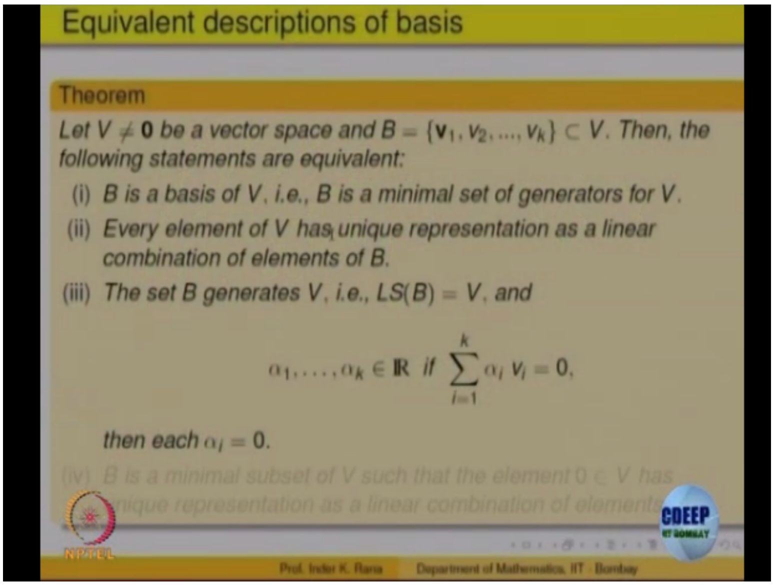
Basic Linear Algebra
Prof. Inder K. Rana
Department of Mathematics
Indian Institute of Technology – Bombay

Lecture - 14
Linear Span, Linear Independence and Basis - II

(Refer Slide Time: 00:39)

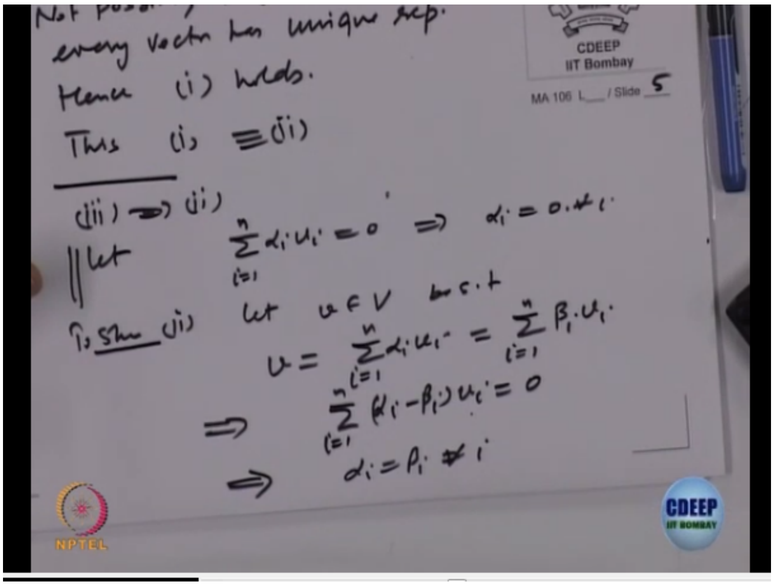


And now we want to prove another one namely saying that every element has got a unique representation is equivalent to saying that the set B generates that is the linear span is B and if a linear combination is 0, then all the scalars must be 0. That is what essentially we did in the previous thing also right. So from II to III right is that obvious II implies III? Because you assume everything, every element has got a unique representation.

And if there is such a representation that sigma alpha i beta i=vi=0, the right hand side 0 is=0*vi summation. So what will uniqueness say? The uniqueness will say that each alpha i must be=0. So what I am saying is II implies III is obvious. Is that obvious to everybody? Yes, because if a linear combination is 0 and if this representation has to be unique, then this alpha i has to be 0 and that is what III is saying, is another way of saying. So II implies III.

Let us see why does III imply II? So that is again obvious. So let us look at III implies II.

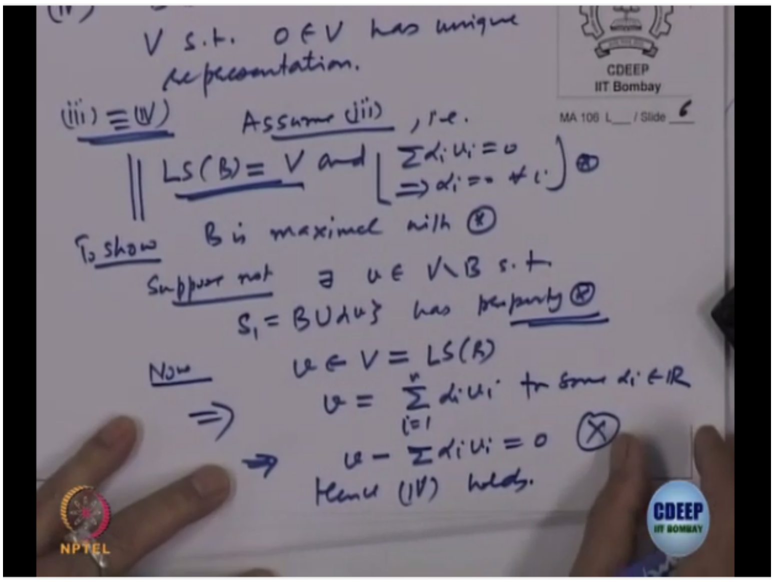
(Refer Slide Time: 01:56)



So let sigma alpha i vi be=0 i=1 to n, so that is III right so let imply alpha i=0 for every i. So this is given to me. This part is given; there is a statement III right. To show II what I have to show, every element has got unique representation. So let v belonging to V be such that v is also=sigma alpha i vi also=sigma i=1 to n beta i vi. Suppose there are two different representations, once again this will imply sigma of alpha i-beta i vi is=0 right.

Take everything on one side, that is=0 and what does that imply? 0 has got a representation by uniqueness for 0 this implies alpha i is=beta i for every i right. So that way both are obvious from each other. So statement III is equivalent to, II is equivalent to III. Let us look at final one which says B is a minimal set. No, sorry this is a wrong thing, it should be maximal. So this let me correct, what is the correct statement of IV, so this is a typo there.

(Refer Slide Time: 03:45)



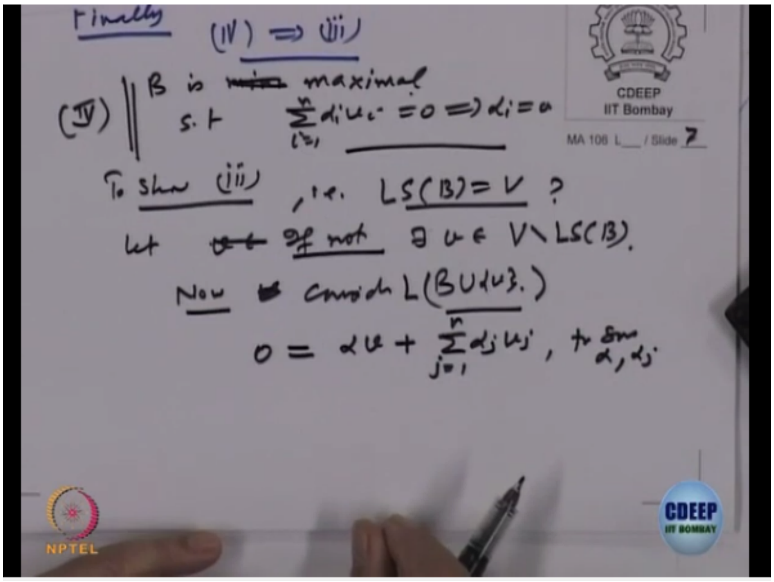
So IV says B is a maximal instead of word minimal it should have been maximal subset of V such that 0 belonging to V has unique representation that should be statement IV. So let us prove that. So we want to prove that III equivalent to IV okay. So assume III, so what does III says that the linear span, so that is linear span of B is=V and sigma alpha i vi=0 should imply alpha i=0 for every i, so that is what is given to me right.

What is to be shown? B is maximal with this property right with this property star, 0 has got unique representation right. Suppose not, suppose it is not right that means what? That means there exists something one more which I can add to it right. So there exists a v belonging to V-B such that if I look at S1 which is=B1 or S1 which is B union this v right. So look at this v has so if not it is not maximal with that but has property star.

If it is not maximal with some property that means I can add one more, make it bigger with that property right. So I am adding a vector v which is not in B to it. Now look at v, v belongs to V and what is V? What is given to me? L of B is right, so this is L linear span of B, so implies what v is a linear combination of for some alpha i right but what does that mean? That means again take everything on one side.

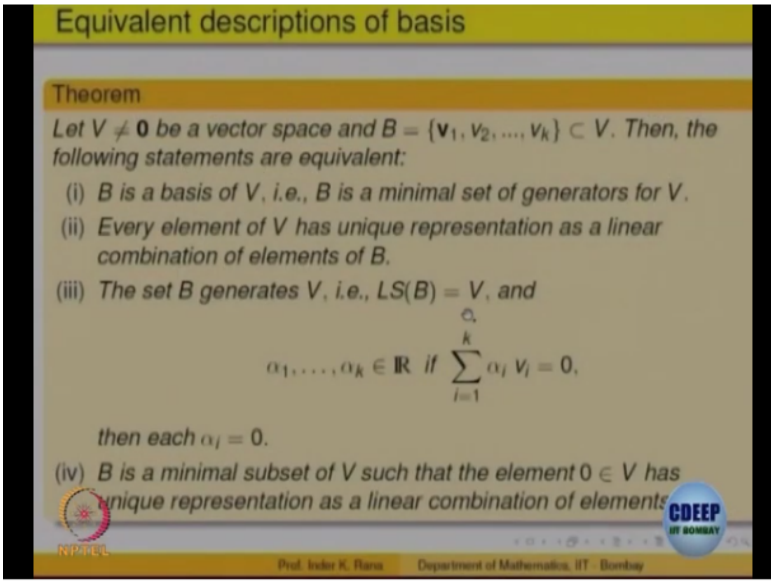
So v-sigma alpha i vi is=0 right, so here is 1*v+something that is=0 that means what? 0 does not have unique representation right, so this property star is contradicted. We said this should have the property but we have written a linear combination=0 where not all coefficients are 0. So not possible, so hence IV holds right. So this is the proof that III implies IV. Let us prove the other way around.

(Refer Slide Time: 07:34)



So finally we want to prove IV implies III okay. So what is IV? IV says B is minimal, sorry alright B is maximal such that sigma alpha i vi 1 to n=0 implies alpha i=0, so that is what is IV and what we have to show right, to show III, so what is it already has the property so what is to be shown to prove III?

(Refer Slide Time: 08:30)

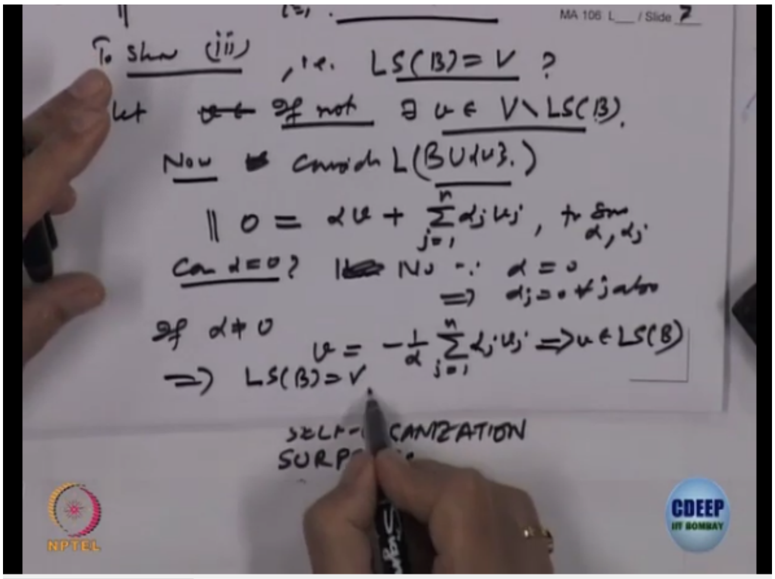


See in III, we want to show that there are two things one it spans right and secondly it has that property, unique representation right. So already unique representation is already there given by IV. So what is left to be shown? To show that is L linear span of B is=V that is what is to be shown right. So let us take a vector v okay, if not if it is not true what will happen, if L of linear span of B is not whole of V that means there is a vector outside it right.

So if not there exists a vector v belonging to V-linear span of B right. Now consider this vector v now okay, so there is a vector v which is outside the linear span okay. Now we want to show that if not there is a vector v so let us consider such that B okay. Till now we have not used it is maximal right. So now let us consider B union v okay and look at the linear span of this thing that will mean what?

That means 0 should be=0 is inside this should be=some alpha times v+right 0 belongs to the linear span always, so 0 should be a linear combination of elements in B and V. So alpha times v+some alpha j vj j=1 to n for some alpha, alpha j. Is that okay?

(Refer Slide Time: 10:33)



Now can alpha be 0? If alpha is 0, then what will happen, sigma alpha j vj is 0 by the given property right. What is the property III? This says for IV it says sigma alpha i vi=0 should imply each alpha i is=0. So if alpha is 0 then can alpha be 0? No, because alpha=0 will imply alpha j=0 for every j also by given hypothesis right. So if alpha is not 0 then what does this equation tell me?

It says I can write v as -1/alpha summation alpha j vj right and what does that mean? v is a linear combination right so v belongs to L of linear span of B okay right. So any element outside also is a linear span that means what, that means there no such v exist that means v is=linear span of B right. So this implies that linear span of B is=v okay. So that basically means let me go through the statement once again.

It says you are given a vector space and you have given a finite set right. We want to know when is the finite set a basis, one is it is a basis means it is a minimal set of generators right. Second, every element in the vector space has a unique representation in terms of elements of B that means every element in V is a unique linear combination, those scalars alpha i's are unique.

And the third says every element is a linear combination and 0 has unique representation. So they say 0 has unique representation. So they are saying that every element has unique representation is equivalent to saying that 0 has unique representation and in terms of maximality and minimality it says B is maximal subset right such that 0 has unique representation right. So these are 4 equivalent ways of describing what is called a basis. All will be useful in constructing basis for vector spaces, so we will see that how these are useful.