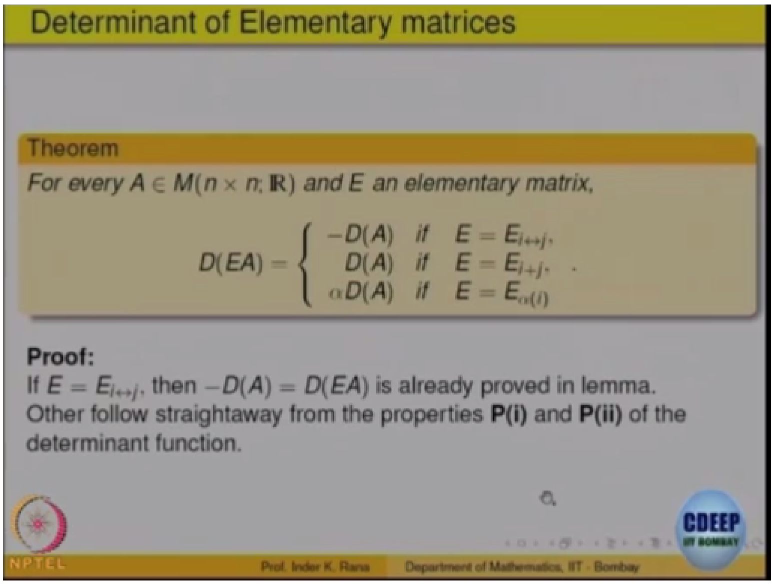

That is same as written again EA = -DA, so that is written again, so we can write that as theorem okay.

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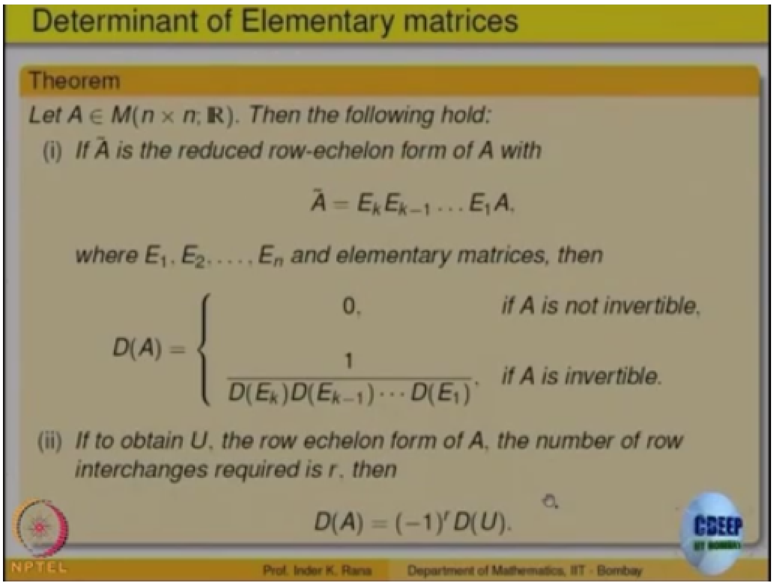

Now let us come to consequences of, supposing a matrix A is changed by row operations 2A tilde that is a reduced row echelon form of A then what it will look like? some elementary matrices multiplied by A, right that we have seen. So what will be determinant of A tilde, that is determinant of the righthand side. So I can write it as Ek multiplied by the remaining one right, inductively I am going to apply the previous theorem now.

So it will be determinant of Ek multiplied by determinant of Ek -1 into up to A again apply the same property. So what will be determinant of A tilde, determinant of Ek, multiplied by determinant of Ek-1 * determinant of A, right is that okay. So from here what is the determinant of A then, I can divide it by those constant right equations, so determinant of A will be 1/determinant of Ek * determinant of Ek -1 * determinant of Ek provided none of them is 0.

So we know that determinant of each Ek is not 0? Yes, just now we computed right, either it is 1 or it is -1 or it is alpha, where alpha is a nonzero right just now we checked. We checked for elementary matrix determinant is either -1 or 1 or alpha where alpha is 1 0. So determinants of elementary matrices are nonzero. So using this we will get determinant of A = 1/determinant of, now here on the right hand side is this numerator is determinant of A tilde in that equation.

What is determinant of A tilde in the reduced row echelon form? what is determinant of A tilde? that depends on whether the matrix is, what is the form of that A tilde either it will be identity it is a square matrix, so one possibility the reduced row echelon form of a matrix A will be identity in which case it is determinant is 1 or there will be a row at the bottom which is 0. If that is the case what is the determinant? that is 0.

So you get 0 right. So the 2 possibilities arise, if A is not invertible then it is 0 right and if it is invertible this is 1, this is 1 over of this, clear. That is just from the form of reduced row echelon form, it is the product of elementary matrices into A, take determinant both sides right and use a property that determinant of A elementary matrix multiplied to a matrix = determinant of the elementary matrix multiplied by the determinant of the matrix.

The product property is 2 power, when one of the product is elementary matrix. So using that we get this, right if A is invertible then determinant of A is 1/determinant of this. Now keep in mind if this elementary row operations involve only interchange of rows say and there are R of them to bring it to, then how many times -1 will be coming? Each interchange will give you a – sign.

If there are R times interchanges required then it will give -1 to the power R and that precisely says if to obtain new row echelon form number of row interchanges it required is R right that means the determinant should be -1 to the power r times determinant of the original one, because each one is going to give you right from here right. So these are useful in computing. So we will see how they are useful in computing, yes the proof of this clear.

It is just saying first step reduced row echelon form is the product of elementary matrices into the matrix A, apply determinant on both side, use the product property and write down. Okay.
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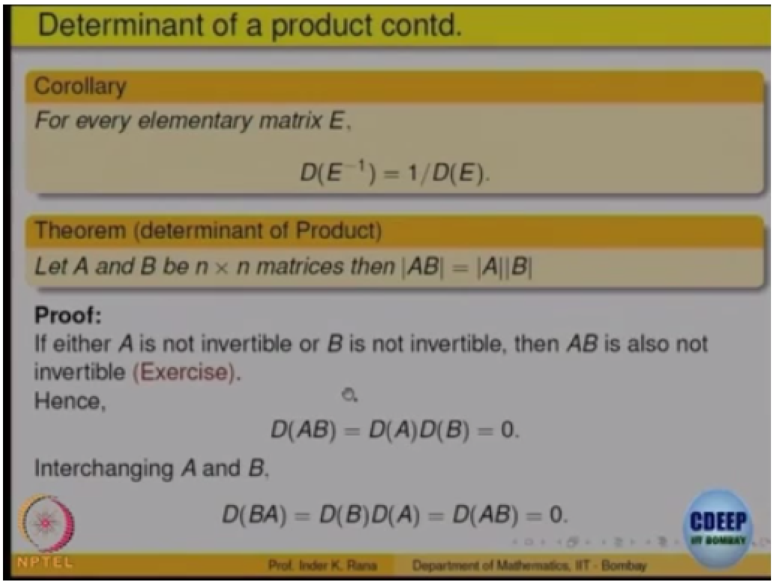

So what is determinant of E inverse for elementary matrix, now the computing the inverses when coming to determinant of the inverse now, there are 2 ways of looking at it. There are 2 ways, one, you can compute what is E inverse. If i and A are interchanged then they were again interchanged right to get the inverse, to get back the identity matrix, so twice right or you can look at E times E inverse is identity right. So here is another easy way of looking at it. Let us look at that is better probably.

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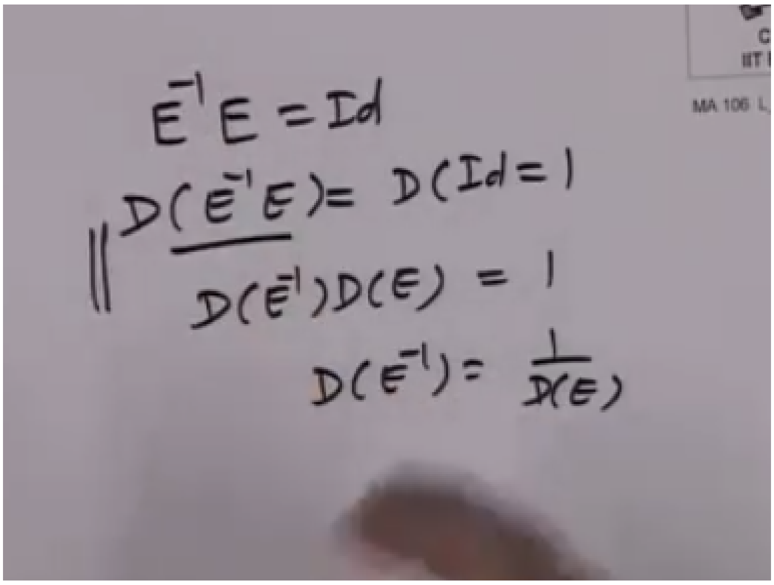

So E inverse * E is identity right, for any matrix anyway A * by A inverse is identity. So what is determinant of E inverse E. So that is determinant of identity and that is 1. Now here is an elementary matrix multiplied. So what is this = so determinant of E inverse determinant of E is that okay. For elementary matrices we already know about product property so that is = 1, so what is determinant of E inverse, that is 1 over determinant of E right.

Same proof later we will work when any matrix A is invertible, but at present we do not know that relation right, that for any 2 matrices determinant of AB = determinant of A * determinant of B. We do not know that yet. We only know for elementary matrices this property is true so we are using that right. So that show you slowly how rigorously we are building up our concepts, our known theorems from simple 3 properties only okay.

So now we come to the fact that determinant of AB = determinant of A * determinant of B right. Proof is quite simple. See there are 2 possibilities one, either A is not invertible right A * B right their square matrices possibly either A is not invertible or B is not invertible if that is the case then the product cannot be invertible. So a simple fact, we are leaving it as exercise for you to check that if either A or B is not invertible then the product cannot be invertible right. So check that.

So in that case what will happen, if AB is not invertible what is determinant, just now we checked, if A is invertible if and only if the reduced (()) (17:00) is identity and determinant = 1 or 0. If invertible right reduced row echelon form determinant will be 1 otherwise it will be 0. So if AB is not invertible, it is determinant = 0 and now at least either B is not invertible, so one of the term on the righthand side is 0 so that is also = 0, right.

So that shows that determinant of AB = determinant of A * determinant of B when at least one of them is not invertible. So what is the second case, both are invertible. If both are invertible then we know AB also is invertible simple algebraic fact, but let us check it anyway. So let us check.

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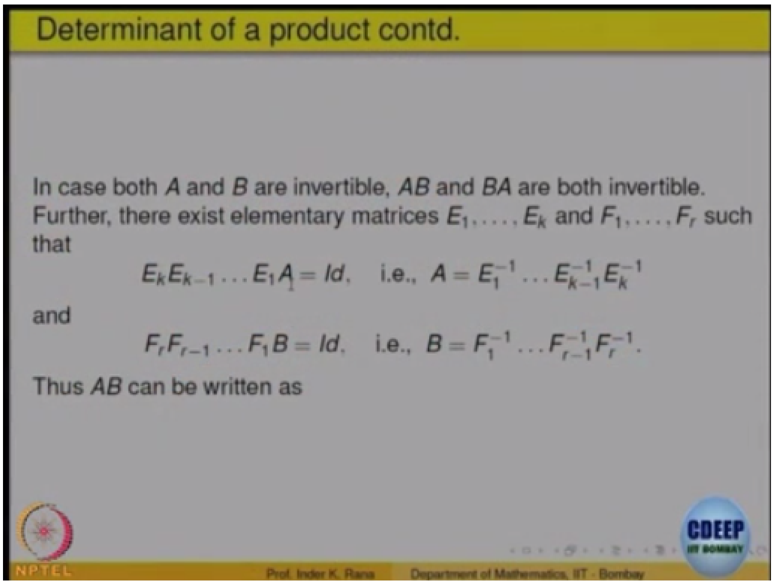

In case both A nad B are invertible AB and BA both are invertible anyway right. Is that okay. We saw AB inverse is nothing but B inverse multiplied by A inverse you can check that is it okay. If A and B both are invertible then the product AB is invertible then the product AB is invertible what is it is inverse B inverse multiplied by A inverse right. So that is okay. So let us write.

If A is invertible so it is reduced row echelon form must be identity so there will be some matrices elementary matrices E1, E2 right Ek so their product with A = identity. So what will be A = i can take on the other side as inverses. So A = E1 inverse multiplied by Ek inverse, Ek inverse right. Similarly, what is B, B is again invertible so it is reduced row echelon form identity, so I should be able to premultiply it by elementary matrices so that it becomes identity reduced row echelon form with identity.

That means B = this, so I know what is A what is B, so what is A * B, this is A, this is B, where I multiply them both right. See if I multiply what I get.

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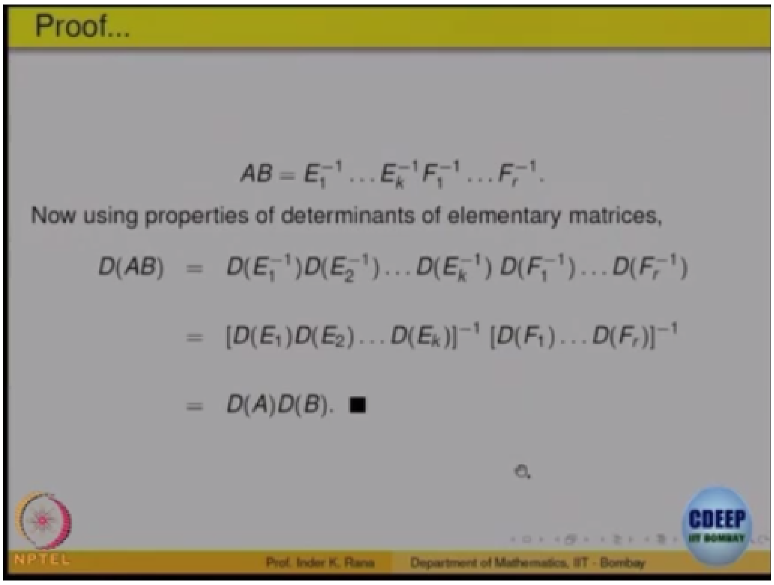

So AB = right this is A E1 inverse up to Ek inverse, F1 inverse, Fr inverse that is B. Now all of them are invertible matrices right. So that is invertible okay. So now let us apply what is determinant of the product now compute determinant of both sides now. Determinant of AB will be equivalent to determinant of right hand side. The right hand side is the product of elementary matrices. All are elementary matrices.

Remember we marked earlier if E is elementary it is inverse is also a elementary matrix right. So these are all elementary so determinant of AB is determinant of the right hand side, but here the product property knows for elementary matrices. So what is the determinant of the right hand side. It is the product of determinant of E1 inverse, product determinant E2 inverse and so on right, is this okay, clear to everybody, yes.

Now this first ones right let us. So what is determinant of E1 inverse, that is 1/determinant of E1 right. So write use that formula to get that this first product of the first k is 1/determinant of E1 E2 Ek inverse, right and similarly the other one okay and what is determinant of E1 * determinant of E2 * determinant of Ek, the product of E1, E2, Ek was A right.

So this is precisely the first term here right inverse is precisely determinant of A, second is precisely = determinant of, see what was A you remember? okay, let me just go back. What was A= this, so what is determinant of A? it is determinant of the right hand side right. So product you can write 1 over that thing so that is what we have done. So that this thing or you can write directly from where that is = determinant of A and the next thing is determinant of B.

So again by, so slowly what we have done is we have looked at what is going to be saying that A is invertible that means we can premultiply by elementary matrices and write it = identity so that is one fact we use and second for elementary matrices determinant is as the product property, determinant of product of elementary is product of the determinants of those elementary components.

Those 2 properties are used just to reduce that determinant of the product AB = determinant of A * determinant of B right. So this is only 1/1 we have used this facts okay. So we have the, now I can write down determinant of if is invertible so what is B, B is A inverse so I can apply that trick again A * A inverse is identity so what is determinant of A inverse 1/determinant of A. I can use this now to deduce that right. Earlier we did it for elementary matrix.

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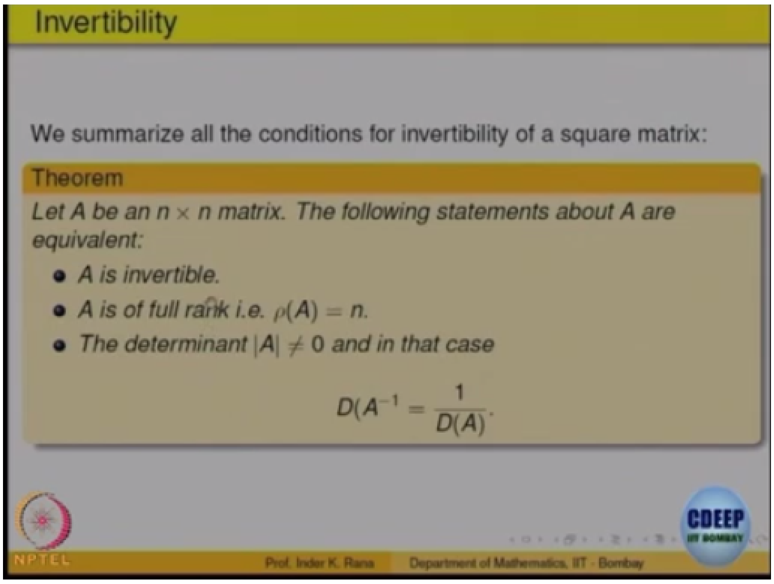

So I can write now equivalent ways of describing invertibility first one was A is invertible, now we showed it is equivalent to full rank right, full rank means what there reduced row echelon form is identity matrix that is equivalent of that and now just we have shown is if determinant is not equal to 0 right we proved A is invertible if and only if determinant is the product of elementary matrix is not equal to 0 and it is determinant of A inverse that is the bracket missing here.

It is 1/determinant of A because of A * A inverse is identity, the product property I can use now. Determinant of A * determinant of A inverse is 1, so determinant of A inverse is 1/that because I have already now proved the product property for determinants of products of arbitrary matrices okay, so that is invertibility.