

Calculus for Economics, Commerce and Management
Prof. Inder K. Rana
Department of Mathematics
Indian Institute of Technology, Bombay

Lecture – 06
Sequences, convergent sequences, bounded sequences

So, we are looking at the problem of the manufacturer trying to change a strategy of producing more chairs. So, in the first week he is able to produce 1200 chairs and he has option of increasing his production by 80 chairs.


(Refer Slide Time: 00:44)

Solution

- To solve the problem let us calculate the output everyday.
Let A_1 denote the out put for week one, a_2 denote the out put for week 2, and so on. Then by option (a),

$$\begin{aligned}a_1 &= 1200 \\a_2 &= 1200 + 80. \\a_3 &= a_2 + 80 = a_1 + 2 \times 80, \text{ and so on.} \\&\vdots \\a_{10} &= 1200 + 9 \times 80 = 1920.\end{aligned}$$

Thus, a_1, a_2, \dots, a_{10} is an ordered collection of numbers with property:
Difference between any two consecutive terms is 80.

 NPTEL (©Inder K. Rana, I. I. T. Bombay)

Calculus for ECM

8 / 45

So, we wanted to generate the data and we said that if this is the first week production then the data generated second week would be 1200 plus 80, third week will be a 2; the second week plus 80 more and so on. So, tenth week it will be 1200 plus 9 times 80; that will be so much, so this gives us a ordered collection.

So, important thing is this data is a ordered collection of numbers. So, this is the first week, second week, third week, tenth week and it has a special property namely the difference between any two of them of say a 10 and a 9 is 80 and the previous one is a 3 minus a 2 is 80 and so on. So, this is the data generated and this is the ordered collection of numbers. You see; if a 10 was equal to this, what will be a 11? So, it will be 1200 plus 10 times 80 and so on. So, that one motivates one to guess that; if I looking at the n th week n could be anything 10, 15, 20 and so on. So, the n th week production will be a 1

plus n minus 1 times d ; what is d ? That is increase per week that we are making; in our case it is 80.

(Refer Slide Time: 02:09)


Solution 1(a):

- This motivates to guess:
$$a_n := a_1 + (n - 1)d.$$

The output will exceed 8000 chairs when $a_n := a_1 + (n - 1)d \geq 8000$. i.e.,

$$a_n = 1200 + (n - 1) \times 80 \geq 8000.$$
$$\text{i.e., } (n - 1) \times 80 \geq 8000 - 1200 = 6800.$$
$$\text{Rightarrow } n - 1 \geq \frac{6800}{80} = 85.$$
$$\text{Rightarrow } n \geq 86.$$

Thus, in 86th week, production will be 8000 chairs. So after 86th week, production will be greater than 8000 chairs.



NPTEL (© Inder K. Rana, I. I. T. Bombay) Calculus for ECM 9 / 45

So; that means, if we want to look at the output to exceed 8000 chairs, then this number n should be bigger than 8000. So; that means, n is equal to 1200 plus n minus 1 into 80 that should be bigger than 8000. And that means, we simplify this equation that is n minus 1 times 80 is bigger than 6800.

So, this means this implies; that means, n minus 1 should be 6800 divided by 80 that is 85. So; that means, n should be bigger than or equal to 86; so after 86 week, the number of chairs produce will be 8000. So, this is how a ordered data is united and a problem is solved; we can try to solve that the problems similarly for the option b.

(Refer Slide Time: 03:10)

With option 1(b)

- In this case, as there is increase the output by 5% per week,

$$a_1 = 1200$$

$$a_2 = 1200 + \frac{5}{100} \times 1200 = 1200 \left(1 + \frac{5}{100}\right)$$

$$a_3 = a_2 + \frac{5}{100} \times 1200 = 1200 \left(1 + \frac{5}{100}\right) + \frac{5}{100} \times 1200$$
$$= \left(1 + \frac{5}{100}\right)^2$$

\vdots

$$a_{10} = 1200 \left(1 + \frac{5}{100}\right)^{10-1}$$



So, in the option b a 1; the first week production was 1200 and there is increase of 5 percent per week; that means whatever amount is produced in a week, next week it is increased by a 5 percent. So, what will be a 2? A 2 will be 1200 plus 5 percent of 1200. So, that will be this; so this is will be the amount that is being that will be produced next week.

And what will be a 3? It will be a 2 plus 500; 5 percent of 1200; so that will be this. So that means, it is that there is a type over here, it should be 1200 multiplied by 1 plus 5 over 100. So, if you continue at the tenth week; this will be the amount of production that we produced. So, again by the given formula that is a 5 percent increase; we are able to find out what is the tenth week?

(Refer Slide Time: 04:13)

Example 1(b)

This motivates one to guess:

$$a_n = 1200 \left(1 + \frac{5}{100}\right)^{n-1}.$$

Hence, output will exceed 8000 chairs when

$$a_n = 1200 \times \left(\frac{105}{100}\right)^{n-1} = 1200 \times (1.05)^{(n-1)} > 8000.$$

which can be solved using log tables:



So, this gives us that a formula probably that; after n weeks, the production will be this much. So, if you want the production to increase, you want to find out when will the; if there is a 5 percent increase every week, when will the production go beyond 8000; then we have to say that a n which is equal to this; is bigger than 8000. So, this equation now one has to solve and to solve that one use log tables, so that can be solved.

But the important thing is in this problem, you are first able to generate an ordered data. A data of numbers a_1, a_2, a_3 ; where there is a order this is the first one, second one and so on and then able to analyze the data.

(Refer Slide Time: 05:05)


Sequences and series

The above discussion motivates the following:

Definition

A **sequence** of elements of a set is an ordered collection of that set. If the first element is a_1 , second element a_2 and in general, the n^{th} element of the collection is a_n , we write this sequence as $a_1, a_2, \dots, a_n, \dots$ or just $\{a_n\}_{n \geq 1}$.

- **Arithmetic sequence** is a sequence $\{a_n\}$, where the difference of any two successive terms is constant, is called an **arithmetic sequence**, i.e., $a_{n+1} - a_n = d$ holds for all $n \in \mathbb{N}$ where d is constant and is called the **common difference** of the arithmetic sequence. the terms of an arithmetic sequence are:
 $a_1, a_2 = a_1 + d, a_3 = a_1 + 2d, \dots$

 NPTEL (© Inder K. Rana, I. I. T. Bombay) Calculus for ECM 12 / 45

So, this motivates the following this definition of a sequence; a sequence of elements of a set is an ordered collection of that set.

So, ordered means there is order you pick up, an element call is the first; call it the next pick up another one; call it the second one. So, on the first element is a 1; let us say a 1, second element you can call it as a 2 and the general n^{th} element as a n . So, this gives us a ordered collection a 1, a 2, a 3 and so on, which is normally written as a curly bracket; a n ; the n^{th} term, n bigger than or equal to 1.

So, a sequence is an ordered collection of numbers and is denoted by this symbol a_n . So, we say sequence is a arithmetic sequence, if the difference between any two consecutive terms is a constant; that means, $a_{n+1} - a_n$ is a same constant d , then we say this sequence is a arithmetic sequence. And that is what happened in our previous case of the option a and there was increase of 80 chairs per week, so this d is called the common difference.


So, the sequence looks like the first term is a 1; the second is a 2 which is a 1 plus d , the third time is a 2 plus d ; that will be a 1 plus 3 d and so on.

(Refer Slide Time: 06:41)

Geometric sequence

- The n^{th} term of such a sequence is given by
$$a_n = a_{n-1} + d = a_{n-2} + 2d = \dots = a_1 + (n-1)d.$$
- If for a sequence $\frac{a_n}{a_{n-1}} = r$ for every $n \geq 1$, where r is some constant, then the sequence $\{a_n\}_{n \geq 1}$ is called a **Geometric sequence** and r is called the **common ratio** of the geometric sequence.
- The sequence $\{a_n\}_{n \geq 1}$, where $a_n = n \forall n \in \mathbb{N}$ is just the numbers 1, 2, 3, ..., n , ...
Similarly, $\{a_n\}_{n \geq 1}$ with $G_n = 2n + 1$, is 3, 5, 7, ...
- **Example:** A manufacturer produces 1200 chairs per week. After week 1, he decides to increase the production for which he has two choices:
 - (a) Increase the output by 80 chairs more per week.
 - (b) Increase the output by 5% per week.

As we saw, the first option gave rise to an arithmetic sequence while the second option gave rise to a geometric sequence.

 NPTEL (© Inder K. Rana, I. I. T. Bombay) Calculus for ECM 13 / 45

We say the sequence the n^{th} term of the sequence will be $a_1 + (n-1)d$; in the arithmetic sequence. We say a sequence is geometric sequence; if the ratio of the two consecutive terms is same. So, a_n divided by a_{n-1} should be equal to r or n bigger than or equal to 1; there is no a_0 . So, better way of writing that will be n bigger than or equal to 2, is a constant; so, this is called a geometric sequence.


So, a_n is equal to for example, if you take the sequence of a_n is equal to n that is arithmetic sequence. And similarly, if I take the sequence say G_n equal to $2n + 1$; odd numbers that is also a arithmetic sequence. So that example of first week, second week; two choices that give us the notion of arithmetic and geometric sequences.

(Refer Slide Time: 07:56)

Another Example

- The amount P_0 invested at a rate of $r\%$ per year with simple interest means that every year, the investor will set $P \times \frac{r}{100}$.

Thus, if P_1, P_2, \dots denote the principal amount along with interest earned till year 1, 2, ... so on, then

$$P_1 = P_0 + P_0 \times \frac{r}{100} = P_0 \left(1 + \frac{r}{100} \right),$$


NPTEL (© Inder K. Rana, I. I. T. Bombay) Calculus for ECM 14 / 45

Let us look at another example which normally students of economics, commerce encounter; this is a interest. Suppose a amount P_0 is invested at a rate of r percent per year with the simple interest means every year, the investor will give; will instead of this a type over here instead of set, it should be get; the investor will get P multiplied by r over 100 amount.

So, if P_1, P_2, P_3 denotes the principle amount along with the interest being earned; then P_1 will be P_0 ; the starting point plus P_0 divided by r by 100; so this will be the amount; what will be P_2 ?

(Refer Slide Time: 08:43)

Example

$$P_1 = P_0 + P_0 \times \frac{R}{100} + P_0 \times \frac{r}{100} = P_0 \left(1 + \frac{2r}{100}\right),$$

and so on. Thus, $\{P_n\}_{n \geq 1}$ is an arithmetic sequence with common difference $P_0 \left(\frac{r}{100}\right)$.

- **Example:**

In compound interest scenario, not only the amount invested earns interest, the principal plus interest earned (if any) also earns interest.

For example, if P_0 is the amount invested at the rate of $r\%$ per annum, then

after one year the amount grows to $P_1 = P_0 + P_0 \times \frac{r}{100} = P_0 \left(1 + \frac{r}{100}\right)$.



So, P_2 will be P_0 plus right that is P_1 ; so P_2 and so on. So, this gives us an arithmetic sequence with common differences P_0 divided by r plus 1. If the amount is calculated with compound interest then the scenario changes. So, r percent per annum; that means, the interest also gets interest on that. So, P_1 will be equal to P_0 plus P_0 by r ; that is P_0 multiplied by $1 + r$ by 100.

(Refer Slide Time: 09:28)

Example

At the end of the second year, the principal grows to

$$\begin{aligned} P_2 &= P_1 + P_1 \times \frac{r}{100} \\ &= P_1 \left(1 + \frac{r}{100}\right) = P_0 \left(1 + \frac{r}{100}\right)^2 \end{aligned}$$

and so on.

Thus, if P_n denotes the total amount to which P_0 has grown after n years, then

$$P_n = P_0 \left(1 + \frac{r}{100}\right)^n, \quad n \geq 1.$$



Thus $\{P_n\}_{n \geq 1}$ is a geometric series with common ratio $\left(1 + \frac{r}{100}\right)$.

P_2 will be P_1 plus whatever we have earned after one year, we will get interest on the whole amount. So, it is P_1 multiplied by $1 + \frac{r}{100}$; so that will be; if I put the value of P_1 that is P_0 into $1 + \frac{r}{100}$; so, this is P_0 multiplied by this and so on.

So, at P_n after n years at the rate of r percent interest, this will be the principle would have grown to this much. So, again this generates a sequence an ordered collection of numbers which in fact, is a geometric sequence with common ratio as $1 + \frac{r}{100}$. So, the principle if the simple interest that gives us a sequence of numbers amount P_n is arithmetic progression, if it is compound interest; it is a geometric progression.

And in each case one would like to know what happens to the terms of the sequence a_n as n changes n becomes large and large. Of course, if it is a interest; the principle will keep on growing and similarly in the example of production of chairs also, it will keep on increasing, but in a general sequence that need not happen always.

(Refer Slide Time: 10:57)

Behaviour of a sequence

- In the example we considered above, in either option the values of a_n keeps on increasing as n increases.
In general, for an arbitrary sequence this may not happen.

Example

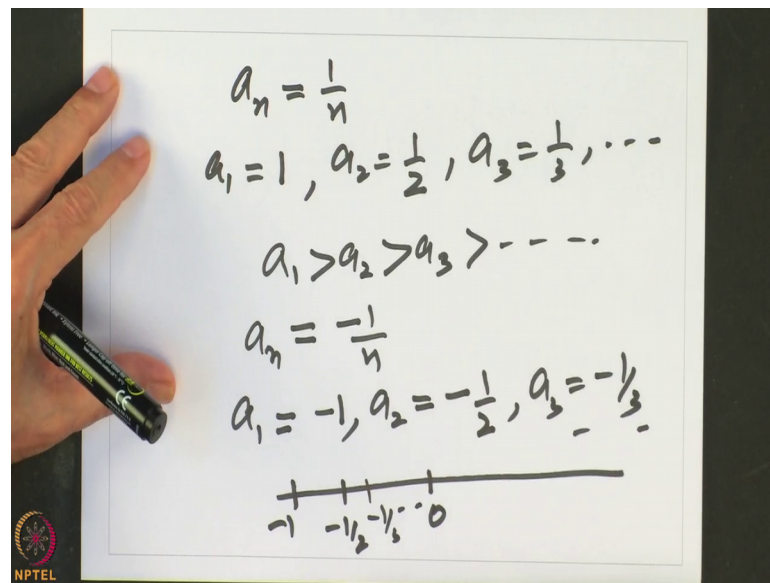
Consider a sequence $\{a_n\}_{n \geq 1}$ where

- $a_n = \frac{1}{n}$.
Terms are decreasing and coming closer to the value 0.
- $a_n = \frac{-1}{n}$.
Terms are increasing and coming closer to the value 0.
- $a_n = \frac{(-1)^n}{n}$.
Terms are fluctuating around 0 and coming closer to the value 0.

NPTEL © Inder K. Rana, I. I. T. Bombay Calculus for ECM 17 / 45

So, let us look at some example; for example, consider the sequence a_n equal to $\frac{1}{n}$ over n , so let us write the sequence.

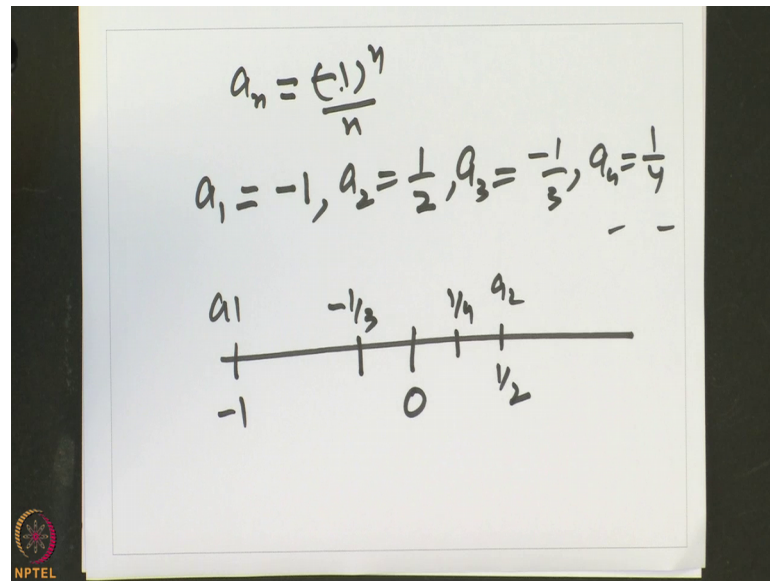
(Refer Slide Time: 11:03)



So, a_n is equal to 1 over n ; so what is a 1 ? So, that is equal to 1 , what is a 2 ? That is 1 by 2 , what is a 3 ? That is 1 by 3 and so on. So, it is quite clear that a 1 is bigger than a 2 , a 2 is bigger than a 3 is bigger than so on. So, it is quite clear that this sequence 1 over n ; is a decreasing sequence the values are decreasing; first value is 1 , second is 1 by 2 , third is 1 by 3 . So, values are decreasing; it is coming closer and closer to the value 0 , let us look at the sequence; a_n equal to minus 1 over n .

So, what will be the terms of the sequence? When a_n is equal to minus 1 by n ; so, what is a 1 ? a 1 will be equal to minus 1 . What is a 2 ? a 2 is minus 1 by 2 , what is a 3 ? Equal to minus 1 by 2 and so on. So, if I try to plot it; it is 0 , here is minus 1 , here is minus 1 by 2 , here is minus 1 by 3 and so on. So, it is quite clear that the sequence is an increasing sequence; as you move the terms are increasing, the values are increasing and coming closer to 0 . Let us look at the sequence a_n minus 1 to the power n divided by n ; what are the value of this sequence?

(Refer Slide Time: 12:44)



So, let us write that again; so, a_n is equal to minus 1 to the power n divided by n . So, what is a_1 ? n equal to 1, so this is minus 1; what is a_2 ? Minus 1 square; so, that is 1 by 2; what is a_3 ? So, minus 1 to the power 3, that is minus 1 by 3; a_4 will be equal to 1 by 4 and so on. So, let us look at; so this is the point 0 and this is a_1 is minus 1, what is a_2 ? a_2 is 1 by 2, what is a_3 ? Is minus 1 by 3; what is a_4 ? Is 1 by 4.

So, again we observe that this sequence is neither increasing nor decreasing; it is fluctuating some alternatively it is becoming positive and negative and, but still it is coming closer to 0. Let us look at one more example of a sequence a_n where a_n is equal to minus 1 to the power n .

(Refer Slide Time: 13:45)

Behaviour of a sequence

Example


Consider a sequence $\{a_n\}_{n \geq 1}$ where

- $a_n = (-1)^n$.
Terms are fluctuating and not coming closer to any value.

• Consider the sequence $\{\frac{1}{n}\}_{n \geq 1}$.
As n becomes large, $a_n = \frac{1}{n}$ becomes small.
In fact, by choosing n larger enough we can make $a_n = \frac{1}{n}$ as small as we want. In a sense a_n is approaching $a = 0$.

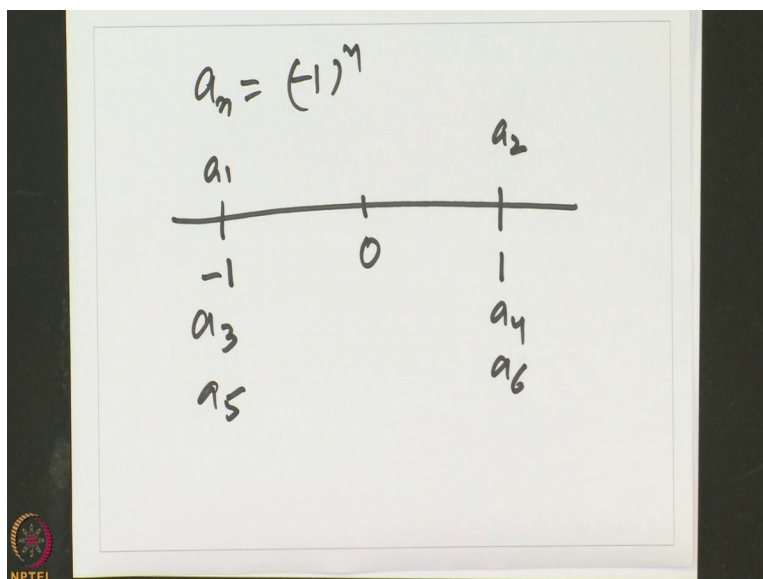
• In fact, given any small number $\epsilon > 0$, we can find n_0 such that

$$|a_n - 0| = \frac{1}{n} < \epsilon \quad \forall n \geq n_0.$$

 NPTEL (© Inder K. Rana, I. I. T. Bombay) Calculus for ECM 18 / 45

So, what is happening to this sequence?

(Refer Slide Time: 13:51)



So, a_n is equal to minus 1 to the power n ; so if I try to plot it, it is 0 here so, n equal to 1; I will get a 1 which is equal to minus 1. What is a 2? a 2 minus 1 to the power 2; that is 1 that is a 2; a 3 will be equal to again minus 1, a 4 will be equal to again 1, a 5; again here, a 6 here. So, all the odd terms will be at minus 1; all the even terms will be equal to 1. So, it fluctuates minus 1, plus 1, minus 1, plus 1, minus 1, plus 1; so, this sequence is neither increasing nor decreasing, it fluctuates at the values minus 1 and n plus 1; it does

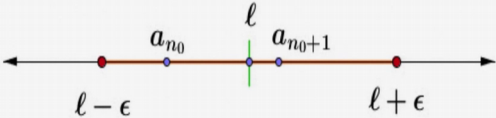
not come closer to any value. So, a behavior of a sequence depends on what is a sequence.

So, let us consider a sequence called 1 over n; n bigger than or equal to 0. As we observed, terms of the sequence becomes smaller and smaller; first term is 1; second is equal to 1 over 2, third is 1 over 3 and smaller and smaller. It is coming closer and closer to 0 in fact, it is approaching; we can say it is approaching 0; it comes close to 0 as close as we wanted. The distance of n from 0; I can make it as small as I want, so this distance is actually equal to 1 over n and I can make it less than epsilon by choosing n large enough.

(Refer Slide Time: 15:36)

convergence of sequences

A sequence $\{a_n\}_{n \geq 1}$ is said to **converge** if there exists $\ell \in \mathbb{R}$ such that




given any real number $\epsilon > 0$,
we can find a natural number n_0 with the property

$$\ell - \epsilon < a_n < \ell + \epsilon \text{ for all } n \geq n_0,$$

i.e., $|a_n - \ell| < \epsilon$ for all $n \geq n_0$.

The real number ℓ is called a **limit** of $\{a_n\}_{n \geq 1}$ and we write it as $\lim_{n \rightarrow \infty} a_n = \ell$.

 NPTEL (© Inder K. Rana, I. I. T. Bombay) Calculus for ECM 19 / 45

Let us formulize this as a definition, a sequence a_n is said to converge to a value l ; in which l is a number. If there exists a number l , we say a sequence converge is if there exists a number l .

So, here is a number l which has the property that given any number epsilon; this is a Greek letter epsilon, given any number epsilon bigger than 0. If I look at l minus epsilon and l plus epsilon then what should happen? All the terms of the sequence a_n should lie between l minus epsilon; l plus epsilon for all n bigger than or equal to 0. So, I can write this also as the distance between a_n and l is less than epsilon for all n bigger than 0.

Let us try to understand this; this is a quite a subtle point. See we want to say that the sequence is approaching this value l . So, intuitively what it means that a_n should come closer and closer to l , so how close? So, I will specify beforehand that the margin of error can be at the most this distance, so this is specified by a number ϵ bigger than 0.

So, it says that I should be able to find a interval around l of length 2ϵ . So, if I look at $l - \epsilon$ and $l + \epsilon$; then all the terms of the sequence are inside this interval for n bigger than or equal to 0. So, that another way of saying that would be that given ϵ bigger than 0; that is a error I can make. You can think of a $a_n - \epsilon$ is a a_n is the value of the sequence, l is the expected value of the sequence; this is the error I am making a $a_n - \epsilon$ mod is the error I am making.

This is a distance that is how far away it is from l ? That is error; that error should be less than ϵ , for which n ? For all n ; that is the important thing that from some stage onwards, everything should fall in between; that means, possibly a 1, a 2 up to a n naught minus 1, these terms are outside somewhere, but after that n naught stage; the tale of the sequence, you can think it as a tale.

That a 1, a 2, a n naught minus 1; a n naught onwards that is a tale of the sequence that should lie in between this interval. Then we will say that the sequence a_n converge to l ; that is a limit. So, we say that a_n converges to l and write it as $\lim_{n \rightarrow \infty} a_n = l$. So, this is only a symbolic way of writing this thing that given ϵ bigger than 0, there exist a stage n naught such that for all elements of the sequence bigger than or equal to n naught, they lie at a distance ϵ from a a_n .


So, convergence of a sequence means a tale of the sequence comes closer to l at the most a distance 2ϵ . And that ϵ is; I will prescribe stage you have to find if you want to show limit of a_n is equal to l . So, this is the convergence of a sequence limit a_n is equal to l .

So, once again let me say we say that a sequence a_n converges; if there is a number l such that for every ϵ bigger than 0, there exist n naught a natural number such that; $a_n - l$ is less than ϵ for n bigger than n naught.

(Refer Slide Time: 20:02)

Convergent Sequences

- **Note:**
Though both sequence $\{n\}_{n \geq 1}$ and $\{(-1)^n\}_{n \geq 1}$ are divergent, they are divergent for different reasons.
 $\{n\}_{n \geq 1}$ is divergent as it keeps growing.
 $\{(-1)^n\}_{n \geq 1}$ is divergent as it oscillates between -1 and 1 .
- **Example:**
Consider the sequence $\left\{ \frac{n}{n+1} \right\}_{n \geq 1}$.



NPTEL © Inder K. Rana, I. I. T. Bombay

Calculus for ECM

20 / 45

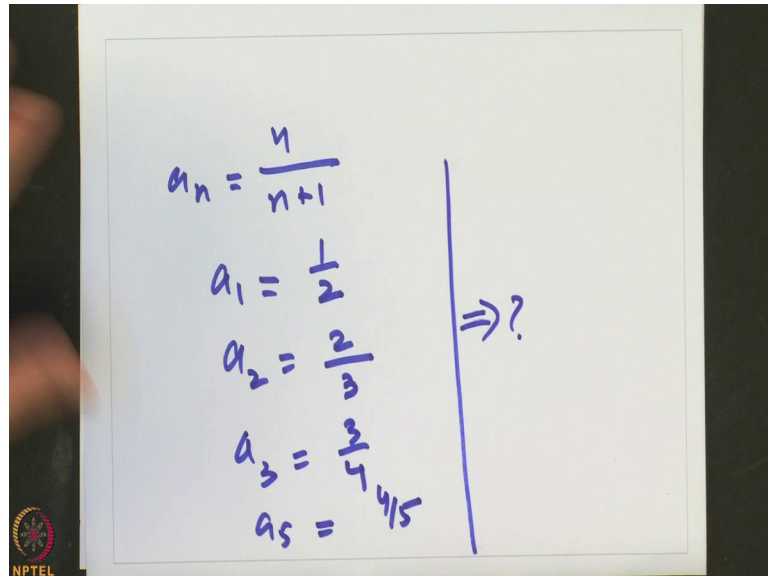
Let us look at some examples to understand this; you note that the sequence we had 1, 2, 3, 4 so on and the sequence minus 1 to the power n ; both are not convergent. Let me just say that when a sequence is not convergent, we said is divergent. I have not said that, so let us say when a sequence is not convergent; we will say it is a divergent sequence. So, the sequence n is not convergent why? Because first term is 1, second term is 2, third term is 3. So, it keeps on increasing it is not coming closer to any value; so it is divergence sequence.

Minus 1 to the power n ; this is again divergent because again it is not coming closer to any value; it is fluctuating between minus 1 and plus 1. One can write a proof of this; that this is a divergent sequence not convergent; just now I said epsilon. So, when it will be slightly quite interesting and logical to write a proof of the fact that minus 1 to the power n bigger than is a divergent sequence, not convergent. Try to write a proof yourself, if not try to read that web course on calculus; that I have said, a complete proof is given there, but intuitively this is it fluctuates between minus 1 and plus 1 and that this is not convergent; it is a property of real numbers intuitively it keeps on increasing; it is not coming closer and this is divergent.

So, these are different reasons for the sequence being not convergent. Let us look at the sequence n over n plus 1; n bigger than or equal to 1. We want to understand whether this sequence converges or not. The one way of analyzing this would be; write a few terms of

the sequence and see what is happening. What is the first term? 1 over 1 plus 2. So, let us just write a first few terms of this sequence; what is happening.

(Refer Slide Time: 22:25)



A photograph of a whiteboard with handwritten mathematical expressions in blue ink. The expressions are arranged vertically on the left side of the board, separated from the right side by a vertical line. The expressions are: $a_n = \frac{n}{n+1}$, $a_1 = \frac{1}{2}$, $a_2 = \frac{2}{3}$, $a_3 = \frac{3}{4}$, and $a_5 = \frac{4}{5}$. To the right of the vertical line, there is a question mark preceded by an implication arrow, $\Rightarrow ?$. In the bottom left corner of the whiteboard, there is a small circular logo with the text 'NPTEL' below it.

So, n divided by n plus 1; so that is a n . So, what is a 1? That is 1 by 2; what is a 2? That is 2 by 3; what is a 3? 3 by 4, a 5; 4 by 5. So, from here can you guess what is happening to the sequence?

So, first term is 1 by 2; second is bigger than 1 by 2; it is 2 by 3, third is bigger; that is 3 by 4, this is bigger. So, it seems it is going further and further increasing, but it is n over n plus 1; it cannot go beyond 1, this is always a fraction less than 1. So, it is increasing towards 1; so, guess is probably it is convergent to 1.

(Refer Slide Time: 23:28)

Convergent Sequences

Since


$$\frac{n}{n+1} = \frac{(n+1)-1}{n+1} = 1 - \frac{1}{n+1},$$

we guess that $\lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right) = 1$.

As

$$\left| \frac{n}{n+1} - 1 \right| = \frac{1}{n+1},$$

for any given $\epsilon > 0$, if we select n_0 such that $\frac{1}{n_0} \leq \epsilon$, then



NPTEL (© Inder K. Rana, I. I. T. Bombay)

Calculus for ECM

21 / 45

So, let us analyze this; so look at n over n plus 1; I can factorize it by this way. So, it is 1 minus 1 over n ; so this part intuitively becomes 0, as n becomes larger and larger. So, this will become smaller and smaller; so this is becoming smaller. So, intuitively this should go to 1; so one guess is that the limit should be equal to 1. But there will have to say why this is happening because this is not a sequence, it is 1 minus 1 over n ; it is something else; this is subtracted from here.

So, to justify this kind of a thing let us look at the distance between; so this is how we guess the limit should be equal to 1 to prove it. Let us look at a n minus 1; so, we have guess 1 is the limit, so a n minus 1, this is this quantity it is equal to 1 over n plus 1 and I can make 1 over n plus 1; small by choosing n large enough. So that means, I can say that this can be made less than epsilon for any given epsilon.

(Refer Slide Time: 24:44)

Convergent Sequences

for all $n \geq n_0 - 1$,

$$\left| \frac{n}{n+1} - 1 \right| = \frac{1}{n+1} < \frac{1}{n_0} < \epsilon.$$

Hence,

$$\lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right) = 1.$$


Question: Can a sequence have more than one limit?

Intuitive answer: NO!

We saw that sequence $\{n\}_{n \geq 1}$ is not convergent as its terms keep growing bigger and bigger.

This motivates our next concept.

▶ Proof



NPTEL (© Inder K. Rana, I. I. T. Bombay) Calculus for ECM 22 / 45

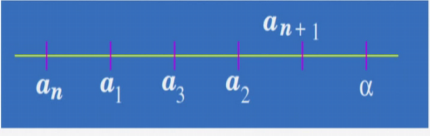

So, this is how rigorously you will prove that this limit is equal to 1; at this stage one can ask a question can a sequence have more than one limit? One can prove that if a sequence has a limit it is unique; there cannot be, intuitively its quite clear that tale of a sequence cannot come closer to two different values; symmetrically it is seems impossible.

But one can prove it rigorously will not go precisely into that and we also saw that the sequence n is not convergent because it keeps on increasing.

(Refer Slide Time: 25:28)

Bounded Sequences

- Definition :
Let $\{a_n\}_{n \geq 1}$ be a sequence.
- (i) $\{a_n\}_{n \geq 1}$ is **bounded above** if for some α ,
 $a_n \leq \alpha$ for all n .

NPTEL (© Inder K. Rana, I. I. T. Bombay) Calculus for ECM 23 / 45

So, this motivates our next definition namely; we say a sequence a_n is bounded above; if there is a number α such that all the terms are less than or equal to α ; that means, the terms of the sequence cannot go on the right side of the α , there is a barrier, so then we say it is bounded above.

(Refer Slide Time: 25:55)

Bounded Sequences

- Definition :
Let $\{a_n\}_{n \geq 1}$ be a sequence.
- (i) $\{a_n\}_{n \geq 1}$ is **bounded above** if for some α ,
 $a_n \leq \alpha$ for all n .
- (ii) $\{a_n\}_{n \geq 1}$ is **bounded below** if $a_n \geq \beta$ for all n for some β .

NPTEL (©Inder K. Rana, I. I. T. Bombay) Calculus for ECM 23 / 45


And similarly we will say a sequence is bounded below; if there is a number β so that all the terms of the sequence are on the right side of it. So, a_n is bigger than or equal to β for all n ; then we say a_n is bounded below.

So, bounded below is there is a barrier on the left; bounded above is barrier on the right. So, bounded below; bounded above if a sequence is both bounded above and below, we say it is a bounded sequence. So, bounded means there is a barrier below; there is a barrier above then we say it is a bounded sequence.

(Refer Slide Time: 26:24)

Bounded Sequences

(iii) $\{a_n\}_{n \geq 1}$ is **bounded** if it is bounded both above and below.



• **Examples:**

(i) $\left\{\frac{(-1)^n}{n^2}\right\}_{n \geq 1}$ is bounded, since for all n ,

$$\left|\frac{(-1)^n}{n^2}\right| = \frac{1}{n^2} < 1.$$

(ii) The sequence $\left\{\frac{2^n}{n^2}\right\}_{n \geq 1}$ is not bounded, since

$$2^n > n^2 > n \text{ for all } n \geq 5.$$

NPTEL (© Inder K. Rana, I. I. T. Bombay) Calculus for EOM 24 / 45

So, examples the sequence is bounded because for all n , absolute value of this is 1 over n square less than 1. So, all the terms of the sequence will lie between minus 1 and 1; so, there is a bound below minus 1, there is a bound below above 1. But boundedness at present does not say anything about the convergence of a sequence; keep that in mind.

So, this sequence 2 to the power n divided by n square is not bounded that one can show with slight amount of work; namely 2 to the power n becomes; grows much faster than n square. So, one has to write something; one has to do some more work, this is not bounded. Those are few who are more in choosiest; try to prove 2 to the power n is bigger than n square is bigger than n for all n bigger than or equal to 5; that means, 2 to the power n will take over n square and it will take over even n for n bigger than or equal to 5; so, it will keep on growing it will never converge.

So, we will stop here for today that having recalled that we are defined the notion of a sequence. Sequence is a ordered collection of numbers which essentially come from a data which is taken a different time point you can think of. And then a sequence may behave differently depending on what is a sequence, it may come closer and closer to a value, it may keep on increasing, it may keep on decreasing and so on.

So, we define the notion of convergence of a sequence; we say a sequence a_n converges, if there is a number l such that the distance between a_n and l becomes small less than epsilon for any prescribed epsilon from some stage and not onwards. That stage n naught

may depend on epsilon, but what should happen is absolute value of a_n minus epsilon should be less than epsilon; for all terms of the sequence n bigger than or equal to N ; that means, the tail of the sequence should come in a neighborhood of l of length epsilon.

So, convergence of a sequence depends on the tail of the sequence. So, we will continue our study of sequences in the next lecture.

Thank you.