Calculus for Economics, Commerce and Management
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Lecture – 07
Limit theorems, sandwich theorem, monotone sequences, completeness of real numbers

So, welcome to today's lecture. We will continue our discussion on the concept of sequences. So, if you recall we are started looking at the notion of a sequence the convergence of a sequence and then we set the limit of a sequence is always unique we gave examples of sequences which are convergent and which are not convergent a non convergent sequence is also set to be divergent. We continue our study; we also looked at last time the notion of boundedness of a sequence namely all the terms of a sequence lie between 2 bounds. So, here is a theorem about convergence and boundedness it says that every sequence a n which is convergent it is also bounded. So, convergent implies it is bounded.

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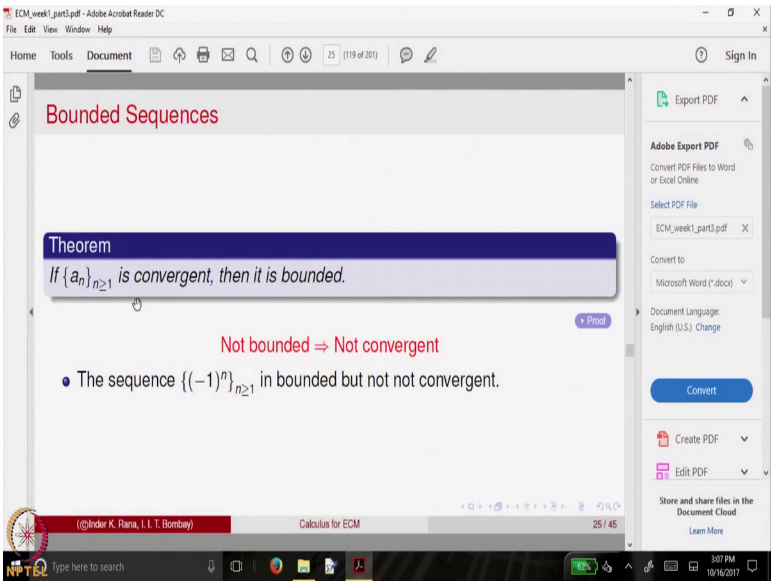

So, boundedness is a necessary condition for a sequence to be convergent equivalently one can say not bounded implies not convergent. So, this is a theorem in calculus, we will not be proving this theorem. So, we will assume the proof of this theorem, those who are interested as I said can always refer to some book on calculus or look at the web course on calculus developed under NPTEL. So, a sequence is convergent, then it is bounded. So, the use of this kind of a theorem is that if sequence is not bounded it cannot be convergent. So, this is this is how necessary conditions are used if a necessary condition is not satisfied then it is not prove that ah it should be convergent so; however, keep in mind that boundedness alone is not enough to say the sequence is convergent.

So, here is a example of a sequence which is bounded, but not convergent. So, this is minus one to the power n is a sequence which fluctuates between minus one to one and it is not convergent. So, let us look at some more theorems about convergence of sequences which help us to analyze the convergence of a sequence and these are theorems in calculus, but can be used.

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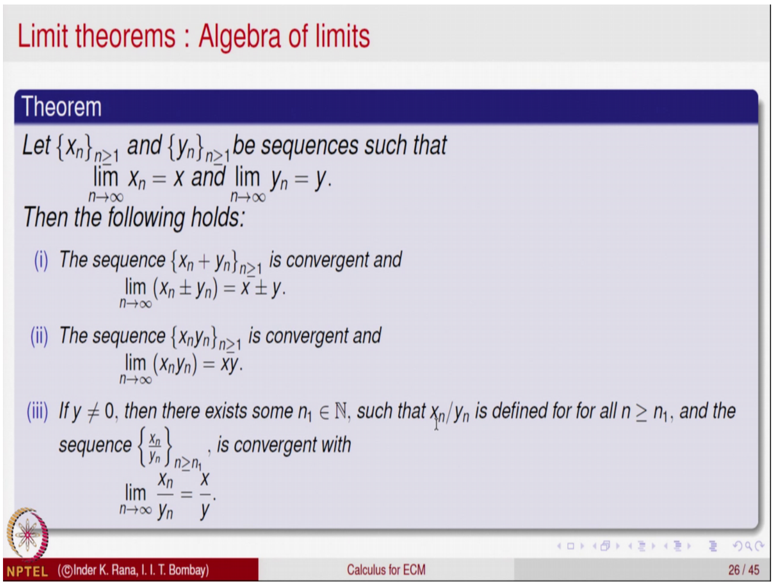

We will be using these theorems and either you can treat it as a rule for analyzing convergence of a sequence or you can if you are very keen as usual you can look at the proofs in the NPTEL web course. So, this says if a sequence a X n and Y n are sequences such that X n convergence to x and a sequence Y n converges to y.

Then the following results hold you can give an X n and Y n you can construct a new sequence called where each n th term of the new sequence is the sum of the n th term of X n and the n th term of y n. So, look at a new sequence whose n th term is X n plus Y n where X n is a sequence which is convergent and Y n is a sequence which is convergent then it says then the following holds the sequence X n plus Y n is also convergent and its result the limit of X n plus Y n is x plus y and similarly the you can have the difference of the 2 sequences. So, you can construct X n minus y n. So, limit of X n minus Y n is same as limit of X n minus limit of y n. So, it just we can intuitively keep it as the limit of the sum of sequence is equal to sum of the limits of the corresponding sequences and similarly for the difference of the 2 sequences.

The next results says about the product given a sequence X n and given a sequence Y n the n th terms can be multiplied together to get the sequence X n Y n. So, n th term of the new sequence is X n Y n. So, if X n is convergent and if Y n is convergent, then the sequence X n Y n the product is also convergent and the limit is the product of the 2 limits. So, it says the limit of the product is equal to product of the limits if the either if the both the sequences is X n and Y n are convergent and the next is. So, we have looked at addition of sequences subtraction of sequences product next we want to look at th quotient of 2 sequences of course, X n over Y n will not be defined if Y n is equal to 0 somewhere.

So, one has to put a extra condition that if the sequence Y n is such that its limit y is not equal to 0, then the result says then for some stage onwards for some n one an actual number Y n will not be 0; that means, X n over Y n is defined for all n bigger than or equal to n one from that stage onwards. So, we have a sequence X n over Y n which starts with n one onwards. So, this sequence is convergent and its limit of X n over Y n is equal to x over y.

So, limit of the convergent sequence is this equal to limit of the quotient of convergent sequences is the quotient of the limit provided the quotient limit is not equal to 0. So, these are 4 basic results about the algebra of limits because we are adding sequences, we are multiplying sequences subtracting and then dividing sequences. So, basically says that the limiting operations preserves the algebra if appropriately defined. So, quotient for the quotient we need y not equal to 0.

So, these results can be used to analyze convergence of sequences. So, there is another theorem which again will not be proving will be using it says the following suppose you have got 3 sequences a n, b n and c n.

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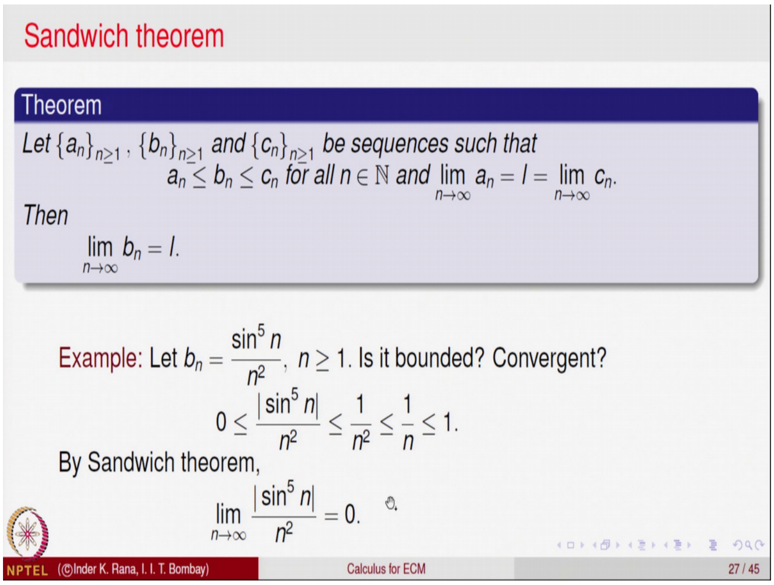

So, there are 3 sequences with the properties that the sequence b n is in between a n and c n for every n; that means, all the terms of the sequence c n, it is not really require that all the terms from some stage onwards if it is true that is also good enough one can forget a few first few terms of the sequence. So, we want a n less than or equal to b n less than or equal to c n. So, that is one condition and is a n and c n both converge to the same limit say l, then the result says then the limit b n also exist the b n is also convergent and is equal to l.

So, basically what we are saying is that if the terms of a sequence are sandwiched between 2 convergent sequences and there are convergence of a n and c n in which it between which it is sandwiched is same both a n and b c n converge to l then b n also will converge to m this is known as the sandwich theorem which is also quite useful again we will not be proving this result. So, let us look at the example for example, let us look at the sequence b n which is sin; sin is a trigonometric function if you know it very good if you do not know it, I think you have look at it in some school book how the trigonometric functions are defined. So, we will not go into the definitions of trigonometric functions only for the sake of examples we will see basically sin of a angle theta is defined.

And this for every theta sin theta is between minus one and one. So, this is basic property that will be using. So, b n is the sequence which is defined as sin to the power 5 of n and divided by n square now since the sequence b n is it bounded is it convergent. So, this is a question to the analyze it is boundedness let us look at the absolute value of b n. So, that will be equal to absolute value of sin 5 to the power n by n square since sin is bounded by minus one to one mod sin is bounded between by 1. So, we get that sin 5 n divided by n square this absolute value of b n is always on negative absolute value is non negative is less than or equal to 1 over n square which is less than equal to one or n. So, which is less than 1. So, this is a bounded sequence and since it is bounded between 0 and 1 over n and 1 over n convergence to 0. So, this b n mod b n is in between 0 and 1 over n.

So, why sandwich theorem we get that this also convergence to 0 now sin this. So, this is absolute value of b n which convergences to 0 and it is again a easy result which will assume that if mod b n convergence to 0 then b n also convergence to 0. So, that will imply the. So, we have looked at some tools which help us to analyze convergence of sequences namely algebra of limits and the notion of sandwich theorem here are some more concepts about sequences which are very useful and important.

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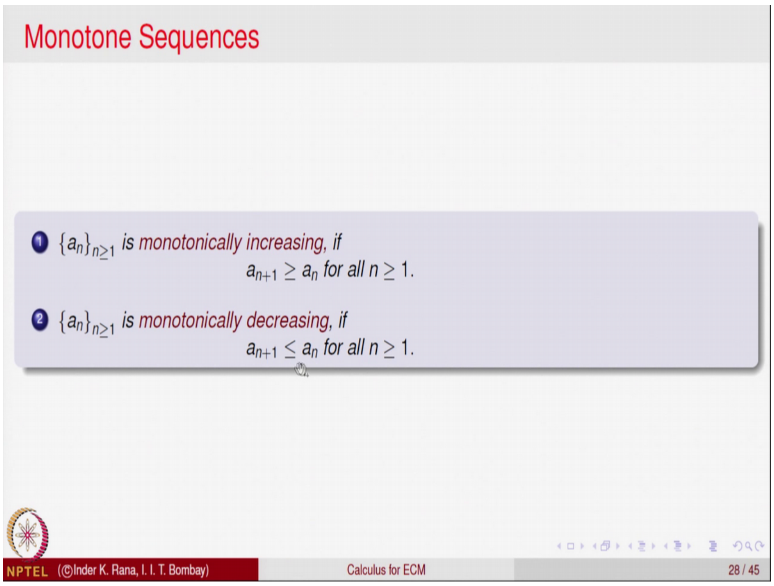

So, we say a sequence a n is monotonically increasing if a n plus 1 is bigger than a n for every n; that means, every term a n plus 1 is bigger than the previous term that is a n.

So, as you as n increases you values of your sequence keep on increasing keep in mind we are saying bigger than or equal to we are not saying strictly bigger, right. So, if it is equal it will be for saying the sequences monotonically increasing. So, we want for every n the next term that is n plus 1 is bigger than or equal to the previous term that is a n. So, then it is we say it is a monotonically increasing sequence. Similarly, we will say a sequence is monotonically decreasing if a n plus 1 is less than or equal to n; that means, as n increases the values of a n decrease right a n plus 1 is less than or equal to again, we are saying less than or equal to we are not saying strictly less than. So, the next term is at the most the previous one; it could be smaller. So, that is called a monotonically decreasing.

So, monotonically increasing and monotonically decreasing if you want to say strictly bigger if you want to have that condition then we will say it is strictly monotonically increasing and similarly we will say a sequence is strictly monotonically decreasing if a n plus 1 is strictly less than a n or all n bigger than n.

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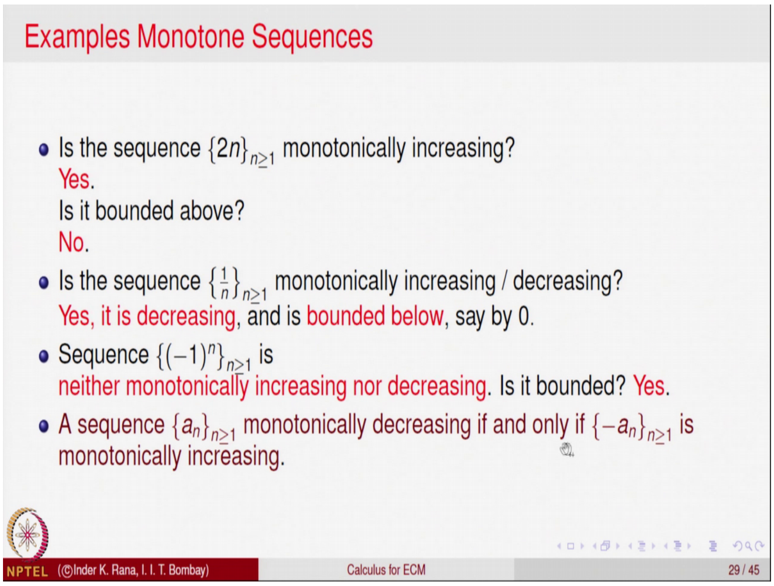

So, for example, if we look at the sequence 2 n, it is strictly monotonically increasing because the first term is 2 the second term is 4 next term is 6 and so on. So, as n increases the values of a n strictly increase, yes, it is monotonically increasing is it bounded above we know it is not bounded above because for n sufficiently large you can make 2 n as large as you want. So, it keeps on increasing it is a sequence which is monotonically increasing and it is not bounded above of course, it is bounded below their all the terms are non negative.

So, it is not bounded above let us look at the sequence 1 over n, we already analyze the sequence first term is one second term is 1 by 2 third is 1 by 3 fourth is 1 by 4 and so on. So, it is a monotonically decreasing sequence which is bounded above by one and bounded below by 0. So, this is the monotonically increasing sequence monotonically decreasing sequence and it is bounded above as well as below; let us look at the sequence minus one to the power n. So, what are the terms of the sequence we have seen it is the fluctuating sequence it is neither increasing nor decreasing because the first term is minus 1 second term is one third term is minus one fourth term is plus 1 and so on.

So, it is a bounded sequence, but it is not monotonically increasing nor it is monotonically decreasing. So, it is a sequence which is bounded, but not increasing not decreasing ah obvious simple fact is that if a sequence a n is monotonically increasing, then negatives of those terms of the sequence will be monotonically decreasing and conversely. So, a sequence is monotonically decreasing if and only if minus of a n, they will be monotonically increasing and vice versa you can also say that a sequence is monotonically increasing if and only if minus a n is monotonically decreasing. So, this is one way of going from increasing to decreasing or decreasing to increasing.

So, here is an important property of real numbers which we said we will state at an appropriate stage.

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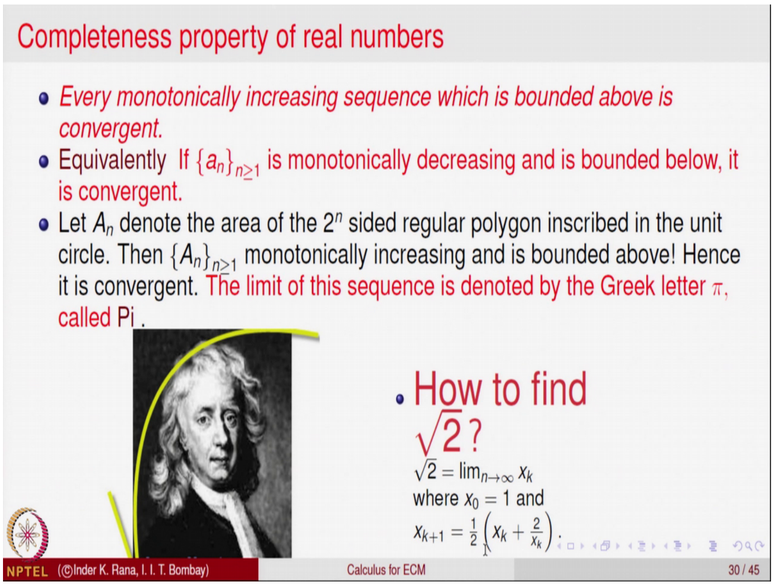

So, we can now state the property of real numbers that every monotonically increasing sequence of real numbers one would specify that every monotonically increasing sequence of real numbers which is bounded above will always be convergent. So, this is a property of real numbers, you can take it an assume for real numbers and when one constructs real numbers from the rational numbers this property automatically comes in. So, this is what is called the completeness property of a real numbers it is a very important property and an equivalent way of saying the same property would be in terms of decreasing sequences every monotonically decreasing sequence of real numbers.

Which is bounded below is also convergent this property is not always true for sequences of rational numbers one can have a sequence of rational numbers which is monotonically increasing and bounded above, but it will not converge to a rational number. So, in the domain of rational numbers this property is not true this property is true for that was actually one of the reasons why one wanted to enlarge the collection of rational numbers to a bigger class of real numbers.

So, real numbers have that property that every monotonically increasing sequence of real numbers which is bounded above is convergent and every equivalently every monotonically decreasing sequence of real number which is bounded below is also convergent some of the applications of this ah property are the following will not be going to the proves of those properties.

Let us look at let a n denote the theory of 2 n sided regular polygon inscribed in unit circle. So, you are given let us look at this property you are given a unit circle.

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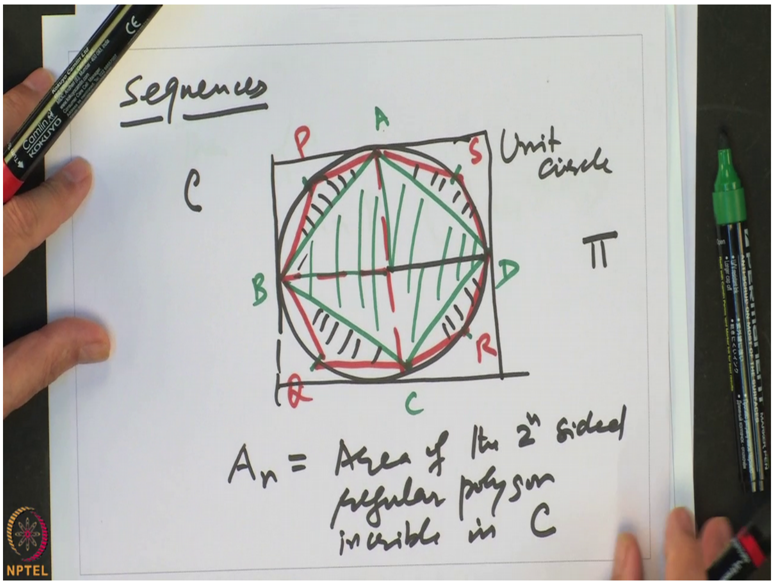

Unit circle means what you are given a circle of radius one. So, this is a unit circle its radius is one this is one unit one would like to find out what is a area of the unit circle normally in your schools school books one assumes the fact that area of the unit circle is a number called pi here is a precise definition of that using the completeness property let us try to estimate the area of the unit circle to estimate this what we will do is let us inscribe inside inscribe inside this unit circle a square. So, this is a square say A, B, C and D.

So, it is a regular 4 sided figure inscribed inside the circle. So, if I look at the area of the circle of the square that we can find out because we know this side is one this side is 1. So, I can find out the area of this 4 triangles using the formula half base into height. So, area of the 4 sided ah polygon regular polygon inscribed in the circle is known. Now let us increase this value let us take the midpoints of this arcs and join and join these points. So, we are joined. So, now, look at let us call this points as P, Q, R and S. So, if I look at the polygon A P B Q C R D S that is a 8 sided polygon inscribed inside the square inside the circle of radius 1. So, earlier the area of the square the square filled up a part of the circle now we added these triangles these 4 triangles.

So, the octagon now fills more part of the circle then compare to the circle, then compare to the square and the area of this octagon is quite easy to find because this side is one.

So, half base into height again you can find out this area. So, what we are saying is if we write a n equal to the area of the 2 n sided regular polygon inscribed in c; c is the circle.

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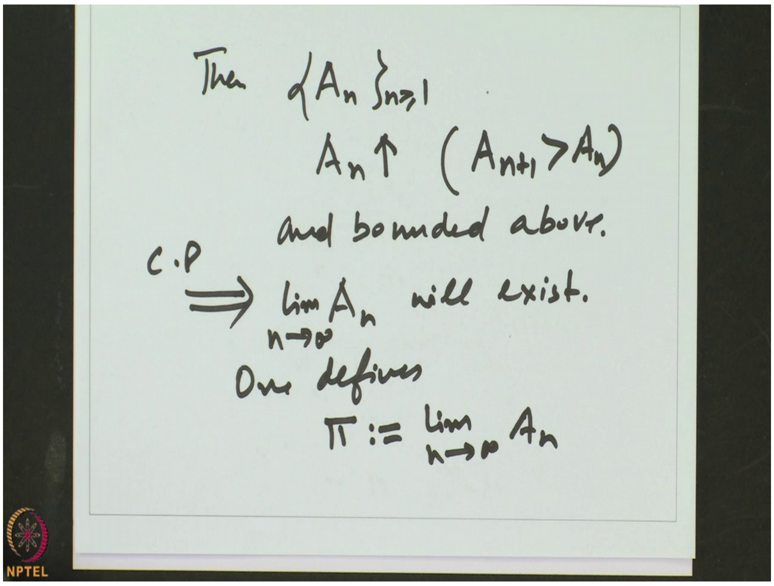

Then this a n is a sequence of numbers and quite clear that a n is monotonically increasing so; that means, a n plus 1 is strictly bigger than a n. So, this is the monotonically increasing sequence of areas and from the figure it its quite clear that if I keep on increasing the number of sides I will slowly and slowly fill up the unit circle ok.

And it is also obvious that if I look at the square outside then the areas of all this 2 n sided regular polygons inscribe they are increasing, but they no never go outside the area of the circumscribed square so; that means, this is monotonically increasing and bounded above. So, implies by the completeness property of real number that limit n going to infinity a n will exist and this limit one defines pi to be the number which is this limit. So, this is the rigorous definition of area of the unit circle namely pi as an application of the completeness property of real numbers.

So, this is one of the applications of real numbers. So, if you take the regular 2 n sides regular polygon inscribed in the unit circle then this is a monotonically increasing sequence which is bounded above and hence will converge. So, the limit is called by denoted by the Greek letter pi that is the area of the unit circle this is one of the rigorous ways of defining area of the unit circle another application of the completeness property how does one finds square root of 2 of course, you will say what is the point we can always divide and then do it right, but can we have a algorithm of for finding this, yes there is possible. This was given by Isaac Newton which said that let us define let us start with the number x 0 equal to 1 and iteratively define x k plus 1 to be equal to 1 by 2 of half of x k plus 2 by x k.

So, once x k is defined x k plus 1 is defined and one shows this sequence of real numbers is convergent using the completeness property and the limit is nothing, but limit exist and that is what is called square root two. So, here is a another way of appreciating the concept of limit of a sequences that square root 2 if you have seen earlier we also mentioned that square root 2 is irrational number. So, its value cannot be found exactly like pi the value of square root 2 cannot be found exactly, but since the sequence x k plus one or x k is coming closer and closer to this. So, for large x k; x k can be taken as a rational approximation for square root 2 and these are the approximations more often they not use in putting algorithms in the calculators and so on.

So, this is a very practical application of convergence of a sequence of real numbers. So, we have looked at sequences of real numbers we have looked at the limit of sequence of real numbers and seen various ways of analyzing convergence and the completeness property of real numbers here are some important sequences which ae useful for example.

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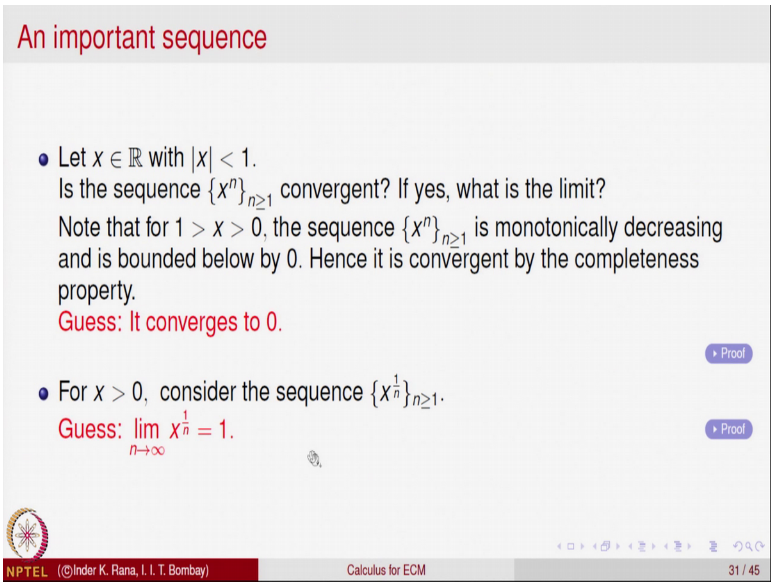

If mod of x is less than 1, x is a early real number with mod X n one then x to the power n of course, this is mod x is less than one this powers will in a some sense come closer and closer to 0 and 1 can prove that the limit exists it is a monotonically decreasing sequence bounded below and it converges and actually it converges to 0, one can prove it rigorously using the tools we have described what will not go and do it.

But we will use this because it is a useful limit for a number between said positive number between 0 and 1; if you take its powers they keep on decreasing and decrease to 0 for x greater than 1; here is another one for x non negative if you take the n th root of x that is x to the power 1 over n like square root cube root there is n through possible for real numbers if you take the sequence then intuitively one denominator n is going to infinities. So, 1 over n intuitively is going to 0. So, intuitively this limit should go to x to power 0 that is one that is not a proof that is only a way of guessing that this limit exist and this equal to 1 n can be proved rigorously there are more ah examples of this.

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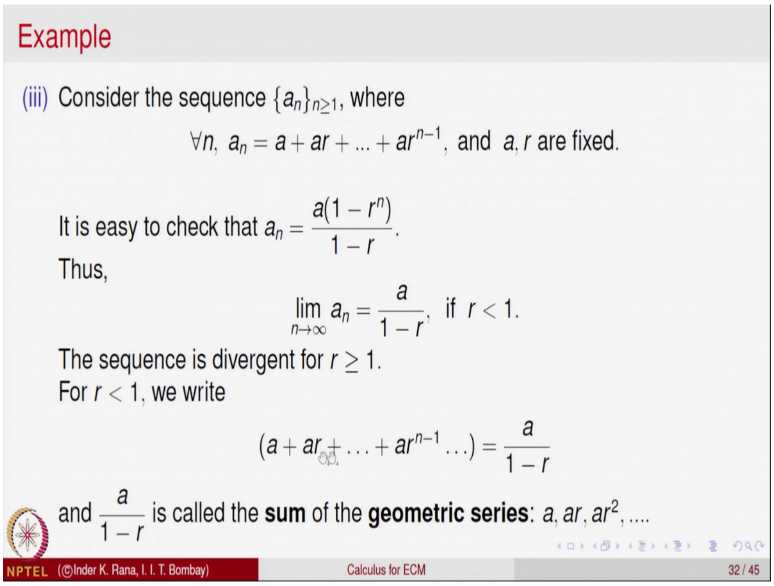

For example you have seen geometric progression in your school a n is equal to a plus a r plus a r n one. So, the geometric progression with first term a and common ratio r then we know that the n th term can be expressed as a into one plus r to the power n divided by one minus r and that is a simple fact that can be reduced now if r is less than one say r is between 0 and 1 and I take the limit this is going to go to 0. So, this says that limit of a n is a divided by one minus r right for r less than 1 if r is bigger than one this is going to keep on increasing. So, it is a unbounded sequence the fact that limit of a n is a over one minus r a r as if r is between one 0 and one that is written as that we write call this as a geometric series a plus a r plus dot, dot, dot a r power n minus 1 plus so on.

This is called a geometric series as a infinite some kind of some they are on left hand side and says is an infinite sum actually exist and is equal to a over one minus r if r is between 0 and 1. So, this is a sum of a geometrics series which is once again an application of the fact that r to the power n when r is between 0 and 1 converges to 0. So, that is another way of looking at.

So, we conclude our lecture today for sequences by saying that sequences provide a intuitive tool for analyzing something happening at different at regular intervals at tie at different tie points and helps us to analyze them eventually when n becomes large what happens to that we will see the applications obvious in the lectures to come.

Thank you.