Basic Linear Algebra
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Lecture - 19
Determinants and their Properties - I

So welcome to today's lecture. We will just recall what we had done last time and then describe what we will be doing today. If you recall last time we looked at how to find inverse of a matrix and we proved a theorem namely that a square matrix is invertible if and only if it is reduced to echelon form looks it is identity matrix and we also give an algorithm of finding the inverse in that case.

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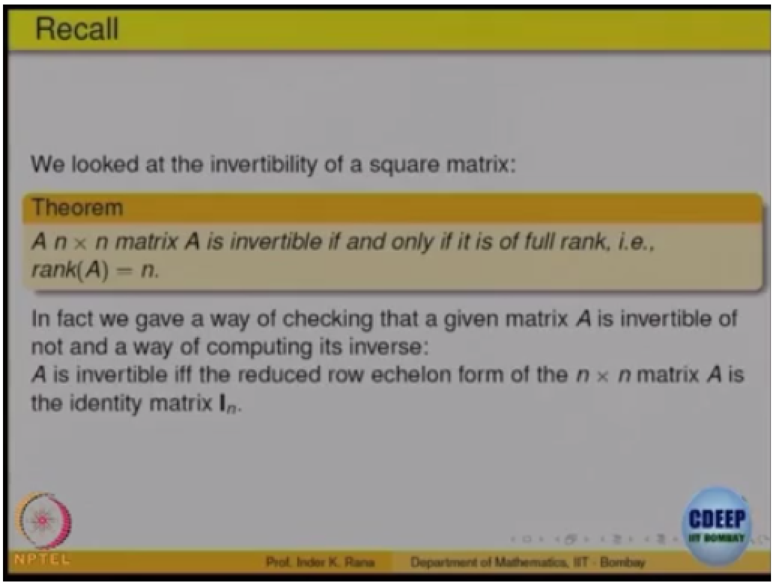

So this was the theorem that we proved last time that n cross n matrix is invertible if and only if it has full rank that is the rank of the matrix is N and that is equivalent to saying that the reduced row echelon form is identity and that process of converting the row, converting the matrix to reduced row echelon form also gives you a way of finding the inverse. So today we are going to look at another concept called determinant of a matrix and which will also related to invertibility of the matrix.

So to motivate what is determinant you all must have been using this concept in your school curriculum and how to compute determinant of something 2 x 2, 3 x 3. So we will start with looking at why determinant has that properties and what are the minimal set of properties that define determinant.

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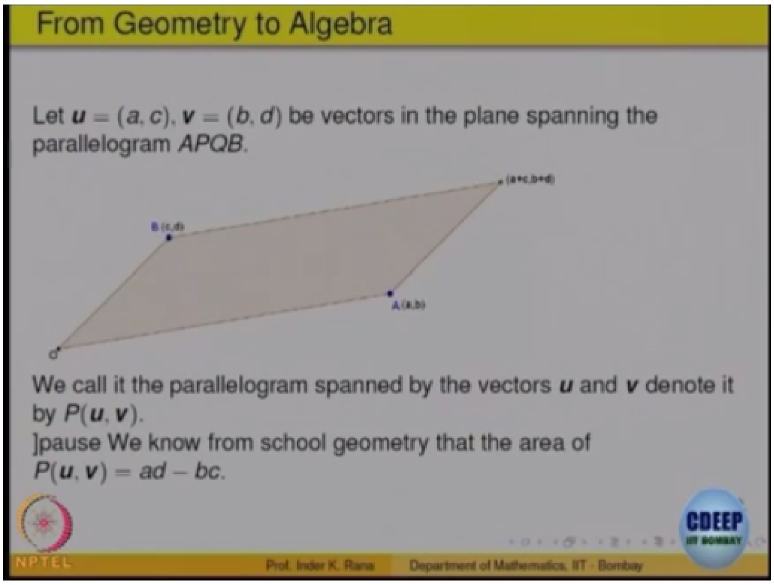

So we will start looking at the concept of geometry namely finding the area of a parallelogram. Now suppose you are given a parallelogram with defined by 2 vertices OA is one vector sorry, OA is one vector and OB is another vector. So let us say one vector is C another vector is AB. So these 2 vectors, this is u and this is v, so AB. So they define a parallelogram any 2 vectors define a parallelogram by drawing a vector parallel to this and when.

So if you recall your vector algebra, the diagonal of this parallelogram gives you the sum of the 2 vectors and it can be proved using purely by geometry that the area of this parallelogram is given by ad-bc, if A has got the vector components AB and B has components CD then the area of this parallelogram purely by geometric considerations by drawing lines perpendicular and so on.

And used looking at congruence of triangles and using the concept that the area of the triangle is half base * height one can come to the conclusion that this area of this parallelogram is given by the ad-bc this is pure geometry.

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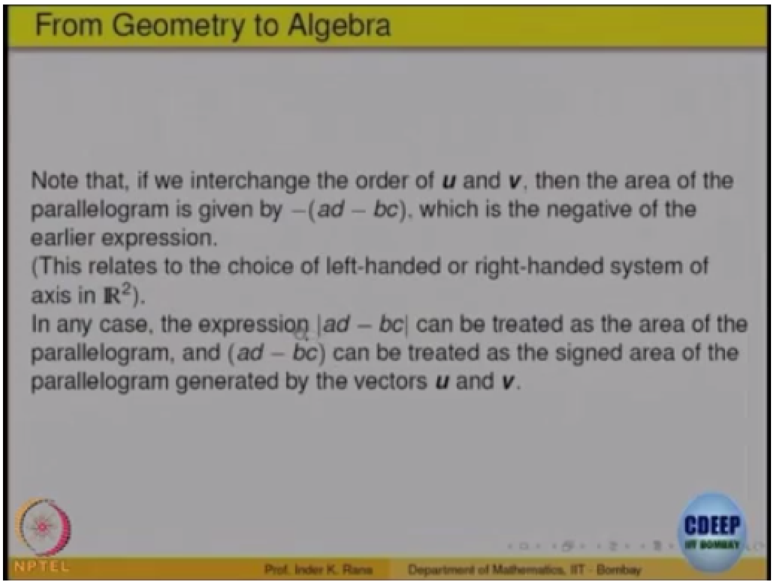

If we interchange the order of u and v you will see that the area comes out as – of ad-bc which was earlier, this also relates to what we call as the left handed system or the right handed system in the plane. We go from left to right clockwise or anti-clock wise. So one can call ad-bc as the signed area because area is normally taken as positive every time. So our starting point is given 2 vectors u and v we have the area of the parallelogram.

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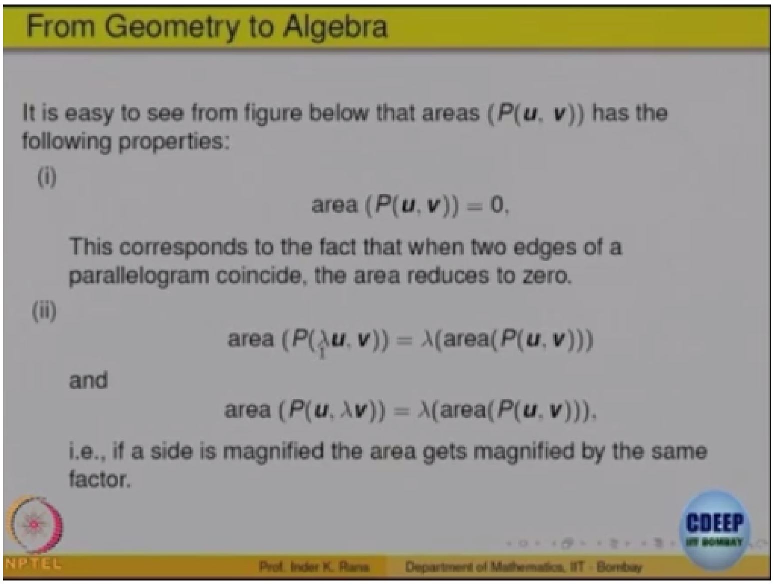

So Puv denotes the area of the parallelogram, now this concept of area, geometric area has very nice properties, the first property is that if the 2 vectors coincide this should be u here Puu if the 2 vectors u and v coincide then there is no parallelogram then the area is 0. So if the 2 vectors u and v are same then this is the typo here it should be Puu = 0. The second important property is that if you scale one of the vectors either u or v.

So multiply it by a scalar lambda. So you stretch right u by lambda then the area also get stretched by the same factor. So algebraically we can say that the area p of the lambda uv is lambda times the area of and similarly the other variable if you stretch v by lambda then the area again gets stretched by lambda. So the scalar comes out, that is the second property.

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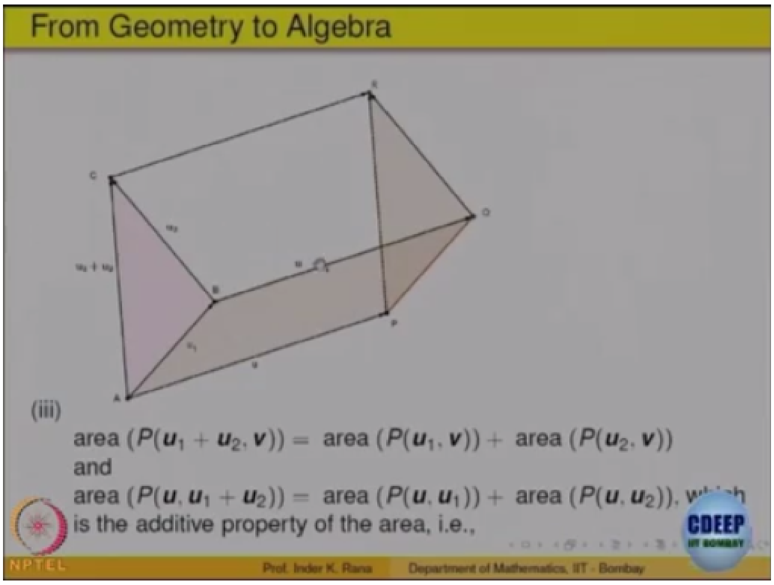

There is a third property namely, if I take 2 vectors u1 and u2 and add them I get any new vector u1 + u2 and I can form the parallelogram given by u1 + u2 and v. Then it is easy to see from the geometry I will just show you that this area is same as area of u1 and v, parallelogram u1 and v and the area of the parallelogram span by u2 and v that is quite easy to see if this is ab is u1 and bc is u2 then the vector sum is a to c, u1 + u2.

And the area of the parallelogram this is u. So what the area of that, what is the parallelogram that is A P R and C, so that is the parallelogram. Now it is quite easy to show that this triangle here is same as this triangle, is congruent to this triangle, I can move it there. So what is this area of the parallelogram, the upper one and the lower one, these are précised area of uv and u1 and v and u2 of v. So that geometrically you can see.

Here we are using the fact that the area, geometric concept of area is additive function if you have 2 non-overlapping regions the areas of the union = the sum of the areas, so that property is used so but basically important thing is so the area of the parallelogram u1 + u2 and v is same as area of u1 v parallelogram and parallelogram u2 v and similarly the other variable you can have addition of vectors in the second variable the same properly which is holding.

So in a sense what you can say, see the scalar lambda came out and sum = the parallelogram span by the sum = sum of the corresponding areas so it is a kind of linearity property which is coming out. So this is what is called the linearity property of.

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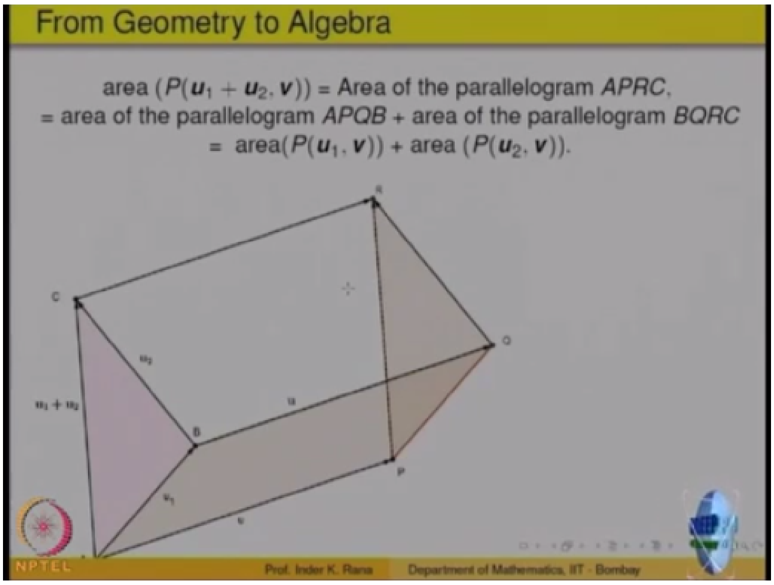

So one says this is the area of span by 2 vectors u and v is linear in each variable u and v in each variable it is linear.

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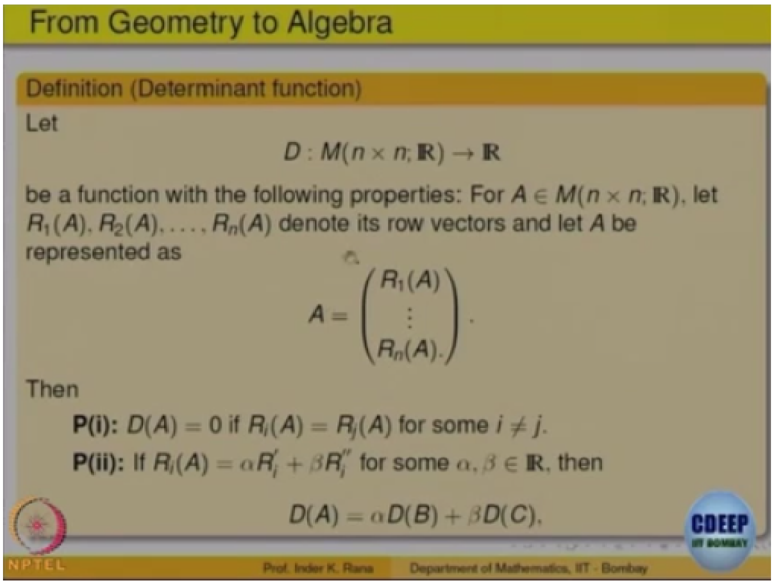

So this way of interpreting motivates us a definition. So you can think of u and v as the row vectors of a matrix so that is 2 * 2 okay. So we generalize this to n * n matrix. So given a matrix n * n n square matrix with real entries, let us say it is rows are R1 A of n Rn A, these are the rows of this n * n matrix. Then the determinant is a function which is defined for every matrix.

It is something like the area of the parallelogram keep that in mind. It is the determinant of A = 0 if 2 rows are identical. If in the matrix 2 rows are identical then one of the defining property says that a determinant should be = 0, it is something saying in the plane if u = v then there is no parallelogram, the area is 0 and the second property is the linearity property that if one row say the Ri'th row, if the i'th row of A is a linear combination of some other row vectors alpha R1 dash Ri dash + beta of Ri dash.

So the i'th row here is the linear combination of 2 vectors then the determinant of A is alpha times determinant of A matrix B + beta time determinant of matrix C, what is B here? B is the same as A except or the i'th place the row is Ri dash and in C is every other row is same except where the linearity is being given that is Ri double dash. So only at i'th place the rows are Ri dash and Ri double dash everything else is same.

So that says that if in a row it is a linearity a linear combination then the corresponding values are linearly added. So if the i'th row is alpha Ri dash + beta Ri double dash then determinant of A is alpha of determinant of B + beta determinant of C.

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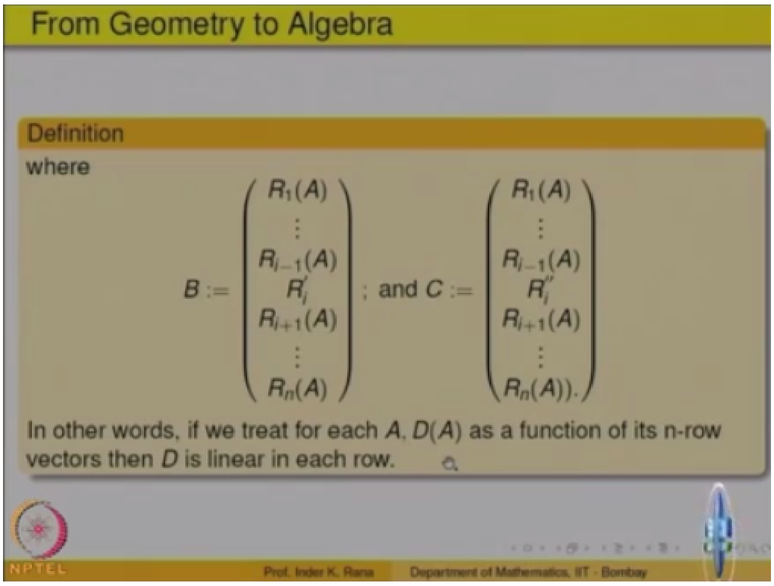

So where as I said this B is the i'th row is Ri dash and in C the Ri'th row is Ri double dash. That is the only difference everything else is same as the rows of A. So this property you can say that there is a linearly in each one of the row vectors. This function is linear in each of the row vectors because determinant of A is dependent on the rows R1 R2 Rn. So if at some place it is a linear combination and the corresponding thing is reflected in the right hand side.

It becomes the linear combination of the corresponding matrices. So that is linearity. So one is the first one was if 2 rows are identical then it is 0 that is something similar to saying that if parallelogram to sides are identical then there is no parallelogram area is 0 and second is linearity in each variable.

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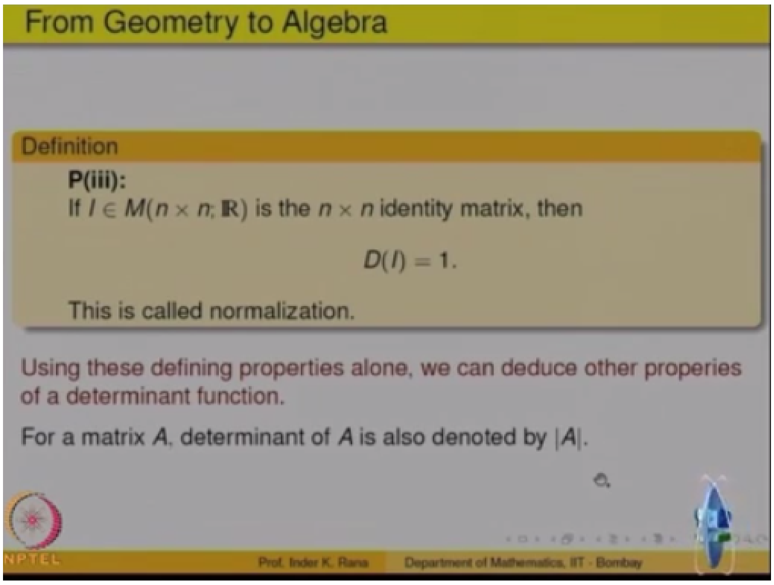

And the third one is what is called a normalization constant that means for identity matrix we define determinant to be equal to 1 right, this is just a normalizing factor nothing more. So you could put anything you like but normally it is good idea to have determinant of identity matrix to be equal to. So there are 3 properties which are defining determinant. One, the first property said that if there are 2 rows which are identical in a matrix then it is determinant = 0.

Second property, if row is a linear combination of some vectors then determinant of that matrix is a linear combination of the corresponding matrices coming from the linear relations and third is normalization namely determinant of identity matrix n * n = 1 right. So what we are going to do is using this defining properties alone these 3 properties we are going to reduce all other properties of determinants that you might have been using earlier right.

So these are the properties which define determinant and everything else can be deduced from these 3 basic properties that is what we want to show. So sometimes that instead of writing the letter D of matrix A one also puts 2 bars around A right this symbol also used to indicate you are looking at the determinant of the matrix A, notation only nothing more than that, is it clear 3 definition?

They are perfectly taken from the concept of area of a parallelogram, area of the parallelogram is 0 if u and v coincide, there is no parallelogram, it is linear in each of the variable if u is u1, u2, u1 + u2 then correspondingly the area is of u1 with u1 and + with u2 and if the scale are stretched each side then the area gets accordingly stretched right. So those are the 3. So that stretching and the sum is put together as linearity, put together as the property of linearity. So these 3 properties. So let us look at the first one.

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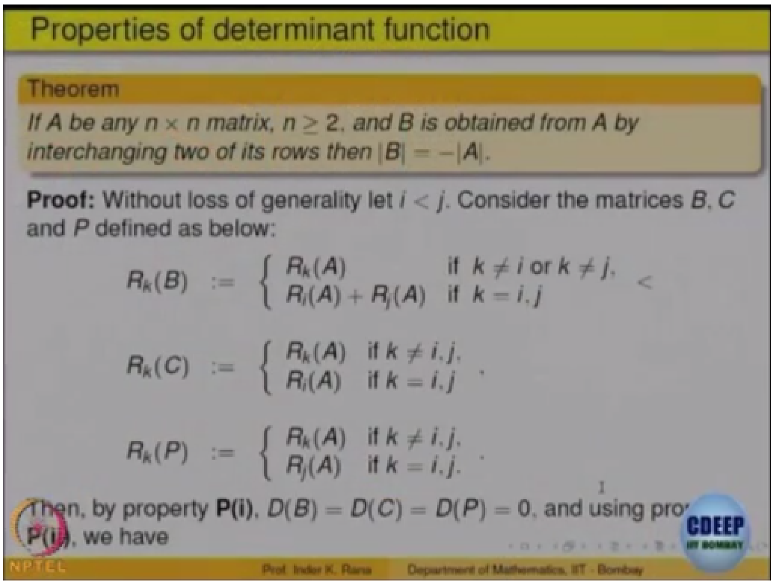

That if A is n * n matrix of course order bigger than 2 for 1 there is nothing to prove and if we interchange 2 rows of the matrix we will get a new matrix right. The claim is that the determinant of this new matrix is – the determinant of the original. You must have used this property for determinants left and right. So what we are going to do is we are going to show that this property is a consequence of those 3 properties.

So let us see how is that. So let us assume i is less than j right, i'th is interchanged with j so one of them will be smaller than the other, so let us say i'th row is smaller than the j'th row. So now let us consider a matrix B, so I am constructing a new matrix B this matrix has all rows same as that of A except for i'th and j'th. What is the i'th row that is Ri + Rj of the original one and the j'th row is also same as that of Ri + Rj.

Ri and Rj were interchange so using that i and j, I have constructed a new matrix right. Now in this new matrix i'th and j'th row are same right. What is the i'th row for this matrix now, Ri+Rj of A. What is the j'th row? that also, so what is the determinant of this matrix by definition, what is the determinant of the matrix B, by the very first property if 2 rows are identical then it should be 0.

So determinant of B is 0 right okay. Let us have a matrix C where i and j'th row are for every k not equal to i and j, the rows are same as that of a right but the ij'th row I am putting both as Ri again there are 2 identical rows right and similarly the matrix p where the j'th row is same identical that is the j'th row of okay. So all these 3 will have determinant 0 right because in each one of them the 2 rows are identical.

Now what is the relation between B, C and P and A let us compute that, what is B = there is a linear combination coming here right. In the i'th as well as the j'th row. There is a linear combination i'th row is Ri + Rj of A and j'th also is Ri + Rj right. So use the property 2 of linearity what I will get? So we will get the following.

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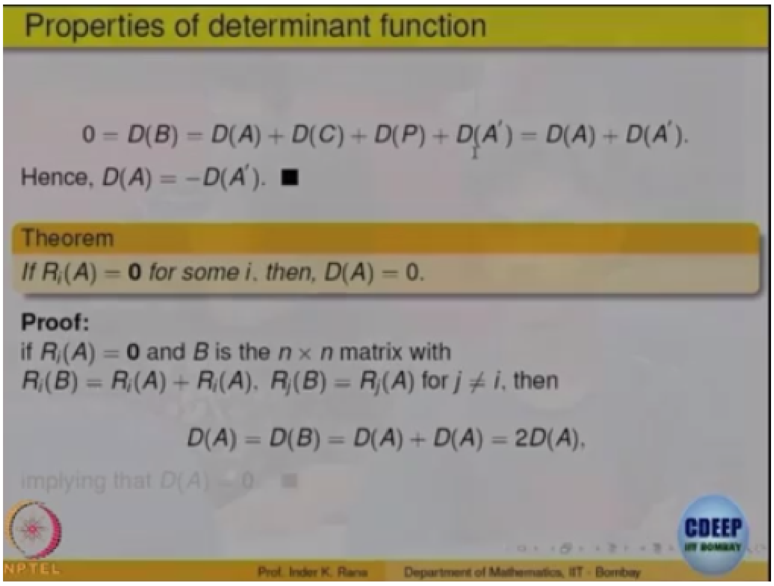

Determinant of B which is 0 right it is same as determinant of A + determinant of C + determinant of P + determinant of A dash. Why is that? At the i'th place Ri + Rj, at j'th place also Ri+ Rj. So how many linear combinations will come? ii, jj, ij and ji, 4 will come. So these are 4 here. A is the original one ij right DC is ii, this is jj, identical and A dash is i and j are interchanged right.

So using property of linearity determinant of B which had i + j and i + j that is 0 so that is = this, d of c is 0, d of p is 0. So what is the D of A + D of A dash if these are 0 so what is D of A, that is – D of right A dash. So we have used the first 2 properties namely if 2 rows are identical then it is 0. If there is a linear combination you get the linear combination of the corresponding matrices.

So using only those 2 properties we have proved the theorem namely if we interchange any 2 rows of a matrix then you get a negative sign, negative sign is introduced in the value of the and this is purely without expanding or doing anything. We have just used those 3 properties right okay. So let us, is it clear, proof? okay. Let us look at 2 which says another property.

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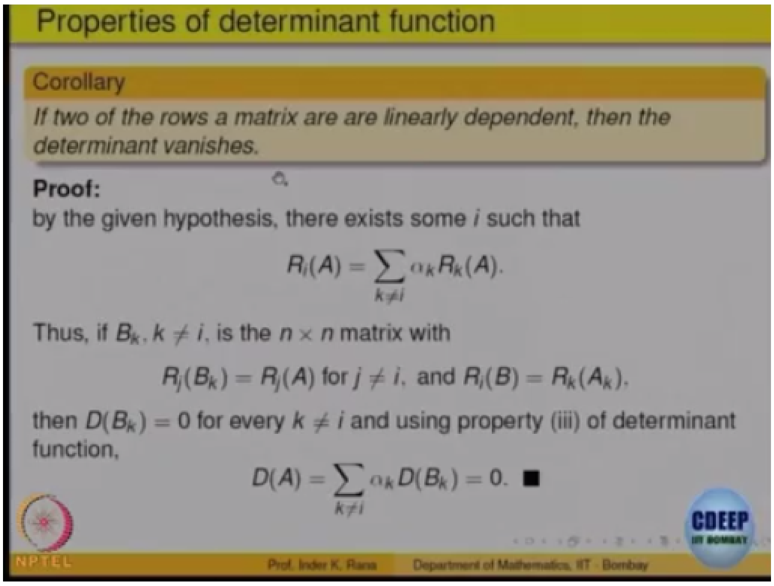

That if 2 rows of a matrix are linearly dependent, matrix is written as row vectors R1 R2 Rn that i'th row will be a linear combination of the remaining one then what will happen if it is a linear combination what will happen? When I expand by using linearity 2 rows will be coming identical right in the expanded one in the linearity right. Supposing the i'th row is dependent on the first and the last 1 and n.

So what will be the Ri= it will be some alpha times R1 + some beta times Rn right. Now when I expand it what will happen, R1 and R1 will be coming and Rn and Rn will be coming in the expanded on the right hand side right using linearly. So the whole will be all 0 right. So determinant of a matrix in which at one row is the linear combination of the other rows vectors that means same as saying it is linearly dependent on the other one than the determinant of that matrix is 0 that is the second property there.

Again using only linearity nothing more, the fact we use is a vector is dependent on the other if it is a linear combination. So if you put that linear combination and expand we will get 2 rows identical in the expanded ones. So all will be = 0. So determinant vanishes. That vanishes means it does not disappear, right that means determinant = 0, a better word is determinant of that matrix is 0 okay.

It is very (()) (20:05) saying that determinant vanishes, okay, it does not disappear it becomes 0. So here is the proof okay. So if i'th row is the linear combination of the remaining ones right then when I expand look at each of the expanded matrix they are the rows, 2 rows will be coming identical in Bk. So all will be 0, so this will be = 0 right. Is it okay, clear or shall I write in 3 * 3 and show you again, yes. You want to know. Okay let us show it, write it in 3 by 3.

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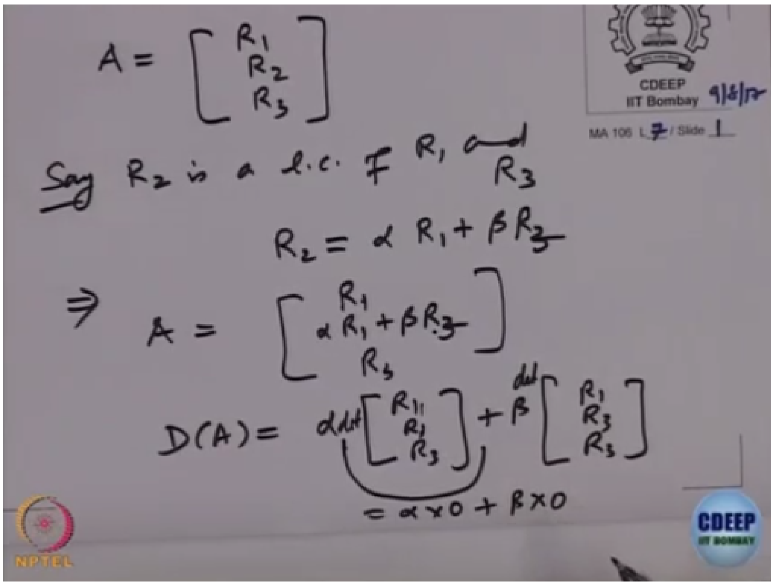

So let us take a matrix A which his R1 here R2 and R3 here right. Let us say R2 is a linear combination of R1 and R3, they are dependent so one of them will be say, let us say R2 is a linear combination, so what is R2 = alpha R1 + beta R2. So what is A = so implies what is A R1, alpha R1 + beta R2, R3 right. So here is a linear combination coming. So by determinant property what is determinant of A = it is alpha times determinant of the matrix R1 R1 and R3 right + beta times determinant of, I should put determinant of the matrix R1.

Sorry R2 was the linear combination it should have been R3, R1 is the combination of R1 and R3, so I wrote a mistake here it should have been R3. So it is R1, R3 and R3 right. So what is this determinant that is 0 because R1 R1, this determinant is 0 because R3 and R3. So I have used the linearity property scalar comes out and this is the linear combination so it is linear combination of R1, R1 R3 + R1 R3 and R3 right, that is linearity property.

So this is 0, so this is = alpha * 0 + beta * 0 and same thing for a general one, one writes this way whatever so the j'th one is the linear combination of the remaining ones. So you construct those matrices here because only 2 over there so we got 2 terms right, otherwise we will get except for K not equal to i all other will be (()) (23:03) and each one of them will be a matrix with 0 determinant because 2 rows will be identical in each one of them right.

So this gives you the property that if 2 rows a matrix are linearly or if any if the set of vectors are linearly dependent then one of them will be linear combination you can write that also, is it clear only those 2 properties are used.