Basic Linear Algebra
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Lecture - 18
Row Space, Column Space, Rank-Nullity Theorem III

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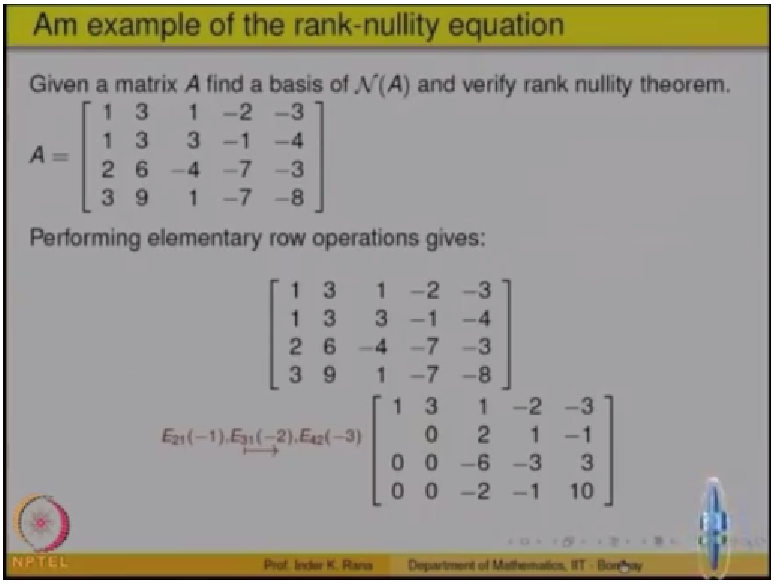

Let us do one more example to get the hang of everything that we are doing okay. So you are given a matrix A, which looks like this. So what is the order? So it is 4 cross 5 right. What we want to do is, I want to verify the rank-nullity theorem. So what is the rank-nullity theorem said for this matrix I should find the rank, I should find nullity, what is nullity? That is the dimension of the null space. Homogeneous system Ax = 0.

Find dimension of that add up I should get = N right, the M cross N matrix I should. I want to verify that, so let us see what are the computations involved. Either case first of all we have to reduce it to the row echelon form. So let us do that. So this is the matrix. So I have written down the operations 0 here which is missing in the typo. So do that, this is what you get.

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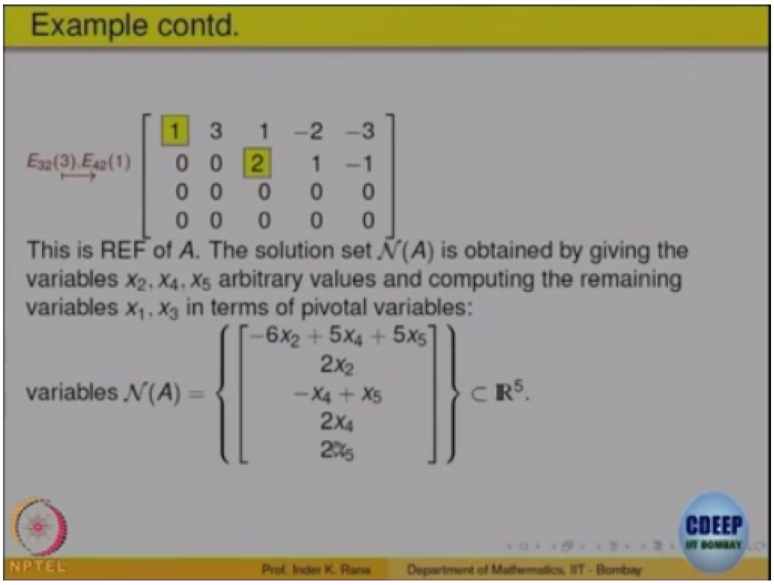

So in this matrix how many pivots are there 2 right, they are in the yellow box. There are first 2 rows which are nonzero. So what is the rank of this matrix, 2 is the rank of this matrix right, a number of nonzero rows okay right. Now what is the next thing I want to do? I want to find nullity. So how do I find the nullity that means I should describe the null space. I should describe all solutions of Ax = 0.

So if I take this matrix and write x here x1, x2, x3, x4 so on =0 what are the equations I will get? x1 + 3x2 + x3 – 2x4 = 0 right. The next one give me 2 times x3, x4 + x4 - x5 = 0 right. So only 2 equations I will get. Rank =2. What is m = 5. So how many independent variables are there? Remember solving the equations if N is the number of variables R is the rank right then the number of independent variables which get arbitrary values is N-R.

And what are those variables? They are precisely the non-pivotal variables, here what are the pivotal variable, x1 and x3. So x1 and x3 will be computed in terms of the non-pivotal variables that is x2, x4 and x5 right and how they are computed in the last equation. So what will be that equation which is last equation we have got, that is 2 x3 + x4 – x5 =0, x4 and x5 are given arbitrary values.

And from that equation I get my x3, x3 is known right everything is fixed. Now compute what is this variable which is free that is the second one. So now x2 gets the arbitrary value right and backward substitution will give me x1. So let us do that. So my general solution looks like this. So when backward substitution when I do right. So last one is solutions will be 2 x5, 2 x4 and this is the solution.

So what are the variables which are free, x4, x5 right and x2. The non-pivotal ones in terms of that I find all the other values right. So this is the null space now I have got a solution right where x1, x2, x4 and x5 are arbitrary, but still I have not got any basis for the null space. I have only described in terms of those free variables. So how do I get a base is, so there is a small trick involved there what do you do.

How many free are there? 3 of them, so these 3 variables are given, we have to give them values right, arbitrary. So what we will do is first choice I will do is as 1 0 0, we have 3 variables, right. The first variable I will give 1 which is free, the second variable 0, the third variable as 0. So I fix the 3 variables which are arbitrary as 1 0 0 everything else will be computed.

The next choice I purposefully give as which is independence for the 3 variable what is the next independent variable, the next value we will get vector 3 variable 0 1 0, first one was 1 0 0, the next one I give 1 0 1 0 and the third choice I gave as 0 0 1. So the three variables are free. So first choice I will give you 1 0 0, the next choice is 0 1 0 and third choice 0 0 1, then automatically I will get the corresponding vectors as linearly independent.

We will check that right. So this is definiteness also I can tell the computer to do that and find out right. So this is definiteness and computation let us do.

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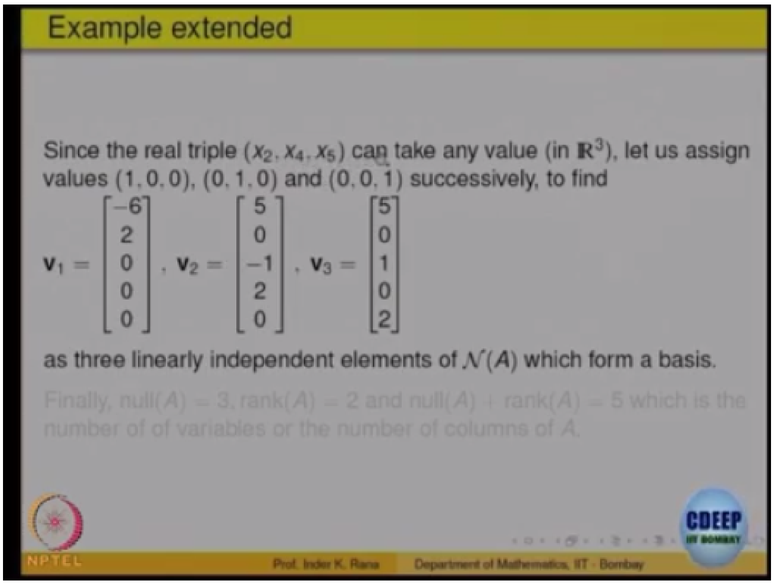

So x2, x4 and x5 that triplet is arbitrary right, the free variables. So what I do first choice is 1 0 0, next choice is 0 1 0 and the third choice is 0 0 1 right. I do that and I compute v1 v2 v3 it was particular solutions with these choices, so that I get these 3 vectors. So claim is these 3 are linearly independent and should generate everything right. See x5 is arbitrary, coefficients are not important 2 and this are not really important.

You got all 2 x5 as x5 idea is for the last right x5 there should be some arbitrary value, x4 is arbitrary, x3 is determined from this equation. Once you have fixed x4 and x5 you will get what is x3 because 2 is there, you have to divide by 2 and solve it. See what will be 2 x3 = -x4 + x5. So what is x3, 1/2. So instead of that fraction just multiply that by 2 just to avoid that fraction nothing more than that.

You can basic idea is you give arbitrary values to those non-pivotal variables and compute everything in terms of them and I could have divided everything by 2. I would have gotten fractions, but that would not change the solutions set right. Scalar multiple would not change the solutions, is it clear now okay. So what I want to do is, want to check that v1 v2 v3 try to check that these are linearly.

So how will you check they are independent. Alpha + beta + gamma = 0 will get a set of equations and check that alpha = 0, beta = 0, gamma = 0, but that is only saying you are getting a linearly independent set. How do you know it is a basis, right? Every solution right, so if you take some other vector which is a solution right of the system then everything should be a linear combination of that.

That also requires verification theorem says it should be so. So what does that mean, every solution should be a linear combination of these right. So let us say x1, x2, x3, x5 is the solution right then a liner combination of that should also give you the solution right. So that means we should be able to find alpha, beta and gamma right, these are 3 vectors. If every solution has to be a linear combination pick up a solution x1, x2, x3, x4, x5 right some particular solution.

We do not know what are those right, but they are fixed. So to find alpha, beta and gamma so that is = alpha times the first one + beta times that means that system is consistent it is solvable that is all I have to say. You understand what I am saying or not okay. Let me just write the last thing here.

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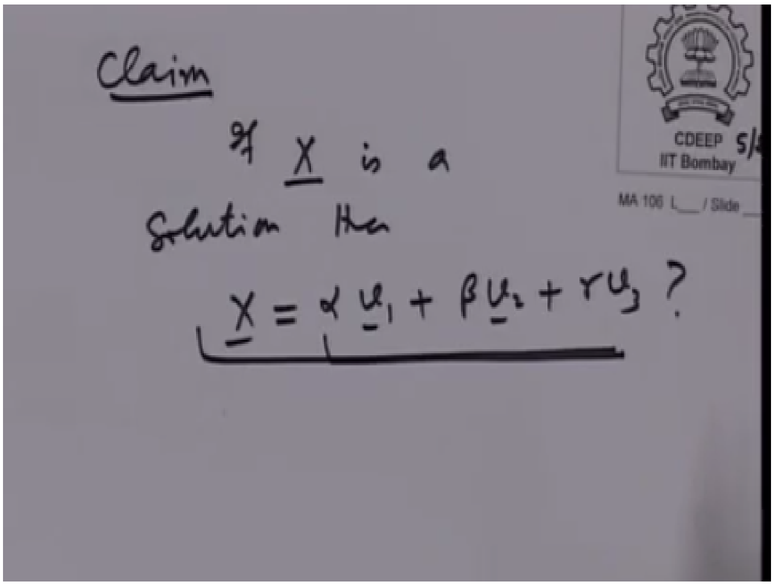

Claim right, if x is a solution then this x should be = alpha times v1 + beta times v2 + gamma times v3 right that is what I have to check. Whatever be this vector x, if that is the solution right. That means what this alpha, beta and gamma are unknowns right. So that means I want to show that this system is consistent basically, this system is consistent right. Is it okay, that you know how to do that how to show a system is consistent or not again going to row echelon form and doing it right.

So I think this is the right stage to recap everything that we have done and because next topic will be something called determinants yeah. So let me just recap whatever we have done till now. We started by looking at what is linear algebra, basically we said linear algebra is study of linearity, these arises in 2 ways one is system of linear equations and doing geometry in algebraic setup.

So we started with looking at systems of linear equations and trying to find their solutions. We looked at geometric way of doing it for R2 and R3. In R2 every linear equation is a line in R3 every liner equation of 3 variables is a plane. So geometrically it says that given a system of equations in the plane 2 lines will either intersect at a point so you need solution or they will be parallel, no solution right.

Or they will be coincidental infinite number of solutions, something similar happens in R3, 2 planes right, all 3 planes may not intersect. If 2 planes can intersect at a line right and those lines can intersect at a point, so all 3 planes can intersect at a unique point that is a unique solution they may not intersect, given 3 planes 2 may intersect somewhere and other 2 may intersect somewhere else, so inconsistency.

Those lines may not cut each other at all right and infinite when all 3 planes coincide then they are infinite or even from one line 3 planes are coming up right like a pages in a book, the center thing right so many planes are coming out of it right infinite solutions for that system also. So different possibilities in R3, but we wanted to know that how the geometry can be converted into algebra.

So solving a system of linear equation first of all we said we can write in the abstract form using matrices the co-efficient of the system and the constant on the right hand side they are important, variables are not important. So a system can be written in the matrix form as Ax = B and Gauss elimination method which we did for 2 and 3 variable says somehow bring the system in a form where more and more number of zeros are coming in each consecutive variables are removed one at a time right, eliminated at a time.

So that mean in the matrix we define what is called row echelon form of a matrix and we looked at what are called elementary row operations. So solving that Gauss elimination method is same as saying you can apply elementary row operations interchanging the equations, multiplying one equation by another, multiplying a equation by nonzero scalar and adding to another one.

By this translated into matrix form is saying that every matrix can be brought to row echelon form and in particular further to what is called the reduced row echelon form and once you do that you are able to write down, the system will have a solution if the number of nonzero rows or the pivots = number of variables then unique solution is there right or if it is less than if it is R that is what we call it as the rank.

If the rank is less than the number of variables that is N then N-R will get arbitrary values and R the pivotal variables will be computed in terms of the other ones. So essentially when you say freedom is there of N-R that somehow looks like saying that degrees of freedom or dimension you can think of, the null space should have dimension N-R that is what eventually comes out to be for us.

So we define the rank then we looked at the rank in terms of basis independence and dependence of vectors right. We looked at vector spaces subset V of RN is called a vector space if you take 2 vectors and add them that is again in that set and a scalar multiple also is back in that set. It is closed under addition and scalar multiplication that is called the vector space.

So the solution set of a homogenous system Ax = 0 forms a vector space that was the main aim why we started looking at vector spaces. So and we said that a solution of Ax =B can be described in terms of solutions of the homogenous system Ax = 0 by what solution of the homogenous system form a vector space. So we will describe that vector space + find out one particular solution of the non-homogenous system somehow or the other and every solution of homogenous system plus this particular solution give you all solutions of the system.

So it became important to describe a vector space as neatly as possible right. So came in the definition of what is called basis of a vector space. So basis is a linearly independent set which would generate everything. Everything should be obtainable form that set. So the property was linear independence and generation or equivalently we said it could be a maximal linearly independent set.

Or equivalently minimal set of generators all 3 are equivalent ways of saying you can get everything from a finite number of vectors by linear combination that is definition of basis and we said any 2 basis should have the same number of elements so we call as the dimension right. Basis can be different. A vector space can have more than one basis, but dimension is always fixed for that, that is the number of elements in them.

Any 2 basis will have the same number of elements then we looked at special cases for a matrix, the null space, the row space and the column space, these 3 are vector spaces, how to find basis for all these 3 for the row space take the row echelon form, the nonzero rows give you basis for the row space, look at the row echelon form, look at the columns right, the corresponding columns right, the pivotal columns corresponding to the original one they are the, they form a basis for the column space.

For the null space to be definite the idea is variable which are free they are given that independence choice namely 1 0 0, 0 1 0 and so on depending on how many variables, if 2 variables are free then we will give 2 choices 1 0 and 0 1. If 3 variables are free 1 0 0, 0 1 0 and 0 0 1 and so on, they automatically generate a linearly independent set for the solution space right. So that is the nutshell till now clear to everybody yes.

So I think we will stop today here because these are important, today was the importing thing because it also gave you algorithm of how to find from a given set a linearly independent set which is maximal and I think as the column vectors right and from a given set finding a basis for the span writing it as the row vectors so do lot of practise for this right so that get used to it okay, thank you.