Calculus for Economics, Commerce and Management
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Lecture – 06
Sequences, convergent sequences, bounded sequences

So, we are looking at the problem of the manufacturer trying to change a strategy of producing more chairs. So, in the first week he is able to produce 1200 chairs and he has option of increasing his production by 80 chairs.

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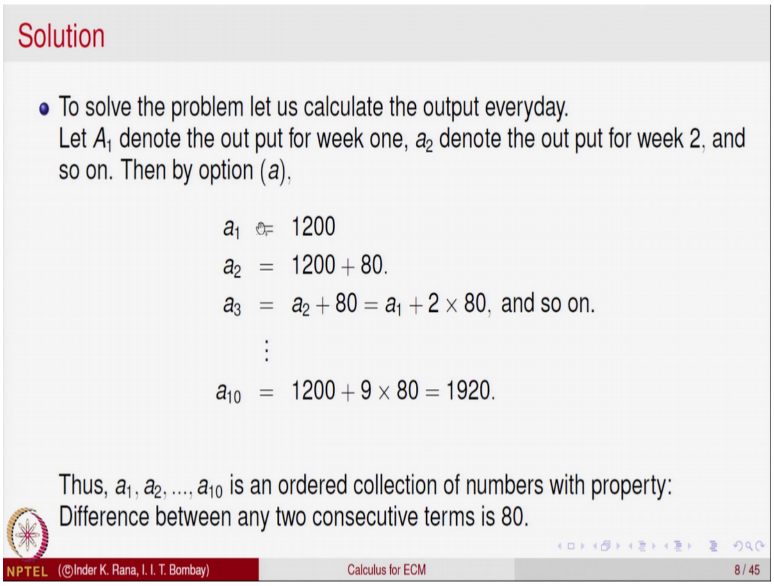

So, we wanted to generate the data and we said that if this is the first week production then the data generated second week would be 1200 plus 80, third week will be a 2; the second week plus 80 more and so on. So, tenth week it will be 1200 plus 9 times 80; that will be so much, so this gives us a ordered collection.

So, important thing is this data is a ordered collection of numbers. So, this is the first week, second week, third week, tenth week and it has a special property namely the difference between any two of them of say a 10 and a 9 is 80 and the previous one is a 3 minus a 2 is 80 and so on. So, this is the data generated and this is the ordered collection of numbers. You see; if a 10 was equal to this, what will be a 11? So, it will be 1200 plus 10 times 80 and so on. So, that one motivates one to guess that; if I looking at the n th week n could be anything 10, 15, 20 and so on. So, the n th week production will be a 1 plus n minus 1 times d; what is d? That is increase per week that we are making; in our case it is 80.

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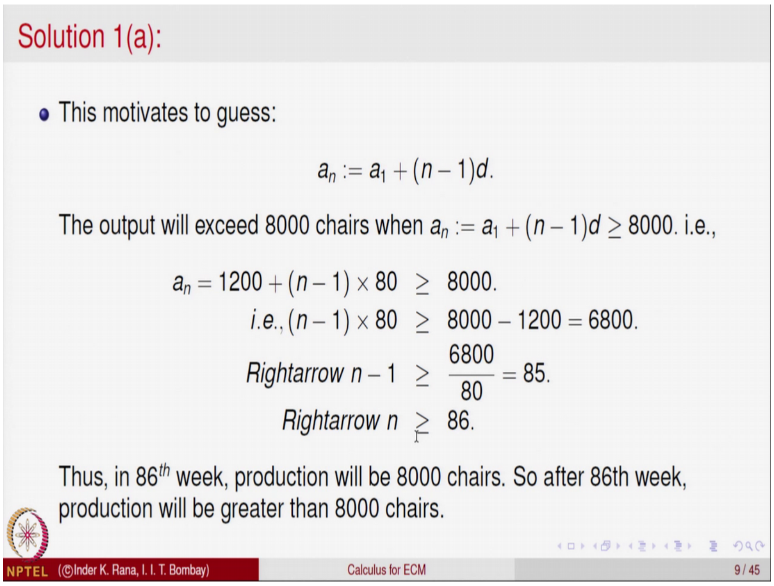

So; that means, if we want to look at the output to exceed 8000 chairs, then this number a n should be bigger than 8000. So; that means, a n is equal to 1200 plus n minus 1 into 80 that should be bigger than 8000. And that means, we simplify this equation that is n minus 1 times 80 is bigger than 6800.

So, this means this is this implies; that means, n minus 1 should be 6800 divided by 80 that is 85. So; that means, n should be bigger than or equal to 86; so after 86 week, the number of chairs produce will be 8000. So, this is how a ordered data is united and a problem is solved; we can try to solve that the problems similarly for the option b.

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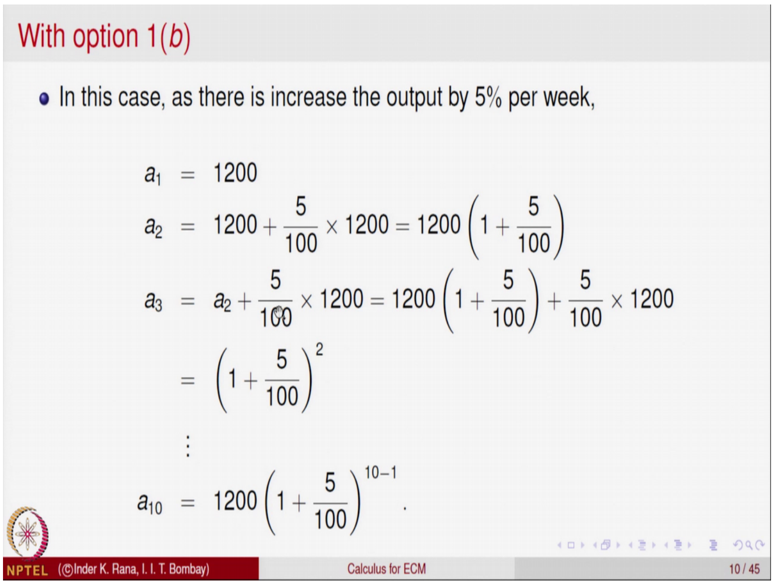

So, in the option b a 1; the first week production was 1200 and there is increase of 5 percent per week; that means whatever amount is produced in a week, next week it is increased by a 5 percent. So, what will be a 2? A 2 will be 1200 plus 5 percent of 1200. So, that will be this; so this is will be the amount that is being that will be produced next week.

And what will be a 3? It will be a 2 plus 500; 5 percent of 1200; so that will be this. So that means, it is that there is a type over here, it should be 1200 multiplied by 1 plus 5 over 100. So, if you continue at the tenth week; this will be the amount of production that we produced. So, again by the given formula that is a 5 percent increase; we are able to find out what is the tenth week?

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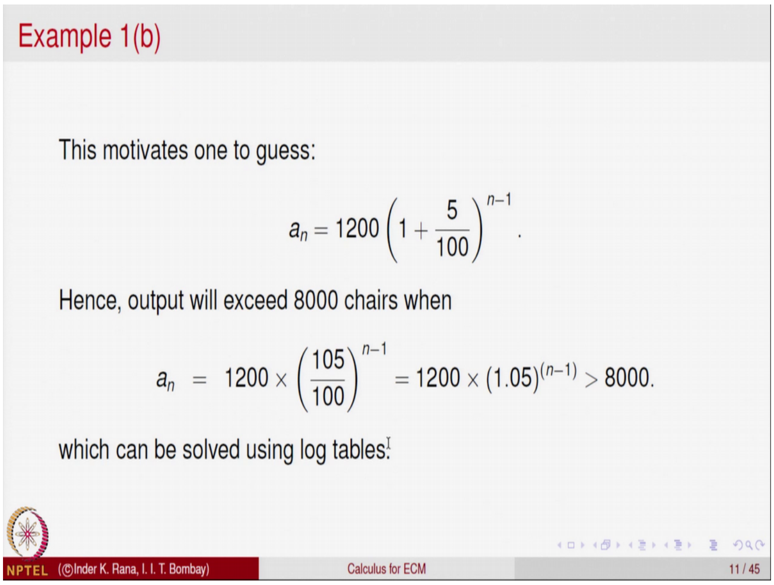

So, this gives us that a formula probably that; after n weeks, the production will be this much. So, if you want the production to increase, you want to find out when will the; if there is a 5 percent increase every week, when will the production go beyond 8000; then we have to say that a n which is equal to this; is bigger than 8000. So, this equation now one has to solve and to solve that one use log tables, so that can be solved.

But the important thing is in this problem, you are first able to generate an ordered data. A data of numbers a 1, a 2, a 3; where there is a order this is the first one, second one and so on and then able to analyze the data.

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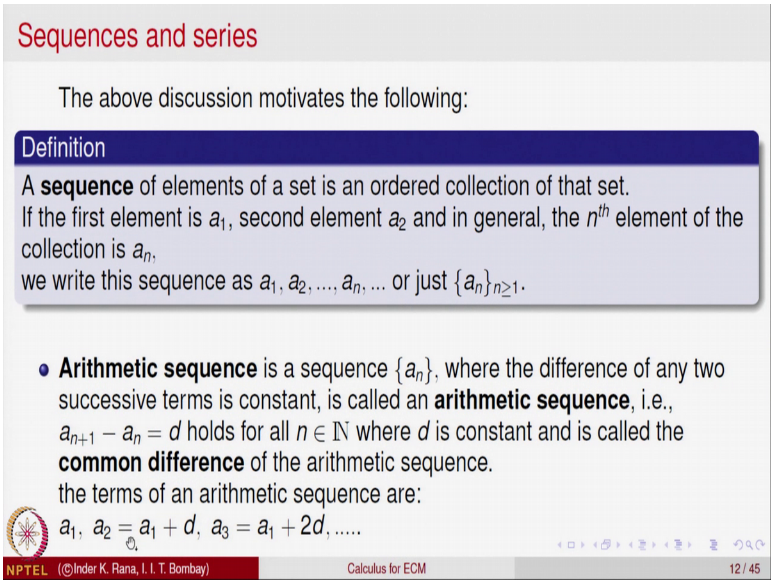

So, this motivatesthe following this definition of a sequence; a sequence of elements of a set is an ordered collection of that set.

So, ordered means there is order you pick up, an element call is the first; call it the next pick up another one; call it the second one. So, on the first element is a 1; let us say a 1, second element you can call it as a 2 and the general n th element as a n. So, this gives us a ordered collection a 1, a 2, a 3 and so on, which is normally written as a curly bracket; a n; the n th term, n bigger than or equal to 1.

So, a sequence is an ordered collection of numbers and is denoted by this symbol a n. So, we say sequence is a arithmetic sequence, if the difference between any two consecutive terms is a constant; that means, a n plus 1 minus a n is a same constant d, then we say this sequence is a arithmetic sequence. And that is what happened in our previous case of the option a and there was increase of 80 chairs per week, so this d is called the common difference.

So, the sequence looks like the first term is a 1; the second is a 2 which is a 1 plus d, the third time is a 2 plus d; that will be a 1 plus 3 d and so on.

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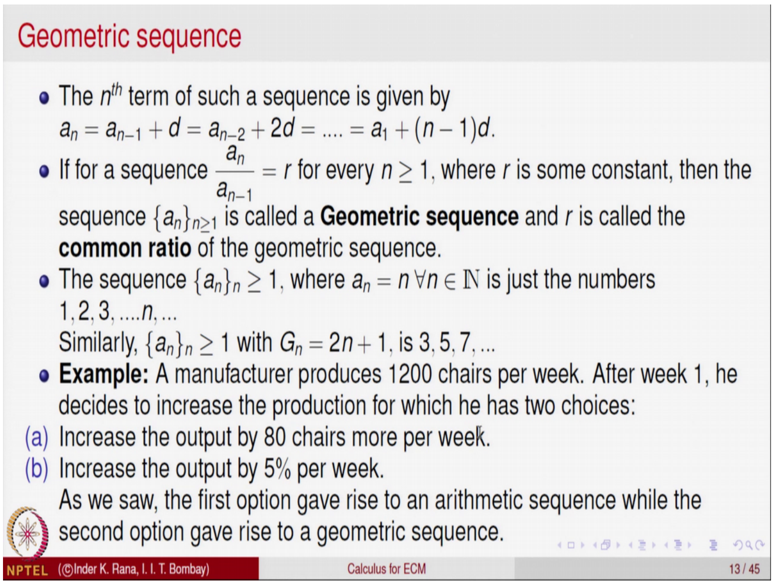

We say the sequence the n th term of the sequence will be a 1 plus and minus 1 d; in the arithmetic sequence. We say a sequence is geometric sequence; if the ratio of the two consecutive terms is same. So, a n divided by a n minus 1 should be equal to r or n bigger than or equal to 1; there is no a 0. So, better way of writing that will be n bigger than or equal to 2, is a constant; so, this is called a geometric sequence.

So, a n is equal to for example, if you take the sequence of a numbers; a n is equal to n that is arithmetic sequence. And similarly, if I take the sequence say G n equal to 2 n plus 1; odd numbers that is also a arithmetic sequence. So that example of first week, second week; two choices that give us the notion of arithmetic and geometric sequences.

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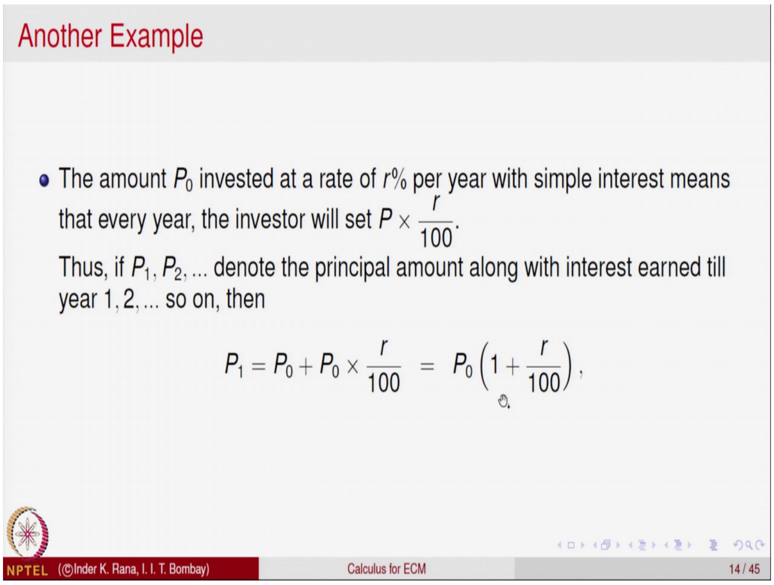

Let us look at another example which normally students of economics, commerce encounter; this is a interest. Suppose a amount P 0 is invested at a rate of r percent per year with the simple interest means every year, the investor will give; will instead of this a type over here instead of set, it should be get; the investor will get P multiplied by r over 100 amount.

So, if P 1, P 2, P 3 denotes the principle amount along with the interest being earned; then P 1 will be P 0; the starting point plus P 0 divided by r by 100; so this will be the amount; what will be P 2?

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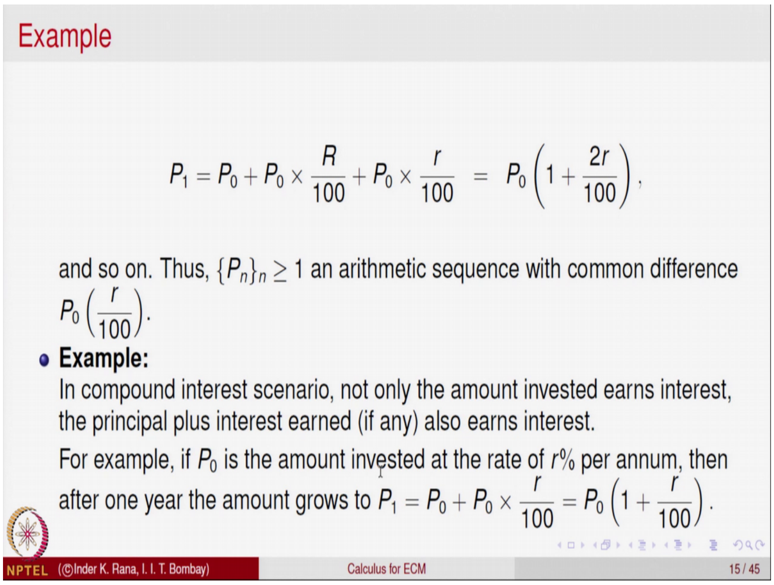

So, P 2 will be P 0 plus right that is P 1; so P 2 and so on. So, this gives us a arithmetic sequence with common differences P 0 divided by r plus 1. If the amount is calculated is the compound interest then the scenario changes. So, r percent per annum; that means, the interest also get interest on that. So, P 1 will be equal to P 0 plus P 0 by r; that is P 0 multiplied by 1 plus r by 100.

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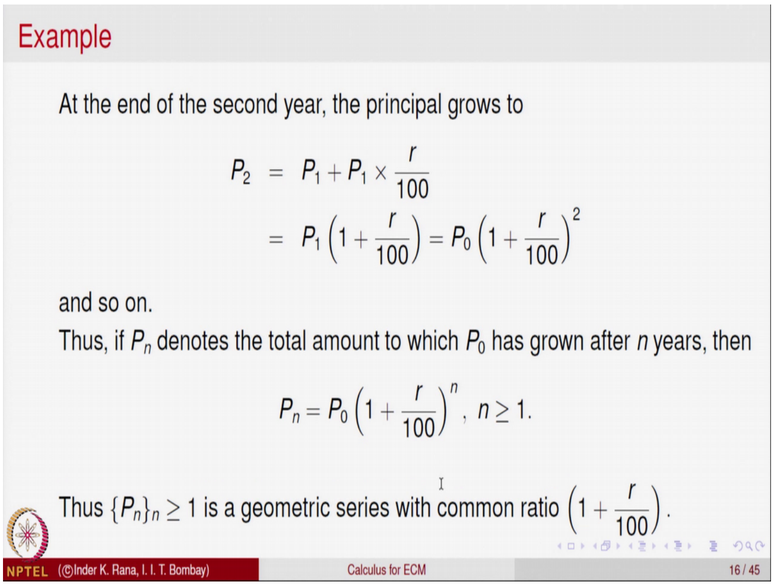

P 2 will be P 1 plus whatever we have earned after one year, we will get interest on the whole amount. So, it is P 1 multiplied by r over 100; so that will be; if I put the value of P 1 that is P 0 into r by 100; so, this is P 0 multiplied by this and so on.

So, at P n after n years at the rate of r percent interest, this will be the principle would have grown to this much. So, again this generates a sequence an ordered collection of numbers which in fact, is a geometric sequence with common ratio as 1 plus r over 100. So, the principle if the simple interest that gives us a sequence of numbers amount P n is arithmetic progression, if it is compound interest; it is a geometric progression.

And in each case one would like to know what happens to the terms of the sequence a n as n changes n becomes large and large. Of course, if it is a interest; the principle will keep on growing and similarly in the example of production of chairs also, it will keep on increasing, but in a general sequence that need not happen always.

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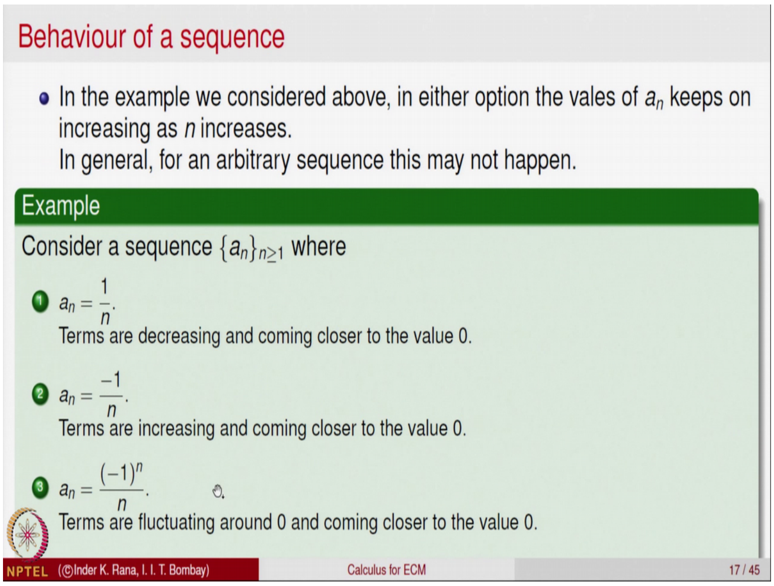

So, let us look at some example; for example, consider the sequence a n equal to 1 over n, so let us write the sequence.

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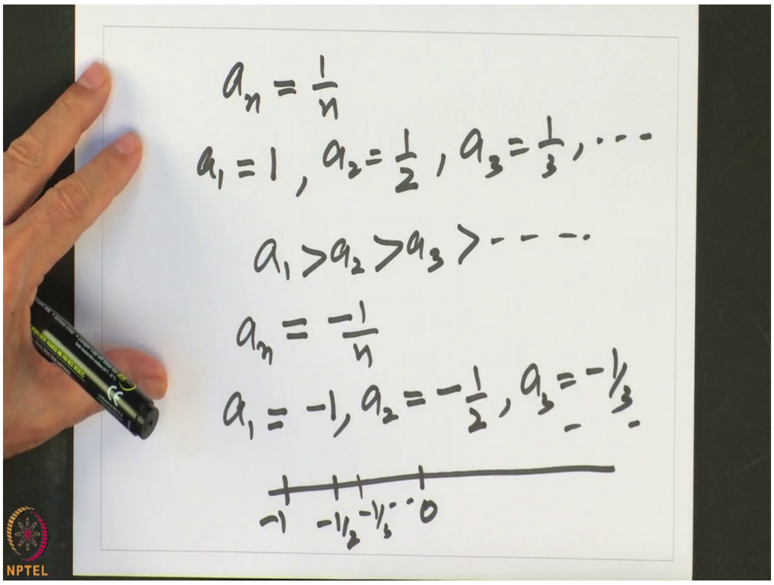

So, a n is equal to 1 over n; so what is a 1? So, that is equal to 1, what is a 2? That is 1 by 2, what is a 3? That is 1 by 3 and so on. So, it is quite clear that a 1 is bigger than a 2, a 2 is bigger than a 3 is bigger than so on. So, it its quite clear that this sequence 1 over n; is a decreasing sequence the values are decreasing; first value is 1, second is 1 by 2, third is 1 by 3. So, values are decreasing; it is coming closer and closer to the value 0, let us look at the sequence; a n equal to minus 1 over n.

So, what will be the terms of the sequence? When a n is equal to minus 1 by n; so, what is a 1? a 1 will be equal to minus 1. What is a 2? a 2 is minus 1 by 2, what is a 3? Equal to minus 1 by 2 and so on. So, if I try to plot it; it is 0, here is minus 1, here is minus 1 by 2, here is minus 1 by 3 and so on. So, it is quite clear that the sequence is an increasing sequence; as you move the terms are increasing, the values are increasing and coming closer to 0. Let us look at the sequence a n minus 1 to the power n divided by n; what are the value of this sequence?

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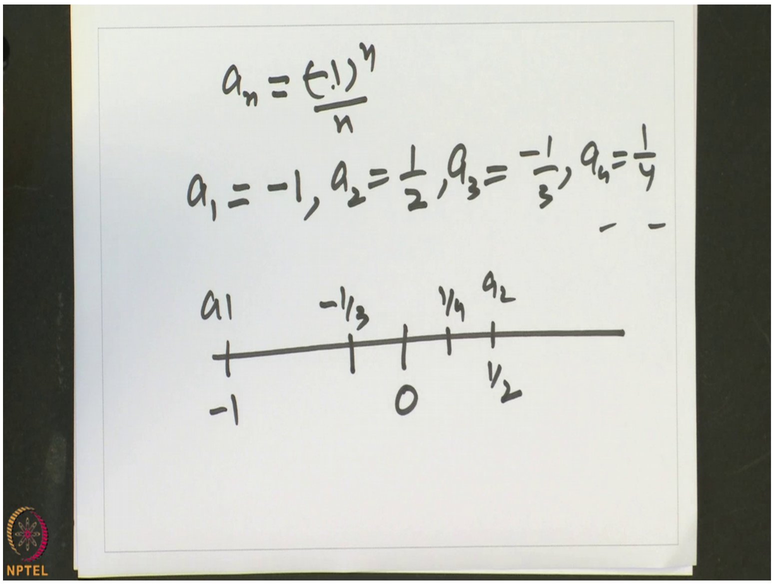

So, let us write that again; so, a n is equal to minus 1 to the power n divided by n. So, what is a 1? n equal to 1, so this is minus 1; what is a 2? Minus 1 square; so, that is 1 by 2; what is a 3? So, minus 1 to the power 3, that is minus 1 by 3; a 4 will be equal to 1 by 4 and so on. So, let us look at; so this is the point 0 and this is a 1 is minus 1, what is a 2? a 2 is 1 by 2, what is a 3? Is minus 1 by 3; what is a 4? Is 1 by 4.

So, again we observe that this sequence is neither increasing nor decreasing; it is fluctuating some alternatively it is becoming positive and negative and, but still it is coming closer to 0. Let us look at one more example of a sequence a n where a n is equal to minus 1 to the power n.

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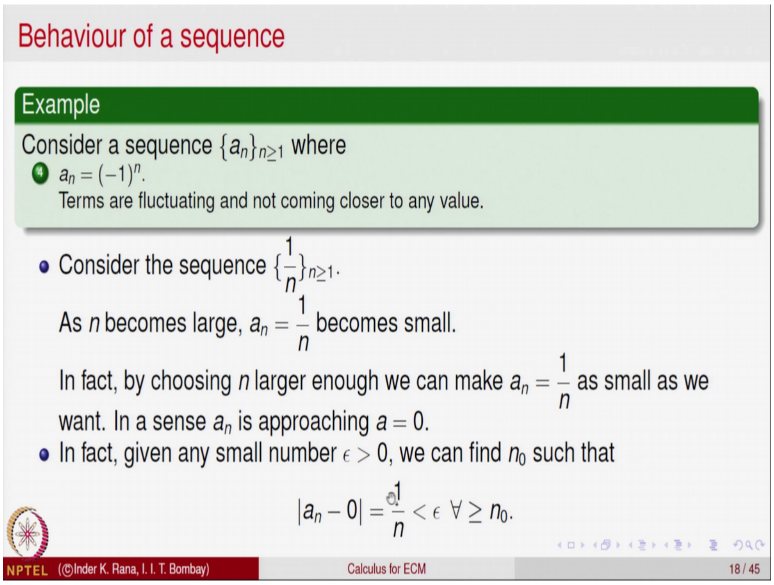

So, what is happening to this sequence?

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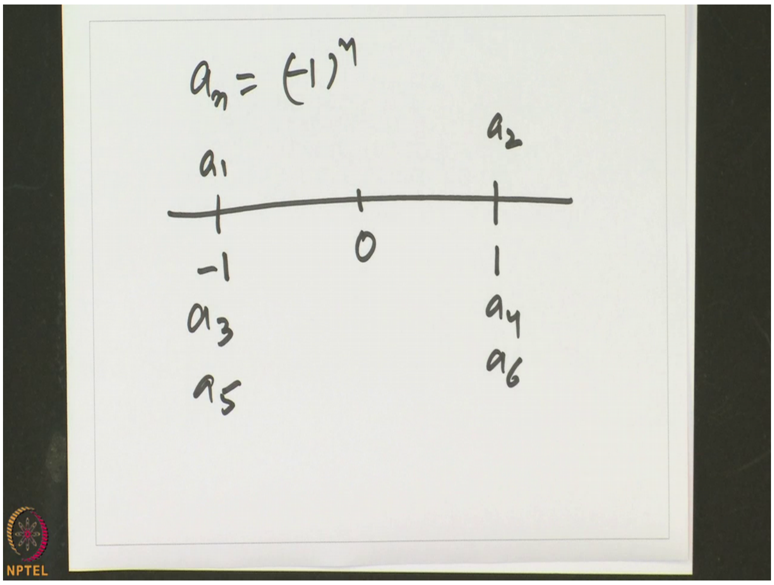

So, a n is equal to minus 1 to the power n; so if I try to plot it, it is 0 here so, n equal to 1; I will get a 1 which is equal to minus 1. What is a 2? a 2 minus 1 to the power 2; that is 1 that is a 2; a 3 will be equal to again minus 1, a 4 will be equal to again 1, a 5; again here, a 6 here. So, all the odd terms will be at minus 1; all the even terms will be equal to 1. So, it fluctuates minus 1, plus 1, minus 1, plus 1, minus 1, plus 1; so, this sequence is neither increasing nor decreasing, it fluctuates at the values minus 1 and n plus 1; it does not come closer to any value. So, a behavior of a sequence depends on what is a sequence.

So, let us consider a sequence called 1 over n; n bigger than or equal to 0. As we observed, terms of the sequence becomes smaller and smaller; first term is 1; second is equal to 1 over 2, third is 1 over 3 and smaller and smaller. It is coming closer and closer to 0 in fact, it is approaching; we can say it is a approaching 0; it comes close to 0 as close as we wanted. The distance of n from 0; I can make it as small as I want, so this distance is actually equal to 1 over n and I can make it less than epsilon by choosing n large enough.

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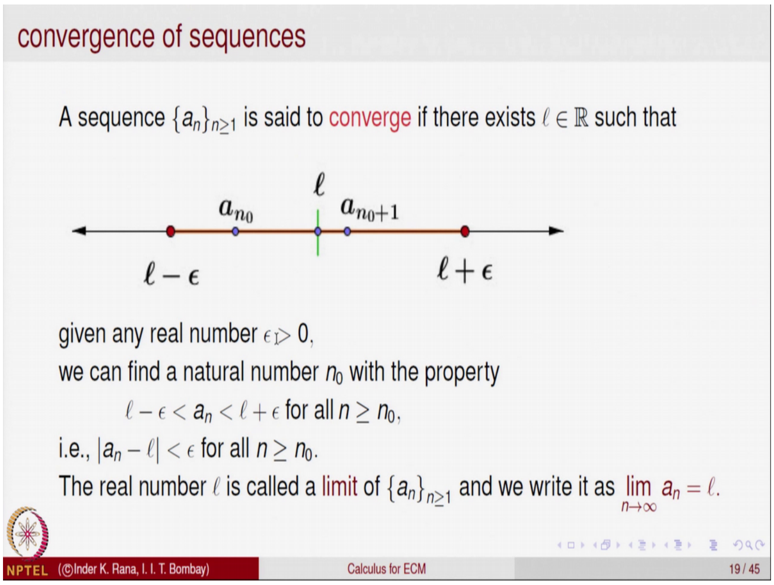

Let us formulize this as a definition, a sequence a n is said to converge to a value l; in which is a number. If there exists a number l, we say a sequence converge is if there exists a number l.

So, here is a number l which has the property that given any number epsilon; this is a Greek letter epsilon, given any number epsilon bigger than 0. If I look at l minus epsilon and l plus epsilon then what should happen? All the terms of the sequence a n should lie between l minus epsilon n; l plus epsilon for all n bigger than or equal to 0. So, I can write this also as the distance between a n and l is less than epsilon for all n bigger than 0.

Let us try to understand this; this is a quite a subtle point. See we want to say that the sequence is approaching this value l. So, intuitively what it means that a n should come closer and closer to l, so how close? So, I will specify beforehand that the margin of error can be at the most this distance, so this is specified by a number epsilon bigger than 0.

So, it says that I should be able to find a interval around l of length 2 epsilon. So, if I look at l minus epsilon and l plus epsilon; then all the terms of the sequence are inside this interval for n bigger than or equal to 0. So, that another way of saying that would be that given epsilon bigger than 0; that is a error I can make. You can think of a n minus epsilon is a n is the value of the sequence, l is the expected value of the sequence; this is the error I am making a n minus epsilon mod is the error I am making.

This is a distance that is how far away it is from l? That is error; that error should be less than epsilon, for which n? For all n; that is the important thing that from some stage onwards, everything should fall in between; that means, possibly a 1, a 2 up to a n naught minus 1, these terms are outside somewhere, but after that n naught stage; the tale of the sequence, you can think it as a tale.

That a 1, a 2, a n naught minus 1; a n naught onwards that is a tale of the sequence that should lie in between this interval. Then we will say that the sequence a n converge to 12; that is a limit. So, we say that a n converges to l and write it as limit n going to infinity a n equal to l. So, this is only a symbolic way of writing this thing that given epsilon bigger than 0, there exist a stage n naught such that for all elements of the sequence bigger than or equal to n naught, they lie at a distance epsilon from a n.

So, convergence of a sequence means a tale of the sequence comes closer to l at the most a distance 2 epsilon. And that epsilon is; I will prescribe stage you have to find if you want to show limit of a n is equal to l. So, this is the convergence of a sequence limit a n is equal to l.

So, once again let me say we say that a sequence n converges; if there is a number l such that for every epsilon bigger than 0, there exist n naught a natural number such that; a n minus l is less than epsilon for n bigger than n naught.

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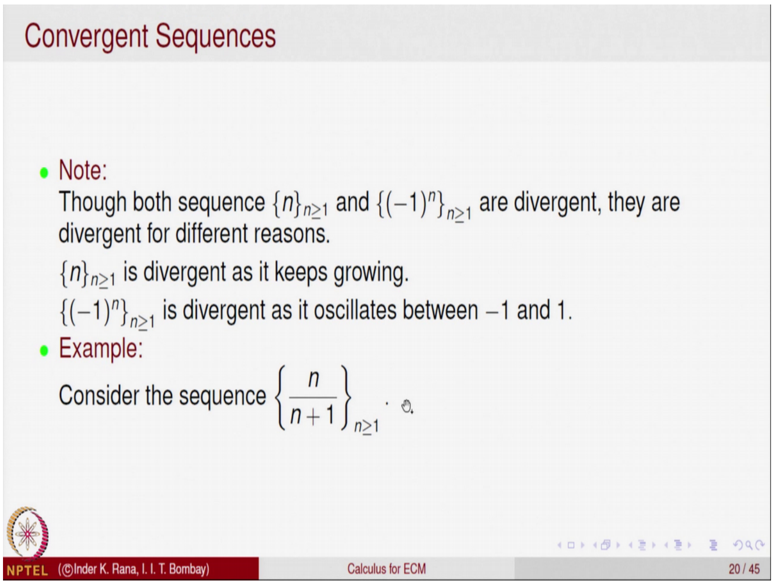

Let us look at some examples to understand this; you note that the sequence we had 1, 2, 3, 4 so on and the sequence minus 1 to the power n; both are not convergent. Let me just say that when a sequence is not convergent, we said is divergent. I have not said that, so let us say when a sequence is not convergent; we will say it is a divergent sequence. So, the sequence n is not convergent why? Because first term is 1, second term is 2, third term is 3. So, it keeps on increasing it is not coming closer to any value; so it is divergence sequence.

Minus 1 to the power n; this is again divergent because again it is not coming closer to any value; it is fluctuating between minus 1 and plus 1. One can write a proof of this; that this is a divergent sequence not convergent; just now I said epsilon. So, when it will be slightly quite interesting and logical to write a proof of the fact that minus 1 to the power n bigger than is a divergent sequence, not convergent. Try to write a proof yourself, if not try to read that web course on calculus; that I have said, a complete proof is given there, but intuitively this is it fluctuates between minus 1 and plus 1 and that this is not convergent; it is a property of real numbers intuitively it keeps on increasing; it is not coming closer and this is divergent.

So, these are different reasons for the sequence being not convergent. Let us look at the sequence n over n plus 1; n bigger than or equal to 1. We want to understand whether this sequence converges or not. The one way of analyzing this would be; write a few terms of the sequence and see what is happening. What is the first term? 1 over 1 plus2. So, let us just write a first few terms of this sequence; what is happening.

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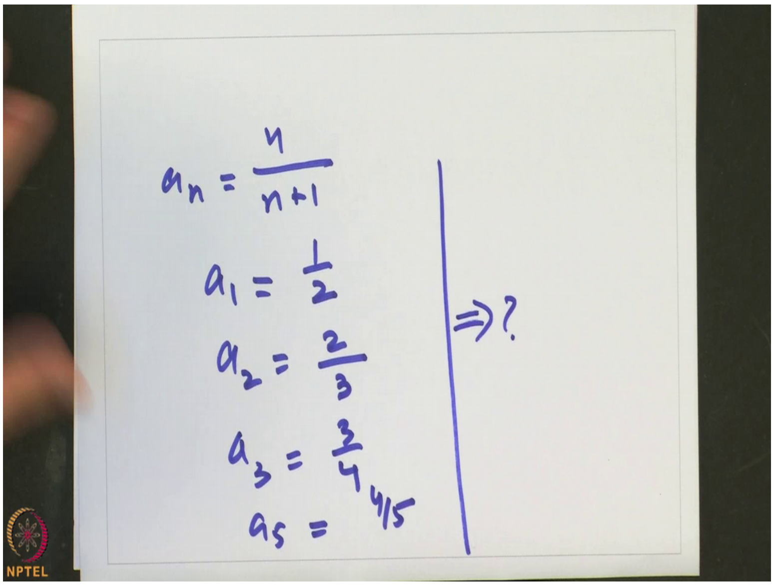

So, n divided by n plus 1; so that is a n. So, what is a 1? That is 1 by 2; what is a 2? That is 2 by 3; what is a 3? 3 by 4, a 5; 4 by 5. So, from here can you guess what is happening to the sequence?

So, first term is 1 by 2; second is bigger than 1 by 2; it is 2 by 3, third is bigger; that is 3 by 4, this is bigger. So, it seems it is going further and further increasing, but it is n over n plus 1; it cannot go beyond 1, this is always a fraction less than 1. So, it is increasing towards 1; so, guess is probably it is convergent to 1.

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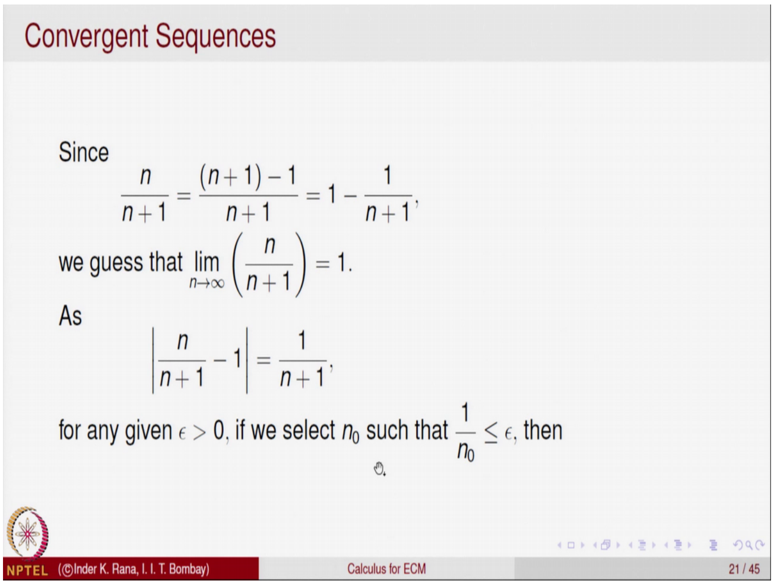

So, let is analyze this; so look at n over n plus 1; I can factorize it by this way. So, it is 1 minus 1 over n; so this part intuitively becomes 0, as n becomes larger and larger. So, this will becoming smaller and smaller; so this is becoming smaller. So, intuitively this should go to 1; so one guess is that the limit should be equal to 1. But there will have to say why this is happening because this is not a sequence, it is 1 minus 1 over n; it is something else; this is subtracted from here.

So, to justify this kind of a thing let us look at the distance between; so this is how we guess the limit should be equal to 1 to prove it. Let us look at a n minus 1; so, we have guess l is the limit, so a n minus l, this is this quantity it is equal to 1 over n plus 1 and I can make 1 over n plus 1; small by choosing n large enough. So that means, I can say that this can be made less than epsilon for any given epsilon.

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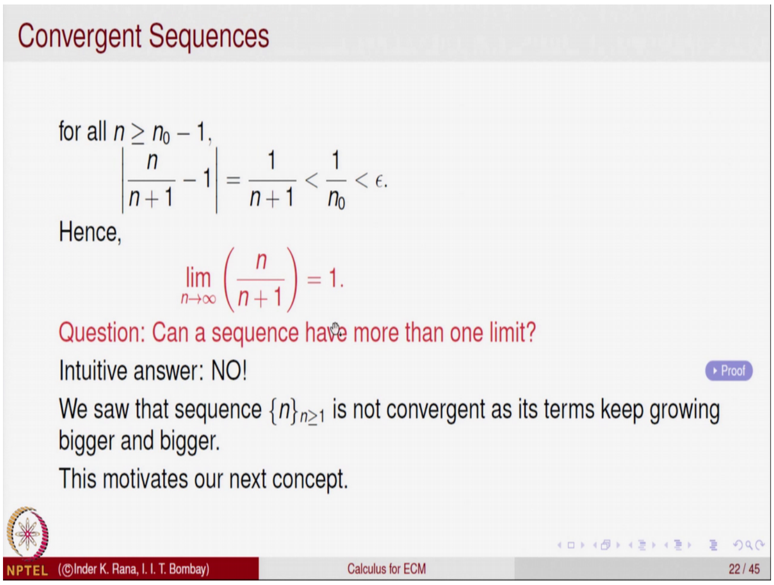

So, this is how rigorously you will prove that this limit is equal to 1; at this stage one can ask a question can a sequence have more than one limit? One can prove that if a sequence has a limit it is unique; there cannot be, intuitively its quite clear that tale of a sequence cannot come closer to two different values; symmetrically it is seems impossible.

But one can prove it rigorously will not go precisely into that and we also saw that the sequence n is not convergent because it keeps on increasing.

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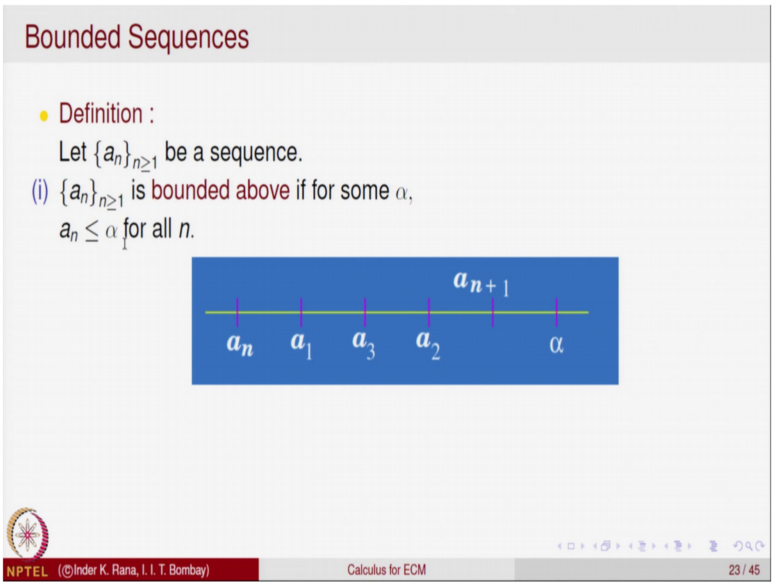

So, this motivates our next definition namely; we say a sequence a n is bounded above; if there is a number alpha such that all the terms are less than or equal to alpha; that means, the terms of the sequence cannot go on the right side of the alpha, there is a barrier, so then we say it is bounded above.

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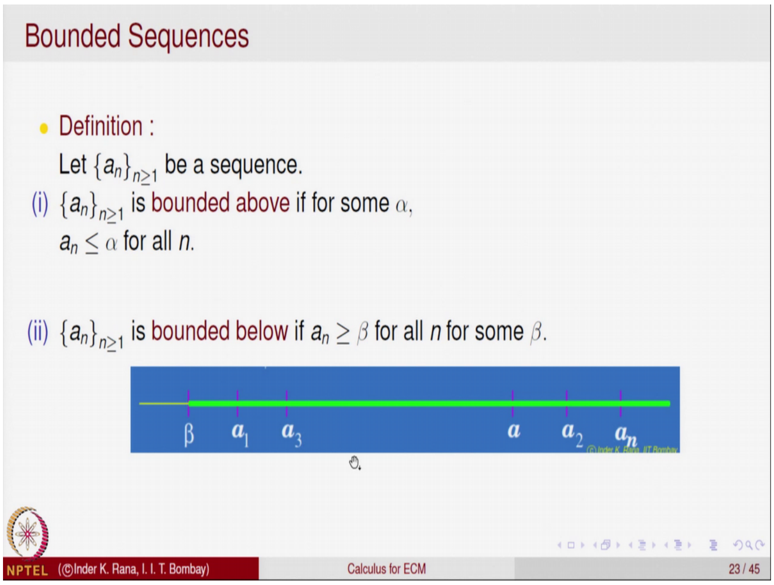

And similarly we will say a sequence is bounded below; if there is a number beta so that all the terms of the sequence are on the right side of it. So, a n is bigger than or equal to beta for all n; then we say a n is bounded below.

So, bounded below is there is a barrier on the left; bounded above is barrier on the right. So, bounded below; bounded above if a sequence is both bounded above and below, we say it is a bounded sequence. So, bounded means there is a barrier below; there is a barrier above then we say it is a bounded sequence.

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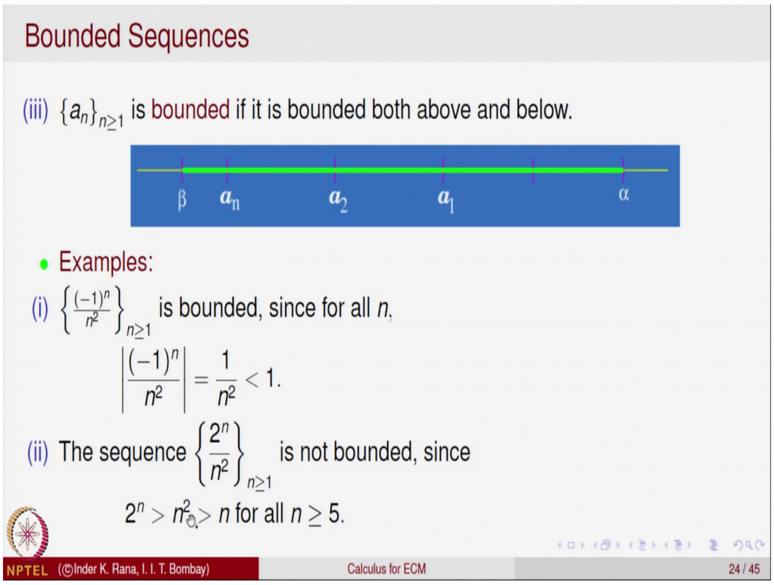

So, examples the sequence is bounded because for all n; absolute value of this is 1 over n square less than 1. So, all the terms of the sequence will lie between minus 1 and 1; so, there is a bound below minus 1, there is a bound below above 1. But boundedness at present does not say anything about the convergence of a sequence; keep that in mind.

So, this sequence 2 to the power n divided by n square is not bounded that one can show with slight mount of work; namely 2 to the power n becomes; grows much faster than n square. So, one has to write something; one has to do some more work, this is not bounded. Those are few who are more in choosiest; try to prove 2 to the power n is bigger than n square is bigger than n for all n bigger than or equal to 5; that means, 2 to the power n will take over n square and it will take over even n for n bigger than or equal to 5; so, it will keep on growing it will never converge.

So, we will stop here for today that having recalled that we are defined the notion of a sequence. Sequence is a ordered collection of numbers which essentially come from a data which is taken a different time point you can think of. And then a sequence may behave differently depending on what is a sequence, it may come closer and closer to a value, it may keep on increasing, it may keep on decreasing and so on.

So, we define the notion of convergence of a sequence; we say a sequence a n converges, if there is a number l such that the distance between a n and l becomes small less than epsilon for any prescribed epsilon from some stage and not onwards. That stage n naught may depend on epsilon, but what should happen is absolute value of a n minus epsilon should be less than and epsilon; for all terms of the sequence n bigger than or equal to 0; that means, the tail of the sequence should come in a neighborhood of l of length epsilon.

So, convergence of a sequence depends on the tail of the sequence. So, we will continue our study of sequences in the next lecture.

Thank you.