Basic Linear Algebra
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Lecture – 10
Solvability of a Linear System, Linear Span, Basis I

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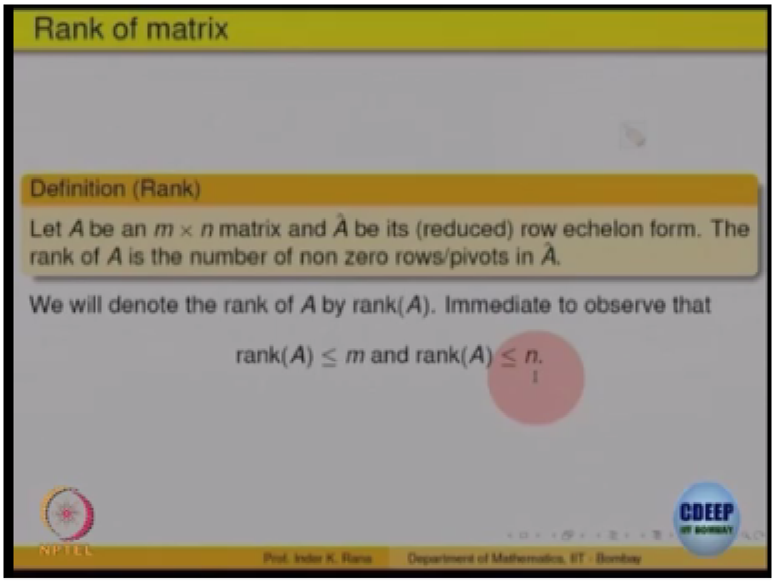

Let us just recall, what we have done in the previous lecture, we have started looking at what is called the rank of a matrix, so we defined for the matrix A which is m cross n, look at its row echelon form particularly you can also look at the reduced row echelon form and the rank of the matrix is defined as the number of nonzero rows in that matrix or the number of pivots, both are same.

So, we had mention that if you take row echelon form of a matrix that may not be unique but the number r which is number of nonzero rows, its same in all row echelon forms, okay and the first nonzero entry was called pivot, so number of pivots is also fixed, once you have row echelon form, so the rank is defined to be the number of nonzero rows in the row echelon form, now reduced row echelon form.

So, clearly the number of nonzero rows, so that it is always < or = m and n, the minimum of the 2, right, so it is < or = both the number of nonzero rows, so the rank is always < or = the minimalism of number of rows and number of columns.

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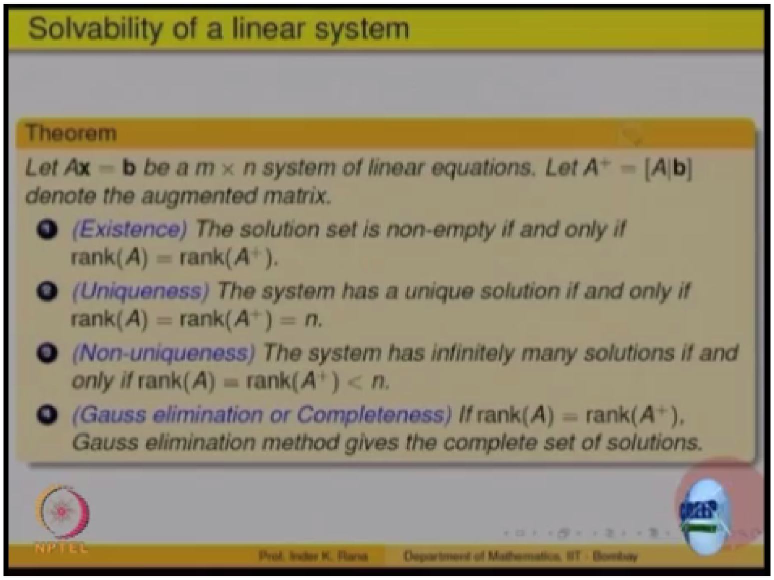

So and we stated the theorem of solving a system of equations in terms of this new terminology rank, so it says the following, first of all, the solution set is non-empty that means the system is consistent if and only if the rank of A = rank of the augmented matrix, the rank; 2 ranks are same that means the pivot, it should not happened that there is no pivot in a row in A but there is a pivot in the augmented matrix, right.

So that will give you an inconsistent equation 0 = nonzero thing, so existence is if only if the rank of A is same as the rank of augmented matrix that means, the number of nonzero rows in A and the number of nonzero rows in augmented matrix should be same, okay, once you have them in row echelon form. So, one is a unique solution possible, n is the number of variables, m is some m cross n, n is number of variables.

So, if there are n pivots exactly, right that means each variable will get a unique value, so the system has unique solution if and only if the rank of A = rank of A + that is consistency and both are = n, the number of variables, then there is a unique solution and non-unique means, actually there are infinite number of solutions, when the rank of A = rank of the augmented matrix, once again consistency going on but it is strictly < the number of variable.

The number of the pivots is strictly less than the number of variables, okay, so then it says there are infinite number of solution and we said that the difference n – r, the rank, right those many variables will get infinite values, their arbitrary, they are free to give any values, right, so you can choose the remaining, so that describes all solutions also, if the rank = rank of A + either there will be unique solution, if it is = n.

If it is < n, there are infinite solutions and although infinite solutions can be obtained by backward substitution, right, putting the non-pivotal variables arbitrary values and calculating the pivotal variables in terms of rows arbitrary values, right.

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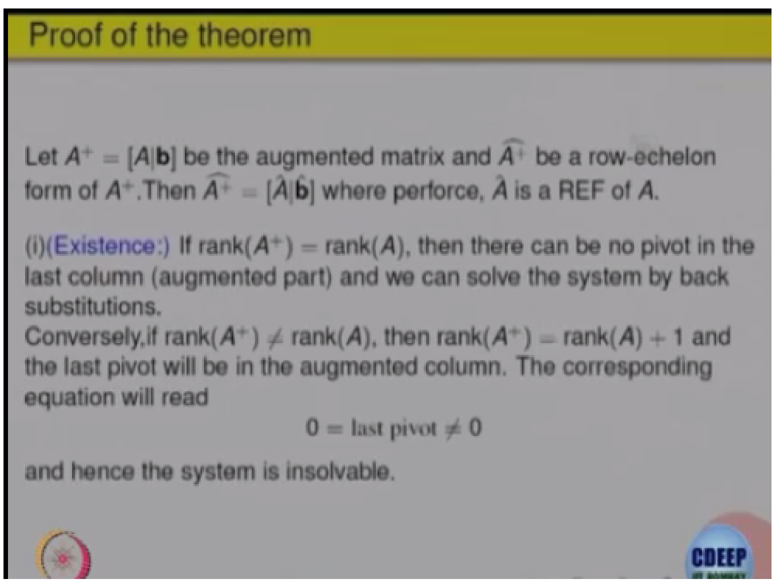

So, let us write a; so the I think proof we have already gone through, the existence and; so let us keep the proof, right.

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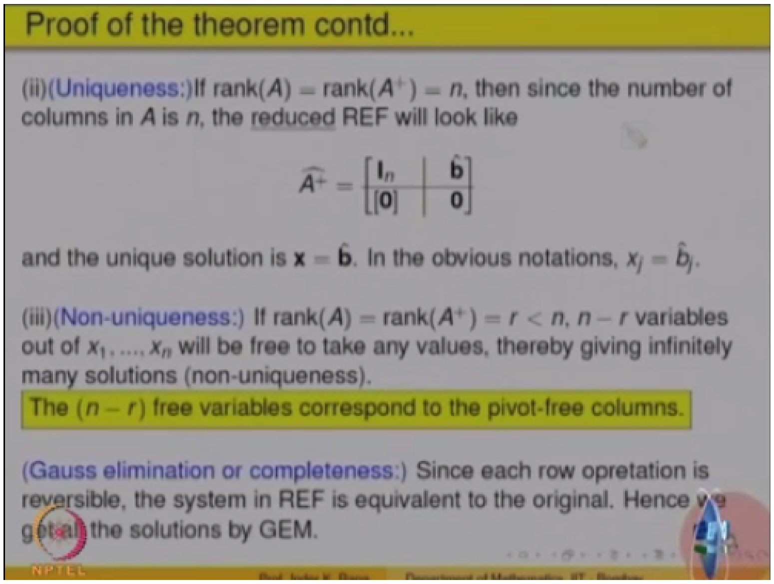

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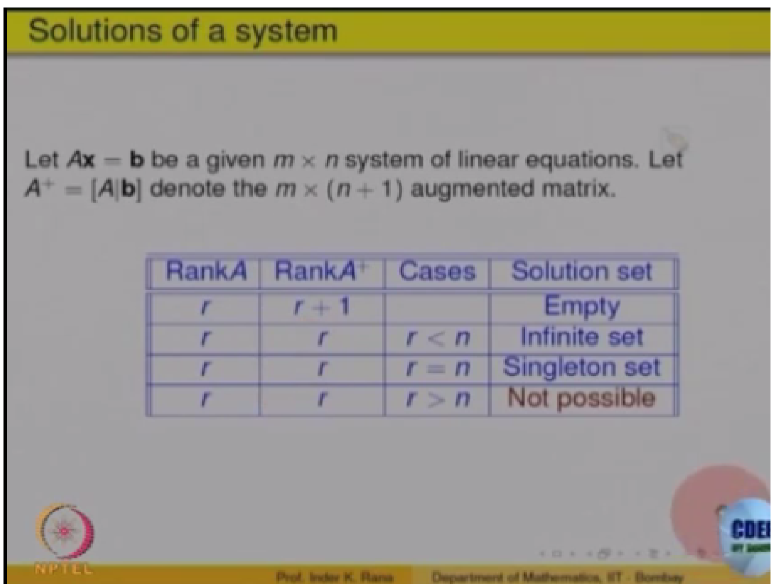

Uniqueness also we gave gone through, so let us; so I am just summarising now, if the rank of A is r, rank of A + is r +1 that is one more column is there in augmented matrix, so possibility is rank of A = r but the rank of the augmented matrix is r + 1, then there is no solution, it is an inconsistent solution, the solution set, right, the set of all solutions is an empty set. If both are r and r is strictly < n, the number of variables, then there are infinitely many solutions possible, right.

And if both are = n, the rank of A is same as rank of augmented matrix = n, then the solution set will be a single point that means there is a unique solution, right, this cannot happen, r cannot be strictly bigger than n, right, okay the rank of the matrix cannot be bigger than number of rows or number of columns, so this possibility does not arise, so these are the only 3 possibilities which can arise.

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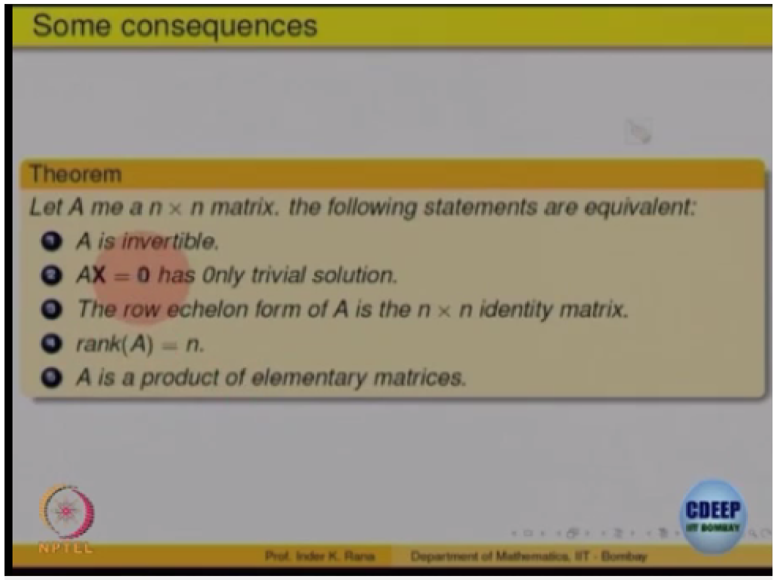

And as a consequence of this, there is a some simple application which we have already discussed, a square matrix, n cross n is invertible if only if Ax = 0 has only trivial solution, right, there should not be any nontrivial solution for this, is it clear because Ax = 0, 0 is always a solution, right and if A is invertible that means only one solution possible, for 0 should be the only one, is it clear to everybody.

Ax = 0, always has 0 as the solution, if we put x = 0 that is the solution, right and when invertible that means there is only one solution possible, right invertible means that was also echelon to the row echelon form is identity matrix, right that was also same as saying the rank = n and we also showed that this is same as saying the product, A is a product of elementary matrices, right, A is invertible if and only if it is a product of elementary matrices, we showed that.

Anyway actually, that gives us also we have computing the inverse in case it is invertible, so all these are equivalent ways of saying that matrix is invertible, Ax = 0 has only trivial solution possible, there is only one solution possible, right, you can also look at this way, if you multiply in this equation both sides by A inverse, what you get; A inverse A x = 0 that means x = 0 that means only solution possible is x = 0, right.

The row echelon form should be identity matrix, right, invertibility is same as saying that rank = n, number of nonzero rows must be same as a number of variables and that is same as a row, the number of rows or number of columns in the square matrix and product of elementary matrices, so there are various ways of saying when a matrix is invertible, right all are same but there is different ways of saying the same thing, okay.

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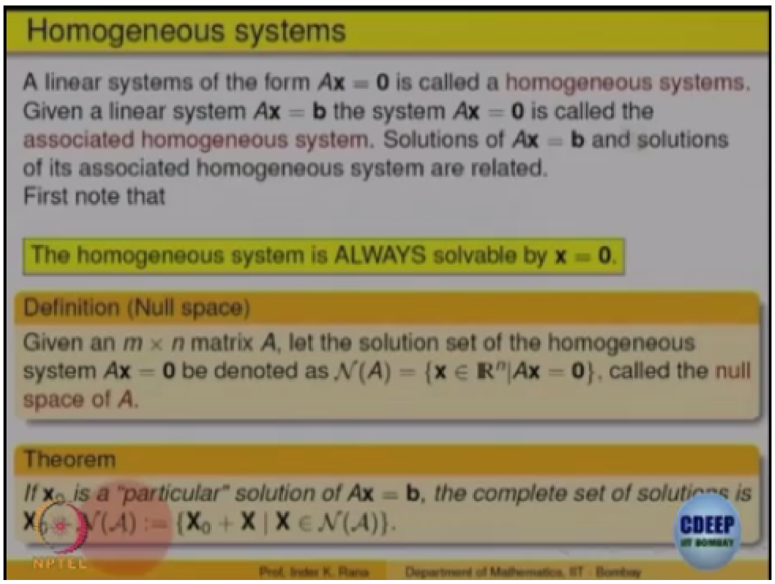

So, let us look at that the previous one said that system Ax = 0 seems to be an important A is to consider, so such a system is called a homogenous system, so system of linear equations were the right hand side that vector is a b is 0 is called homogenous system, right, so given below, we want to relate it with the; so, given any linear system; Ax = b, if you put b = 0, will get a homogeneous system, right.

So, given a system of linear equations Ax = b, if you put b = 0 that is called the homogeneous system associated with the given system of linear equations, right we are putting, so what is the advantage of doing that and what is the use of it; so, let us; the observation is the solutions of the general system are related with the solutions of the corresponding homogeneous system, in what way, we will describe that.

So, first of all let us is note that the homogeneous system always have a solution, right, Ax = 0, so 0 is always the solution of that okay, so let us define all solution of the homogeneous system has a set, so let us call that as null space given a matrix A, you are given a matrix m cross n, so look at all possible, the vectors x, so I will want to get Ax = 0, so what will be the order of the vector x, Ax; you have to multiply A with x, A is m cross n.

So, vector x ha to be n cross 1, right so you are looking at all vectors x in Rn say that Ax = 0, so look at that side, so collection of all vectors which are killed by A, you can think it that way that is called the null space of the matrix A, right and that is same as a solution space of the homogeneous system Ax = 0, right in terms of set theory, we call it as null space denoted by n of A, right that is a solution space of the homogeneous system Ax = 0 for a given matrix A.

So, for every matrix we are associating is space with it, a set with it, the null space which is the solution set of the homogeneous system, okay. What is the advantage of that? So, the advantage is look at, we want to solve the system Ax = b, right, the theorem says if the solution set of Ax = b is same as; you find at least one solution of the system Ax = b and add to it all solutions of homogeneous systems, you get all solutions of the system.

So, finding a solution for the non-homogeneous case or the general system is find at least one, find all solutions of the homogeneous case, take 1 from there and add it to it, we will get all, one by one pick and add, right, so the solution of the; solution space of Ax = b is same as X0 + X, where X is in the null space that means Ax = 0 and what is X0; some particular solution, if I were able to find at least one solution of the system Ax = b, add to it a solution of the homogeneous system, you get a solution of the general system, right.

So, what it says is that the solution of the homogeneous system is important thing to consider using that we can describe all solutions of the system Ax = b provided we can find the null space, the solution space of homogeneous system plus one particular solution, right.

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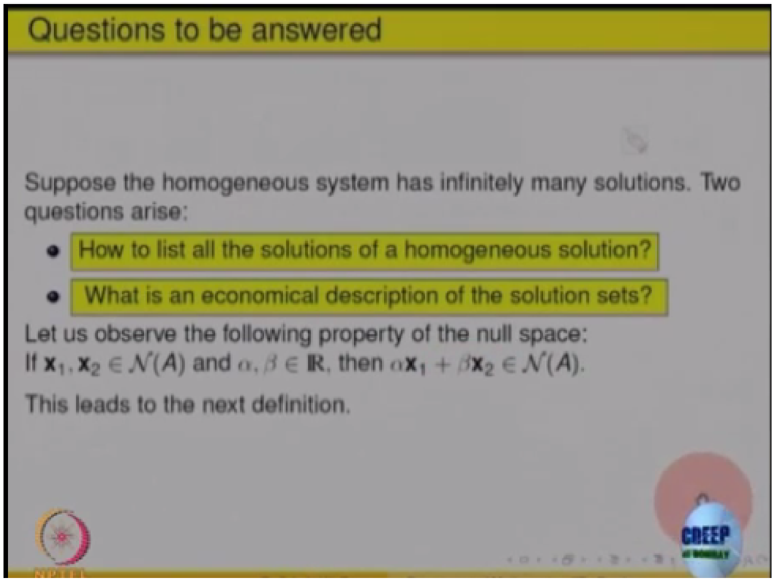

So, we will concentrate on this, so questions are; how to list all solutions of a homogeneous system? So, the problem; we are reducing the problem to the homogeneous case in a sense and what is the best economical way of telling what are all the solutions, you can say this is the solutions, right but can I give a method of generating all the solutions of the homogeneous system, right what is the most economical way?

So, let us observe something that is X1, X2 are solutions of the homogeneous system, Ax = 0, then alpha X1 + beta X2 also is a solution, if X1 and X2 are the solutions of the homogeneous system that means AX1 = 0, AX2 is also = 0, then the claim is alpha X1 + beta X2, where alpha and beta are (()) (13:13) is also a solution of the homogeneous system because we multiply by A, then what is a product?

A multiplied by alpha X1 + beta X2 that is same as alpha times AX1 + beta times AX2, why the distributive property of multiplication of matrices, right but AX1 is 0, AX2 is 0, so A apply 2 alpha X1 + beta X2 is also = 0, so it says that this null space, right, the solution of the homogeneous system has a nice property that if I take 2 elements in it and take a combination, alpha X1 + beta X2 of that that again also is a solution of it, right.

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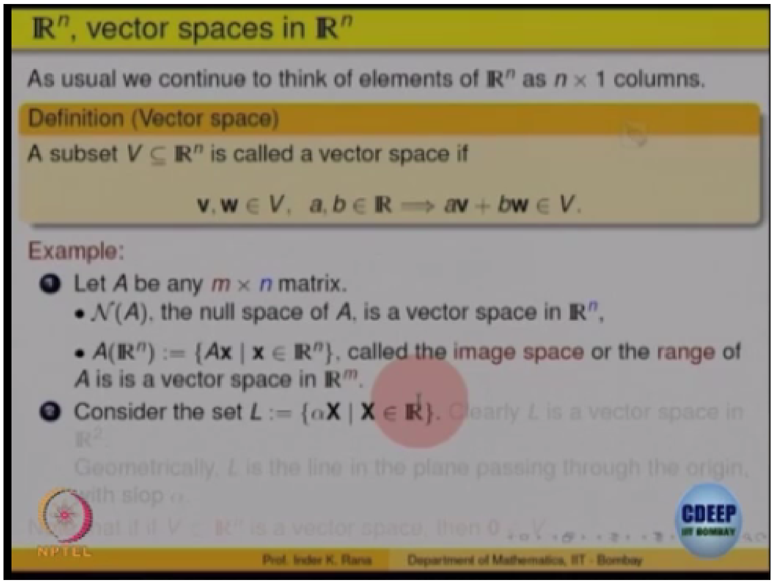

So, this property if we want to make it as a definition, so let us make it a definition, a general definition law for which particular case NA will have that property, you say that a subset V of Rn, right is called a vector space if it has that property may be, take any 2 elements in V, take 2 any scalars A and V and take the linear combination av + vw, right, multiply v with a, multiply w with b, take their sum that is called a linear combination that also should belong to V, right.

So, if the set V has this property, then we say it is a vector space, so in our example for any matrix say the null space is the vector space because it had that property, right that is what motivated is to define, so let us look at examples; for any m cross n matrix, look at the null space, null space is a subset of what; Rm or Rn? Matrix is m cross n, we are going to do Ax, so x should be of the same order as n.

So, null space is a subset of Rn and it is a vector space, right, let us look at one more example, look at A times; this is the notation ARn, so what is it; look at A times the x, the multiplication, where x varies over all of Rn, A is a matrix, multiply any vector x with it, we will get a new vector, so take the collection of all such vectors, okay that is called the image space of the matrix or also called that the range of the matrix A that again is a vector space.

Is that a vector space? For example, let us look at Ax1 is one element of the set, Ax2 is another element, so let us look at alpha times Ax1 + beta times Ax2, we want to show it is also in ARn but what is alpha times Ax1 + beta times Bx2, by distributive property again other way round, it is A apply 2 alpha x1 + beta x2, so it is A of some other vector but that vector is again in Rn, right, clear to everybody.

If I take 2 vectors of this type Ax1 and Ax2 and take alpha times Ax1 + beta times Ax2, right then this addition what is this equal to? This is equal to A apply to alpha x1 + beta x2, so it is apply to; A apply to some other vector, A multiplied with some other vector in Rn, so that is again inside this set, it is of the type; same type again, A apply to something, right, so this also is the subspace of Rn but this is a subspace of what, is it of Rn or Rm?

Here, A apply to something, A is m cross n, x is n cross 1, when you will apply, you will get a vector of m cross 1, multiplication, so it is a subset of Rm, so this is the subspace of Rm, the null space is the subset of Rn but the range is a subset of Rm, we will describe later the matrix as a kind of the map, it takes vectors from Rn to vectors in Rm, right, which are killed that is in the domain, the range is right where it they go, right.

So, Rn to Rn, so we will come to that and we describe linear transformations right, so at present just keep in mind because of matrix multiplication, the range is a subset of Rm and the null is the subset of Rn, right, they have different sub spaces, you have different Rn or Rm. So, let us look at one more example. Consider the set L, which is alpha X, there is a typo error, it should be R to the power 2, I take a; okay, I take some vector, right, a vector in X2 R2, take some fixed vector, okay.

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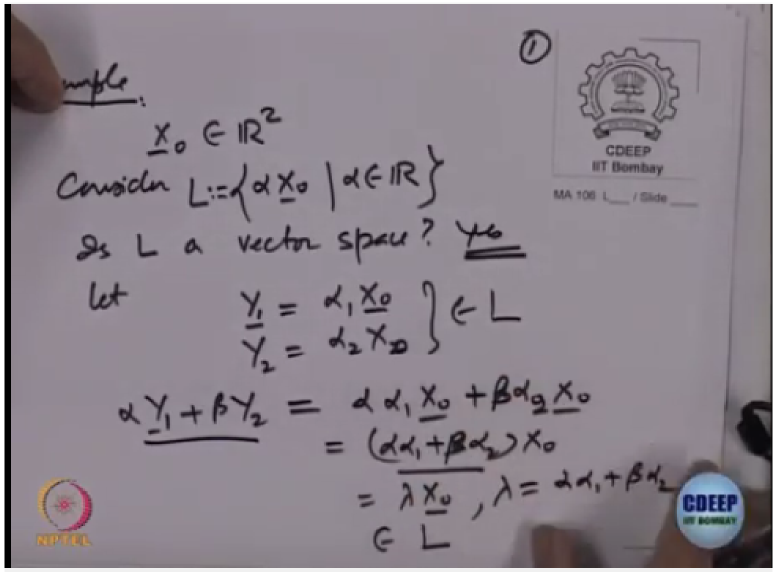

So, let us take a vector X0 in R2, is a fixed vector, okay, consider the set of alpha X0, where alpha is a vector in R, okay, have a fixed vector X0 and I multiplied with scale; different scalars, so the question is; is call it as L; is L a vector space? We want to know whether it is a vector space or not, so how do I check it? Take 2 elements, so let us call it Y1 = alpha 1 X0 and Y2 = alpha 2 X0, both belong to L, right.

Take 2 vectors, then what is Y, what is say alpha Y1 + beta Y2, what is that =; sorry this is X0, so what is alpha Y1 + beta Y2? So that is alpha, alpha 1 X0 + alpha, alpha 0; alpha 2 of X0, right, is that okay, so what is that equal to? Alpha, alpha 1 + alpha, alpha 2 of X0, so I should have written beta somewhere, this is beta here that is beta here, right, I take 2 elements in that set L; Y1 and Y2, 2 vectors, take alpha times Y1 + beta times Y2.

So that gives me; so this is = some lambda times X0, what is lambda; that is alpha X1 + beta alpha 2 that means a linear combination of Y1 and Y2 is again belonging to L, right, is that okay for everybody, so this L is a vector space; yes, this is a vector space, can you tell me geometrically, what does it look; what is this vector space in R2, you can visualise geometrically, what is R2; R2 is the plane, right, so R2 is the plane.

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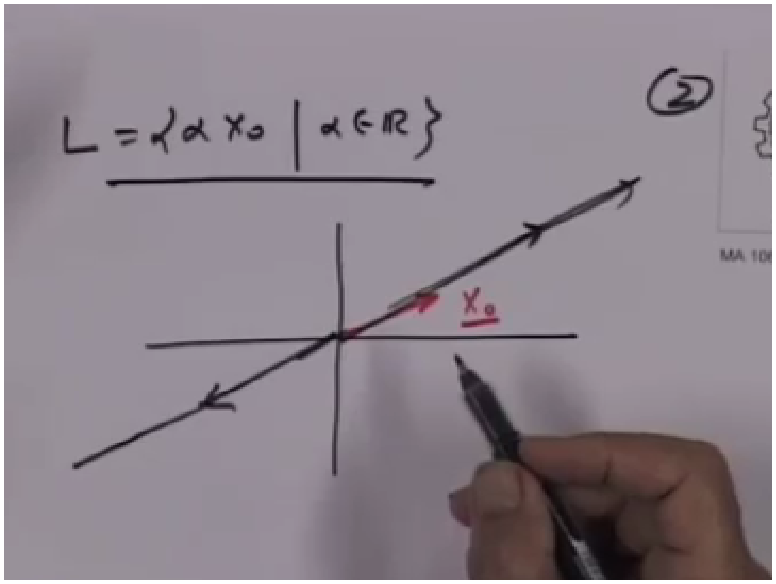

So, visualising; so what is L, so that is = alpha X0; alpha belonging to R, let us try to draw a picture of it, so what is the picture; X0 is the vector, right, so I can draw that vector, okay, let us draw it from here, so this is a vector, okay, so that is my vector X0, what is alpha X0, what I am doing? I am taking a fix vector and only stretching it, right, so when I stretch it, its length will change only, right, so this will give me, right, the vector could be anywhere here from here to here or here to here or here to depending on whether it is positive or negative.

So, what is this; this is a line through the origin right in R2, is that okay, right, so this is a line through the origin that is a vector space, okay.