Basic Linear Algebra

Prof. Inder K. Rana
Department of Mathematics
Indian Institute of Technology – Bombay

Lecture - 15
Linear Span, Linear Independence and Basis - III

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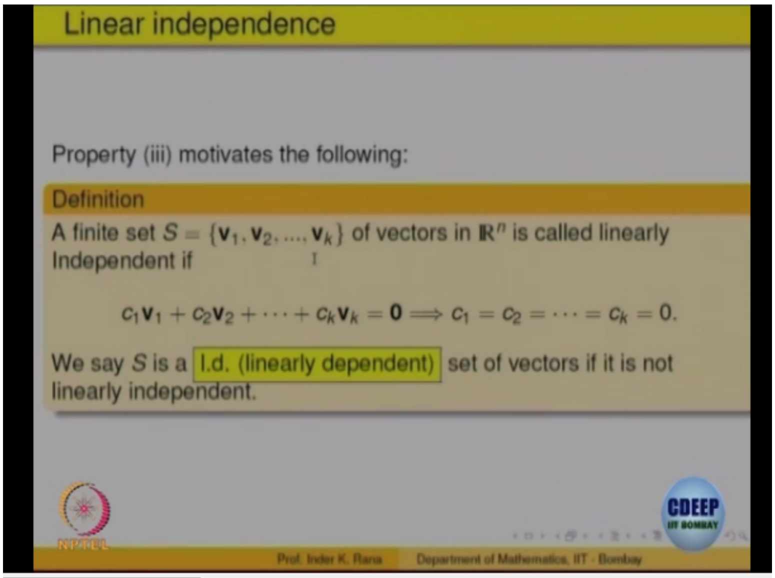

What was property III? That said every element has got a representation and 0 has unique representation. That unique representation property is an important one which will let us signal it out okay right. It says a set of vectors in Rn are called linearly independent, so that property that 0 has unique representation meaning what, it sigma alpha i vi is 0 then each alpha i must be 0, so that property is called the linear independence right of the vectors v1, v2, vk.

So set of vectors which called linearly independent if a linear combination of them=0 implies all the scalars are=0 right. So and if they are not, what is the opposite of it? That means there is a linear combination which is 0 but not all the scalars are 0 that is called that property is called that set is linearly dependent right. So opposite of independent we say it is dependent. Independent means 0 has unique representation and dependent will mean that 0 does not have.

That means sigma alpha i vi=0 for not all alpha i=0 such a relation should be possible. Let us look at some examples.

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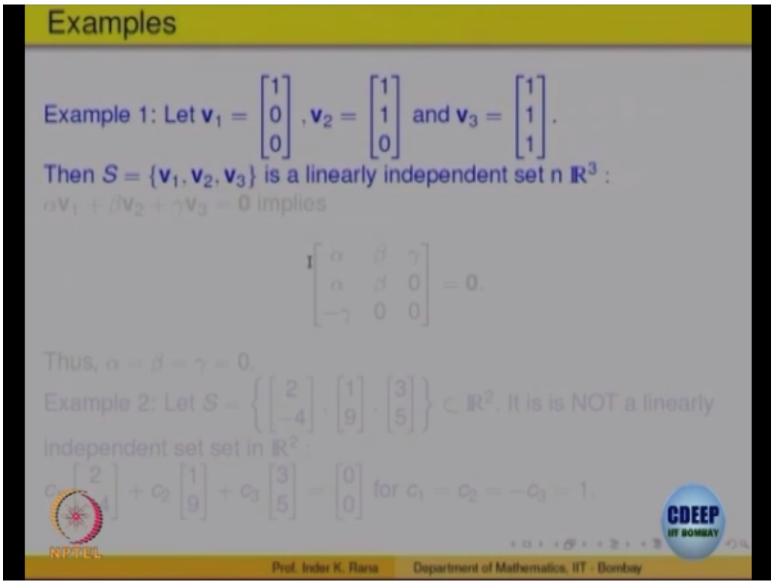

Can I say the vectors v1, v2, v3 in R3 right, there are 3 components, the vectors are v1 which is 1 0 0, v2 is 1 1 0 and v3 is 1 1 1. I want to check whether they are linearly independent or not. So how will you check something is linearly independent, uniqueness of representation for 0. So take a linear combination which is=0, does it imply that all the coefficients involved are 0 or not, so let us check that.

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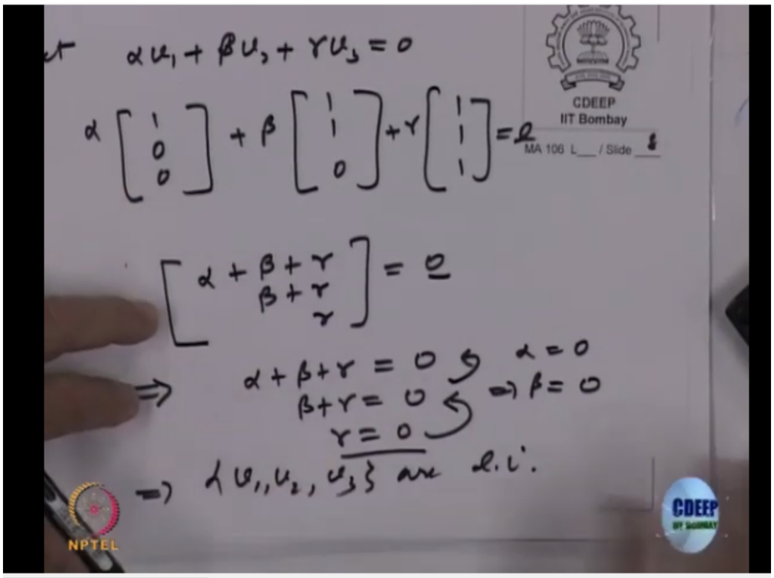

So let alpha v1+beta v2+gamma v3 be=0 right. So what does that mean? Alpha times that is 1 0 0+beta times 1 1 0+gamma times the third vector 1 1 1 is=0. Let us simplify, what does it mean? It means alpha+beta+gamma that is the first element right. What is the second? That is beta+gamma and there is third one that is gamma right should be=0 right. I just added those vectors but that means what?

Alpha+beta+gamma was the first equation that should be 0. Second, beta+gamma should be=0 and third says gamma should be 0 right. So these 3 equations must be satisfied and now thus gamma=0 if I put it in the backward here gamma is 0 that means implies beta=0, now alpha and beta both are 0, put the values here that says beta and gamma are 0 that implies alpha is also=0 right.

So that says that these 3 vectors in R3 are linearly independent so okay right. So this implies v1, v2, v3 are linearly independent right.

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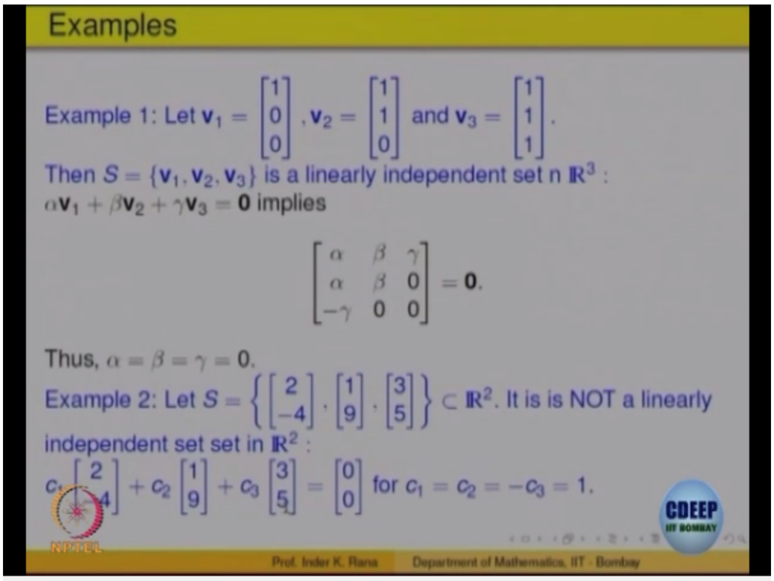

So that is a proof that I guess now here that the linear combination must be=0. Let us look at another example. Let us look at S is the set of 3 vectors right. First vector is 2 -4, second is 1 9 and third is 3 5 right. This is a subset of R2. Each vector has got two components and got 3 vectors. We want to know whether they are independent or not? Here it is quite easy to see that we can represent a linear combination of it=0.

So how will you make the first one 0? So the first component for example 2 right 2+1-3 that will give you first 0 right. So it says the possibility right, so let us if you look at c1 times the first vector+c2 times the second+c3 times the third one, your c1=c2 and last one is -1, this linear combination is=0 and the coefficients involved not all are 0. In fact, none of them is 0 right. So I have got a linear combination of the 3 vectors=0.

So 0 does not have a unique representation in terms of these vectors 3, so what does it mean? The meaning is that these 3 vectors are not linearly independent right. They are linearly dependent vectors. This phenomena will observe, this is going to happen very often in the sense that see these are 3 vectors of 2 component each. So whenever you will have a collection of vectors where the number of components is<right.

Number of components is<the number of vectors, they are going to be linearly dependent vectors always. Can you see the reason why it should be? Let us just write and see whether why that should be true.

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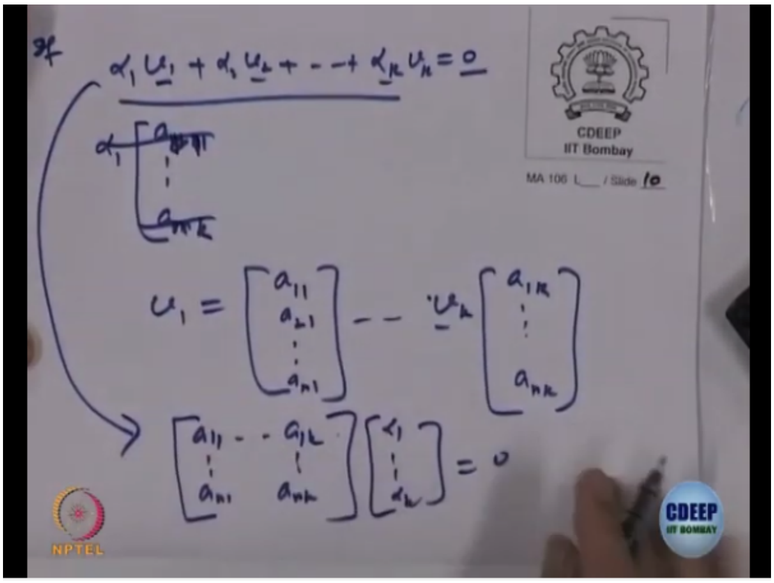

If alpha 1 v1+alpha 2 v2+alpha k vk is=0 right that means alpha 1 of a11 a1n right. Probably, as you write other way around, v1 it has got n components right okay let us write what is v1. So first component of v1 right, so we want to write as the first row okay a11, second row so a21 okay and we have got n so an1. Is it okay? Right, so and vk will be a1k, ank right. So then this equation is just writing it as a11 an1 a1k ank applied to alpha 1 alpha k right is=0.

So this equation is same as this equation right. So now how many equations are here? n equations in k variables.

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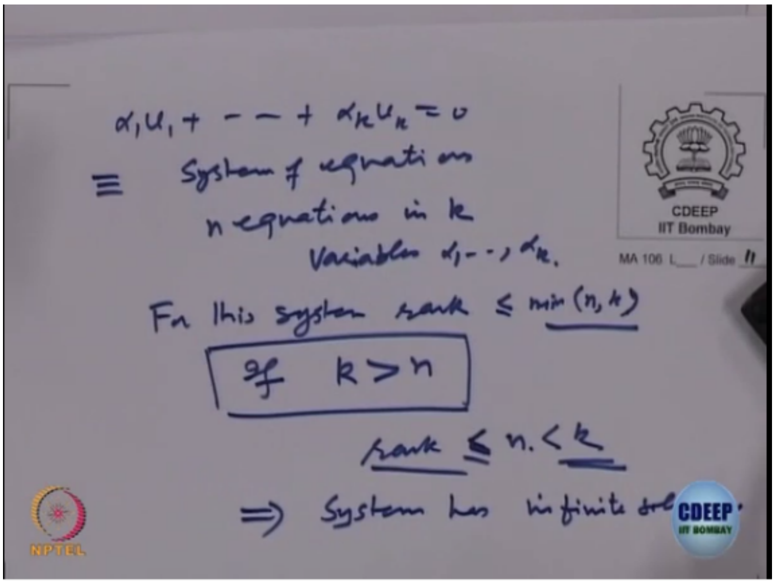

So this is equivalent to so alpha 1 v1+alpha k vk=0 is equivalent to giving a system of equations right. So we have got n equations in k variables. What are the variables? k variables alpha 1 up to alpha k are the variables right. So n equations and when does the system of equations has a nontrivial solution is homogenous system. It always has a solution 0. When does it have a nontrivial solution?

When the rank should be<the number of right rank should be less not the full rank right. Here what is the rank possible? For this system, the rank it should be<the minimum of n and k, it should be minimum of n and k right, minimum of number of rows and number of columns. So if number of variables, what is the number of variables? If k is strictly>n so this is assumption we are making if the number of variables that is k is>n, then what will happen?

Then, the rank will be always<is a minimum. So what is the minimum of k and n? That is n, rank will be<or=n right which will be<of course k. The k variables right, how many variables are there? k alpha 1 to alpha k and the rank is strictly<k. So system of equations is consistent with actually infinite number of solutions right. When the rank is strictly<so implies system has infinite solutions. Is it okay?

Going back to system of linear equations, if the rank is strictly<the number of variables then how many variables get arbitrary values? Number of variables-the rank right. So k-the rank number of variables will and k is strictly bigger, so that it will be at least one variable getting arbitrary values right could be more. So that means there are infinite number of solutions

possible that means what that means in this linear combination I can have infinite many solutions of alpha 1, alpha 2, alpha k such that this is=0 right.

That means if I have got more number of vectors than the number of components then that set is always linearly dependent and that is what is happening in this case. I have got components 2 right, number of vectors is 3, so this has to be linearly dependent and we verified that actually there is a possibility here right but that is general solution and that is quite useful writing later on that if I have given a set of vectors right first see the components and the number of vectors.

It may be obviously linearly dependent if possible right the number of vectors is more than the number of components.

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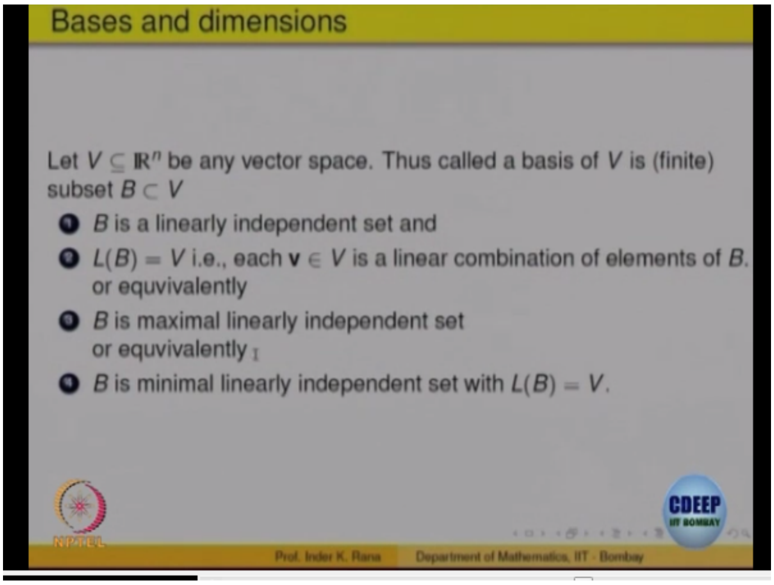

So let us I always recaptured all this in this slide. B is linearly independent set right and this 2 is redundantly this would not be there right, so B is linearly independent and the span is V right? So item wise you should remove, B is maximum linearly independent or minimal this should be a minimal set right minimal not again independent. I think this all is wrongly typed thing. Is it clear?

Anyway wrong thing also helps you to understand what should be the right thing. So the first is B is linearly independent and spans V that is 1. Second, maximum linearly independent and the third is the minimal set which generates. You do not have to put linearly independent there okay.

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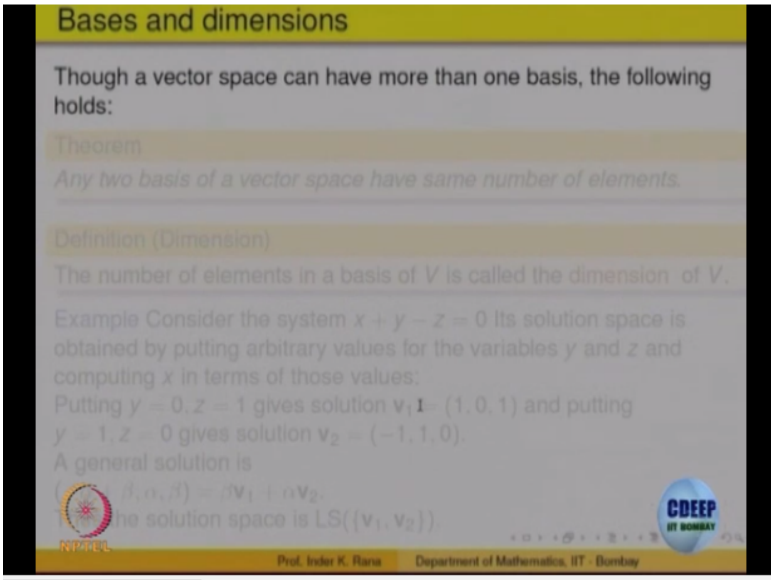

So now let us we said number of vector space can have more than one basis we saw that. So let us construct one more example to say that a vector space okay.

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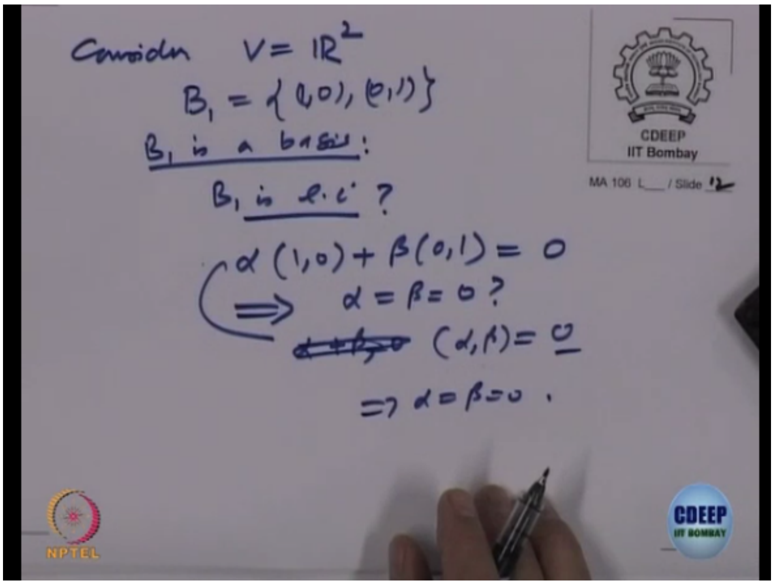

So let us look at so consider V=R2 itself, I think we have done for lot of examples for R2 but it is good let us V=R2. Let us look at the set B1 which is our 1 0 and 0 1 right. So why it is a basis? B1 is a basis. Why it is a basis? One, I should be able to check either by either of the definitions right. So let us check that B1 is linearly independent. It is a linearly independent set. So how do I check it?

So alpha times 1 0+beta times 0 1, if it is=0 should imply alpha=beta=0 right. So what is this equation? That is same as alpha+beta gamma, no sorry this is same as alpha gamma beta=0 right. Alpha 0+0 beta so that is alpha gamma beta and that implies alpha=beta=0. So linear independent right. I should check either it is maximal right or everything generates either of it is okay.

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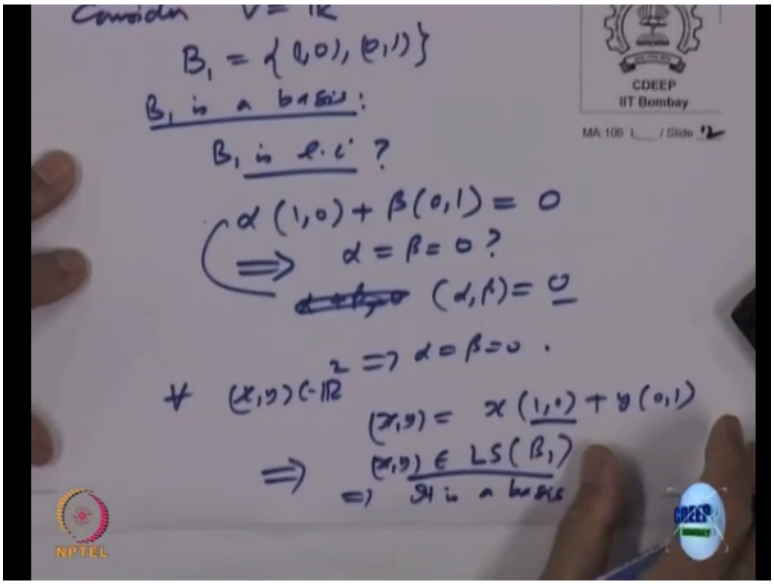

So now let us look at for every x, y belonging to R2. I can write x, y is=x times 1 0+y times 0 1 right. So what does this imply? That means x, y belong to any vector in R2 belongs to linear span of B1 right. Is it okay? It belongs to linear span of B1 because this is 1 vector 1 0 that is other vector. So linear span of B1 is the whole space and it is independent. So it generates L of B is=S and linear independence that implies it is a basis. So implies it is a basis. Let us construct some other example.

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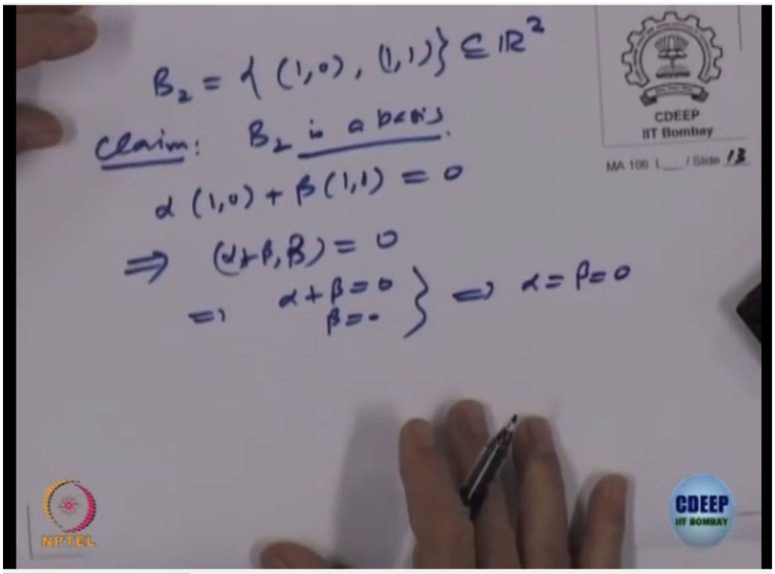

Another basis, so let us look at B2, say let me keep 1 0 and let us take 1 1 right. That is another set in R2. Claim B2 is a basis. I can use any one of those 3, so let us take independence first of all if you can check independence right. So alpha of 1 0+beta of 1 1=0. What does that imply? A linear combination=0, what does it imply? It implies alpha, is alpha+beta, beta is=0 right. So that means alpha+beta is=0 and beta=0 so that implies alpha=beta=0 right.

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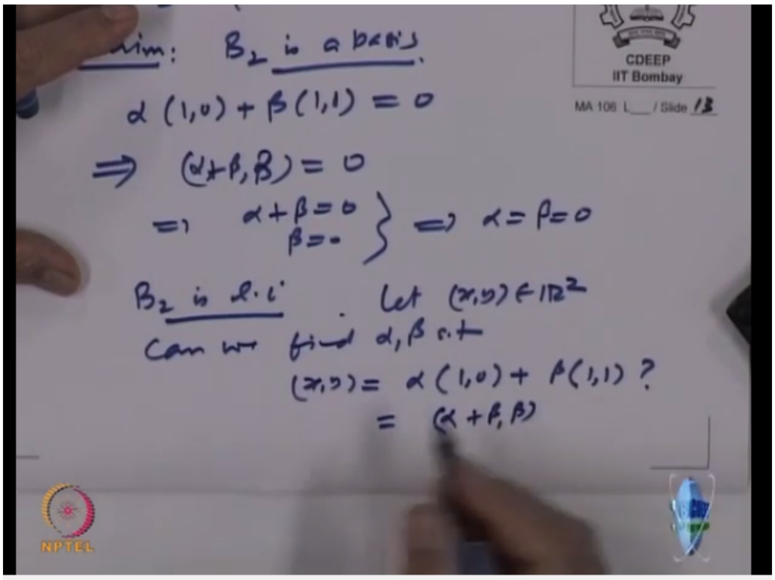

So the set B2 is linearly independent. Next, what should I check if I want to show it is a basis either I should show it is a maximal or it generates, either of it. So let us take let x, y belong to R2. Can I find alpha beta such that x, y is=alpha times 1 0+beta times 1 1. So that is a question right. If I can find that means this will have a x, y will be a linear combination of these two. So what does it mean? That is alpha+beta, beta so that means what?

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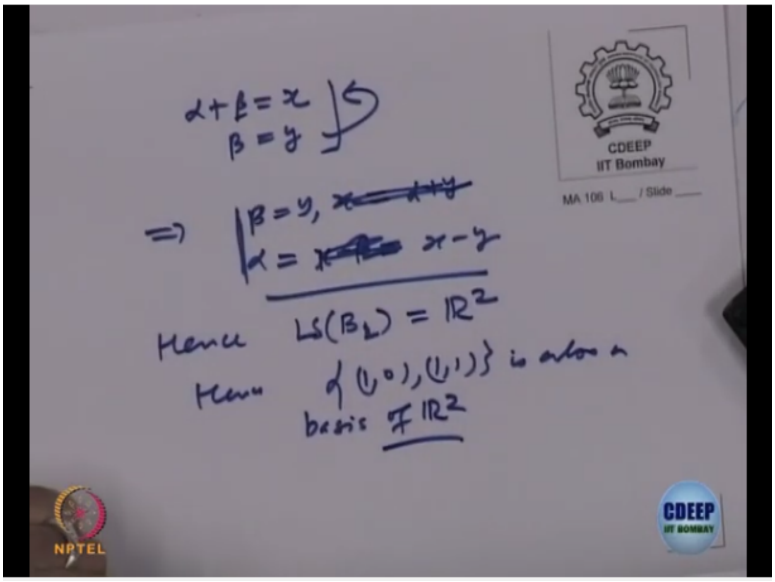

So this is same as saying that alpha+beta is=x and beta is=y right. What is given to you? x and y are given, that is vector given to me, I want to find alpha, beta. So this implies that this beta is=y, so what is x? So that is x is=alpha+y. I put back the equation here. So what is x? Oh, I have to find alpha right, so alpha is=x-beta and that is what was what I am trying to look for? Alpha, we want to find alpha beta say that this is equal to so.

So alpha+beta is=x, beta is=y okay. So x-y right, x and y are given to us, we want to find alpha, beta. So I put in this back equation okay, beta is y so it goes that side, I have found. So for this choice, so hence what does it imply, hence the linear span of B2 we called it is also=R2 right. So we have verified both the things. Linear span is=R2 and it is linearly independent, so hence the set 1 0 1 1 is also a basis of R2 right.

So given a vector space there could be more than one basis of that vector space possible. So that is what the remark says.

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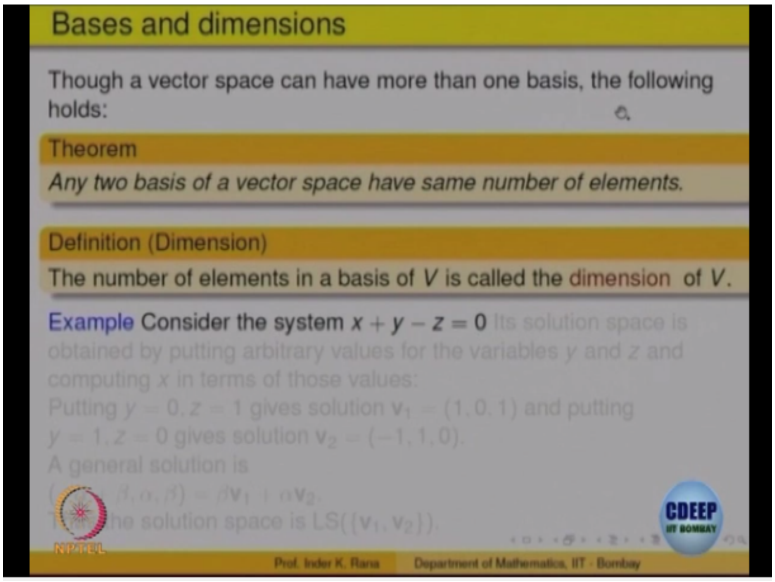

A vector space can have more than one basis; however, a theorem in our subject says that any two basis of the vector space will have same number of elements. The number of elements in two different basis has to be same, it cannot be right one cannot have say in R2 we said we have B1 and B2 both had two vectors right. One of them alone will not be able to generate, we want to see why?

Let us see B1, what was B1? 1 0 and 0 1 right. If I remove one of them, what do I get? Only 1 0 and what will it generate, linear combination x, 0. So there is only x axis right. Other one alone will give only y axis. So I cannot make it smaller anyway. If I make it bigger, if I add a third element to it, then it becomes linearly dependent, independence is gone then, just now we saw, 3 vectors of 2 component each will be dependent, so it cannot be that right.

Similarly, in the other one in the other example if I look at this I cannot remove either of it, if I remove 1 1 I get only x axis, if I remove 1 0 what do I get, x=y right, the line y=x. So that is not the whole of plane right. If I add one more, then it does not remain independent again the problem right. So in R2, the number of elements in a basis different basis are possible number of elements is same that is two.

So we will say the dimension of R2 has vector spaces 2, so you can define it for anything now. The number of elements in any vector space right V in the subset of Rn which is a vector space, so in any vector space is the unique number that is a number of elements in the basis and that is called the dimension of the vector space right. So one we showed every vector space has a basis right.

And secondly this theorem we are not proving that any two elements in the number of elements in the basis is unique. That means any two basis will have the same number of elements we are assuming that. As a consequence of it, every vector space you can associate a unique number, the number of elements in any basis of that which exist is called the dimension of it.

In some sense, that is the size you can think of it the size of right. So let us look at some more examples. So let us look at this. Consider the system x+y-z=0.

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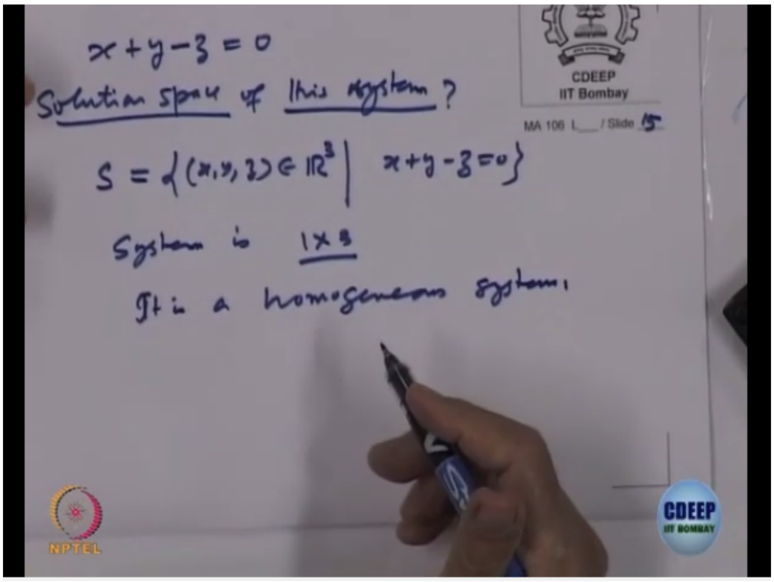

So x+y-z=0, so we want to look at the solution space of this system. What is the solution space of this system? What a solution space mean, it is a set S right of all x, y, and z if I write it as elements of R3, there are 3 variables such that x+y-z is=0 right, as I said I can write this. So what is the order of the system? What is the order? 3 variables, 1 equation, so 1 x 3 right is 1 x 3 so okay.

So what is going to be the rank of, to find solved solutions is a homogenous system, so it is a homogenous system right.

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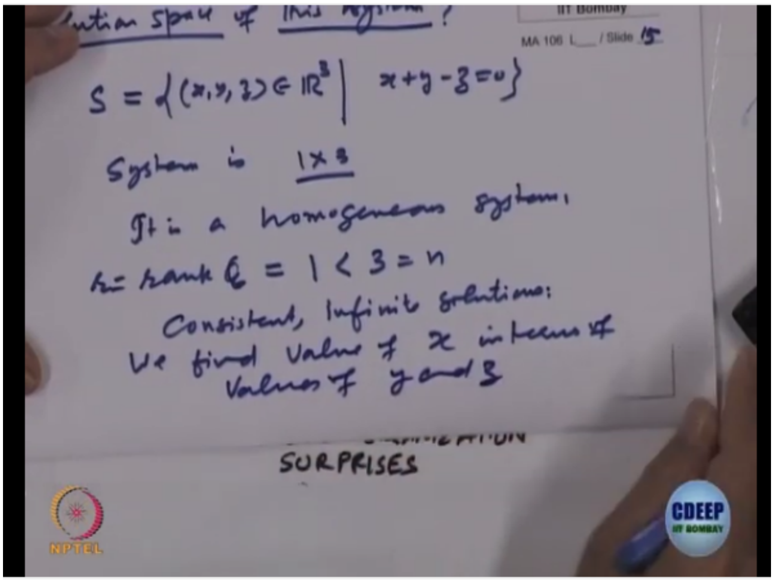

So what is the rank of this? The rank of the system, the rank is=how many pivots are there 1 which is<3 the number of variables n, so that is r and that is=n, so how many solutions are possible? Yes, consistent infinite solutions right and we gave a method of saying how do we find the infinite solutions but where is the pivotal column? Pivotal variable is x right. The pivot comes in the very first stage x.

So value of x should be found in terms of y and z, so we find value of x in terms of values of y and z. So let us do that, so let us put arbitrary values.

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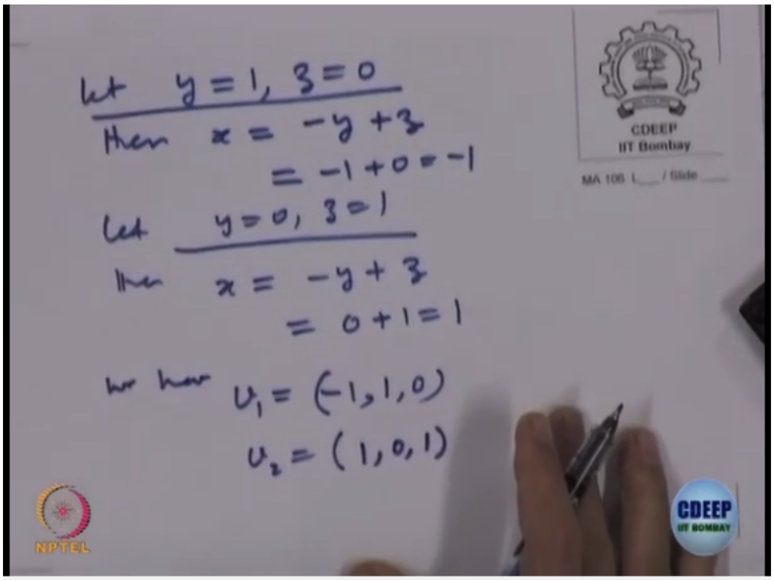

So let y be=1, z be=0 then what is x is=1 -1, so what is y? x=-y+z, so what is that, -1+0 that is -1. So that is for this thing, so let us take another value. Let us take y=0 and z is=1. Then, what is x is=so it is –y+z, so that is=0+1 that is=1. So I got two solutions, so we have v1. So what is v1? So v1 is=we got y=0, x=-1, so -1, 1 and 0 that is one solution. Second solution is v2, we got x is=1, y=0, z is=1, I got two solutions right.

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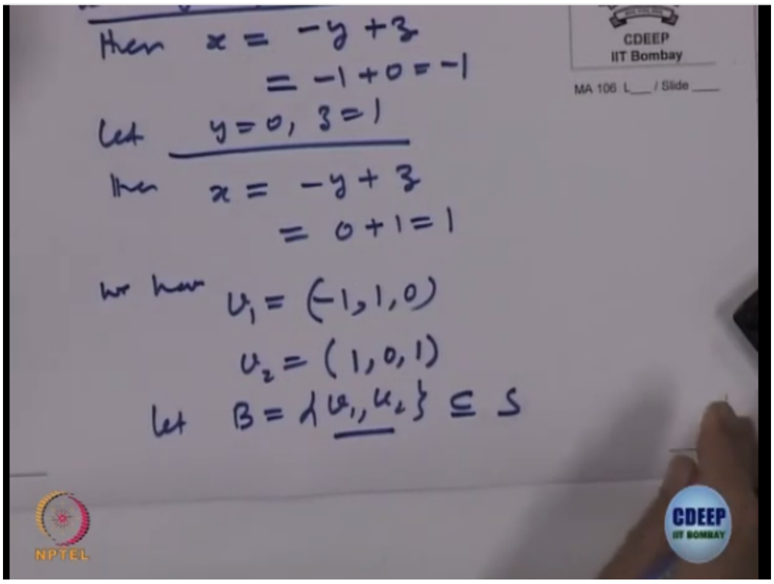

So let us form the set, let B be the set v1 and v2 okay. So if I can show that this set B right this is a subset of S that vector space S all solutions. If I can show that this is generating everything and it is linearly independent and this will form a basis for the solution space right. This will form a solution for the, this will form the basis for the solution space, so how do we check the independence? So let us check independence.

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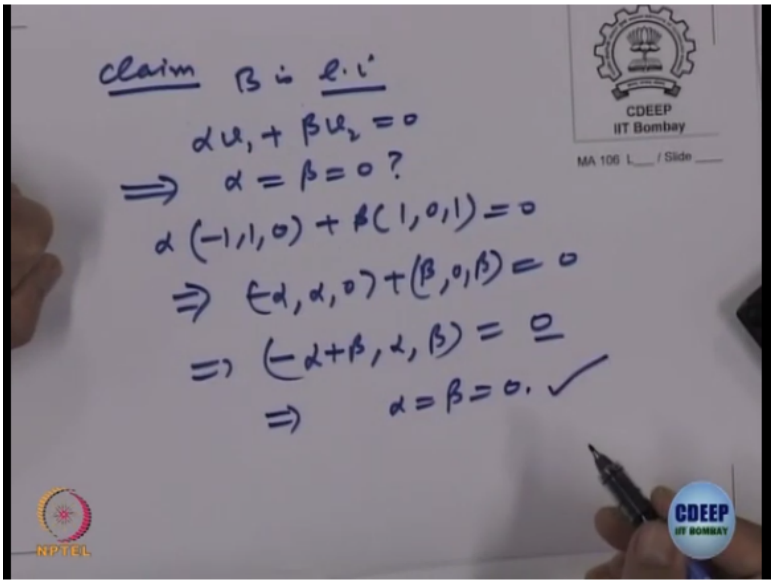

So claim B is linearly independent, so what does it mean? Alpha v1+beta v2=0 should imply alpha=beta=0 right. So what is it? Alpha times-1, 1, 0+beta times 1, 0, 1 so that is=0 that implies –alpha, alpha, 0+beta, 0, beta that is=0, so that implies –alpha+beta, the second one is alpha and third is beta=0 right. So that clearly implies that alpha=beta=0, so linear independence okay. Does it generate everything?

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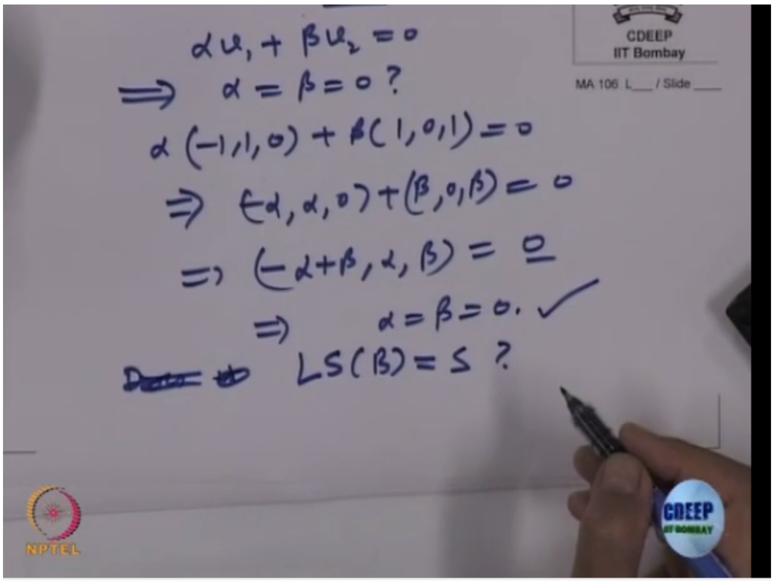

So does it, does that means can I say L linear span of B is=S, all solutions are linear combinations of this right. So that is we have to check. So let us check that.

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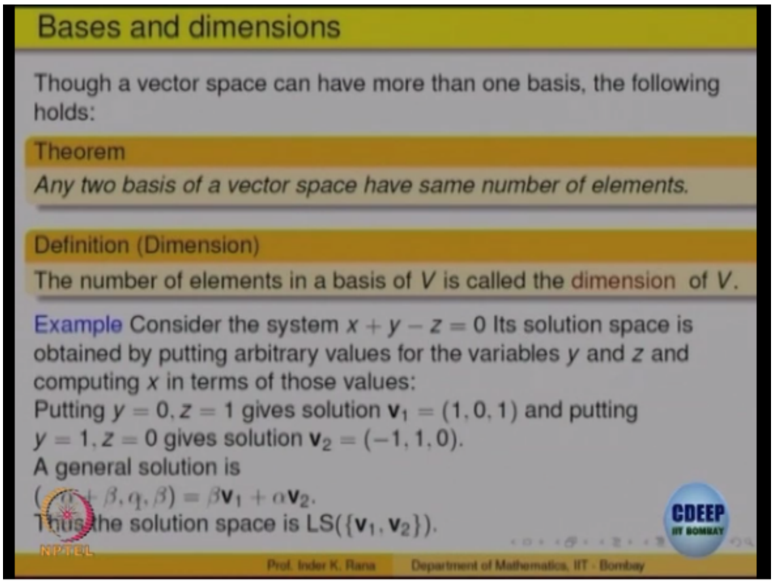

You can write a general solution as here are called v1 and v2, so what does a general solution look like? Okay let us say what is a general solution?

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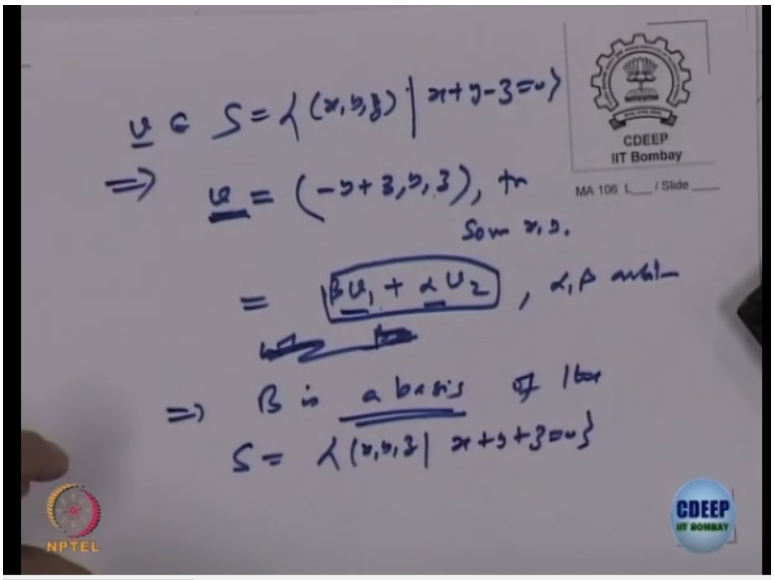

So if a vector v belongs to S implies what? What was S=hat is x, y and z such that x+x-y x+y what was it? x+y-z=0. That means v is=so what is the solution, what is x is=I can compute x in terms of y and z. So what is x from here? x is=-y+z, y and z. Is it okay? v should be of this form where x and y are some constants I do not know, for some x and y right. So that means so let us write this so that is equal to can I write this in terms of v1 and v2?

I want to write it as beta v1+alpha of v2 where see this is what I have written there where what is beta? What is alpha and beta? We will put arbitrary; it does not matter. We can put alpha and beta arbitrary. If you want, if you know v, you can get alpha and beta, you can compute that from the system of equations right. If v is known to you, then you can write down the system of equation and get the solution in terms of alpha and beta in terms of that components of v.

But that itself says that v should be equal to this right where alpha and beta are that means what, every solution is a linear combination of v1 and that says it is a linear combination of v1 and v2. So this is linearly independent that set B right. Those two elements are linearly independent and span everything, every element is a linear combinations, so implies B is a basis of the solution of the set S which is all x, y and z such that x+y+z=0 right.

So what we have done is given a system of equations which had infinite number of solution, there were two free variables, we put them special values and got a basis for the solution space that means the solution space for this particular homogeneous system is written as a linear combination of only two vectors right. So that is the advantage of basis. So we will continue this in the next lecture.