Calculus for Economics, Commerce & Management
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Lecture – 05
Real numbers, sequences

In the previous lecture, we had seen the set of real numbers. So, we defined real numbers as a set with binary operations of addition and multiplication with certain properties. And we also said that we can represent the real numbers on the line, when we try to put integers, rational numbers on the line each one of them goes and occupies a unique position on the line. Still some points on the line are left out which are not represented by any rational number. These are the points; for examples the square root 2, square root 3 and such numbers are not represented which are on the line but they are not represented by some rational number.

So, there are positions on the horizontal line where all the rational set and still gaps are left out. So, these gaps are filled by defining new numbers and adding them to the bag of rational numbers and constructing a new one. So, this construction process is quite long and requires more mathematical maturity, so we are not going to deal with it. So, as and when required for the geometric purposes, for understanding some properties of real numbers; we will treat that each point represents an a real number and every real number is represented by a point in the line.

So, algebraically it is a set with certain properties, geometrically it is a points on the line. So, these are the two views will keep of the real numbers for our further decisions. The geometric view point will be used as and when we want to exhibit some properties of the real numbers. So, that is why the points on the horizontal line are called the real line. So, that set of all points on the line are called the real numbers and the line is called the real number line.

So, now, I am going to describe a very important concept for real numbers; namely if the points are on the line and so; if 0 is a point, we know the 0 is somewhere. And if I take any point x on the real number, it will be somewhere on x on left or on the right. So, there is a notion of distance physically I can measure; geometrically measure the distance between 0 and the point x; it may be on the left or on the right. So, how does one make it more precise mathematically? What is this algebraically? What is this number? So, this is called the absolute value of a real number.

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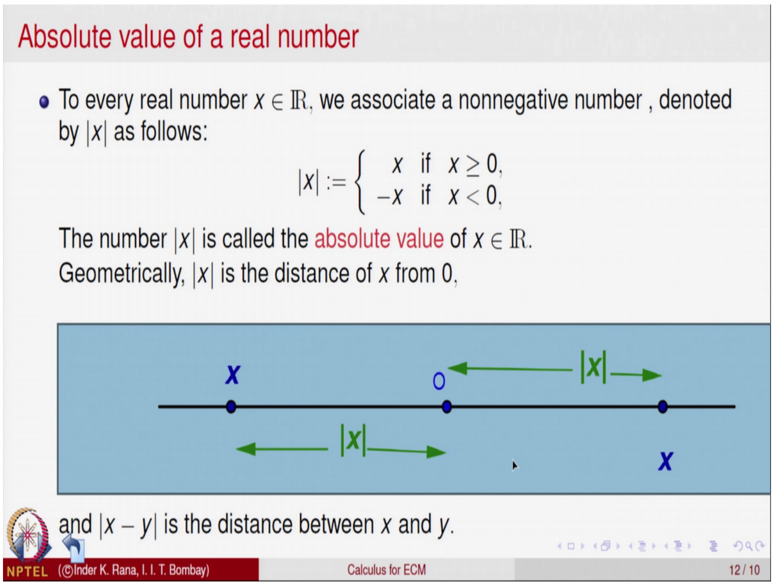

So, for every real number x we associate a non negative number and this non negative number is denoted by two bars.

And x it is called the absolute value of x or called modulus of x and it is defined as follows. So, modulus of x is defined to be x itself; if x is non negative number, if x is bigger than or equal to 0 and it is written as minus x; if x is less than 0. So, note for a non negative number; mod x, absolute value of x is x itself; whereas, if it is negative then absolute value is minus x. So, absolute value of the number is always a non negative quantity is a non negative real number.

So, mod x is equal to x; if it is non negative and is minus x, if x is less than 0; so, this is called absolute value. So, geometrically if this is 0 and this is the point x we want to measure this distance from 0 to x. So, that is mod x because if x is negative then this distance; because distance is always measured as a positive quantity.

So, this is mod x and if x is on the positive side; on the right side then mod x is x itself. So, this mod x is x itself; if x is positive and if x is negative this distance is nothing, but minus of x. So, algebraically it is defined as mod x absolute value of x is x; if x is bigger than 0 and minus x if x is less than 0. So, this is called the absolute value of a real number and geometrically it signifies the distance between 0 and x.

And using this one can easily define what is a distance between two points x and y? So, it is nothing but the absolute value of the number x minus y; that gives us a notion of distance between x and y. So, we define the distance between two real numbers x and y; we are not saying x is bigger than y or y is bigger than x. Take any two real numbers x and y; look at x minus y that will be a number, take its absolute value that will be a non negative quantity and that is called the distance between x and y.

So, distance between two points x and y is the absolute value of the number x minus y.

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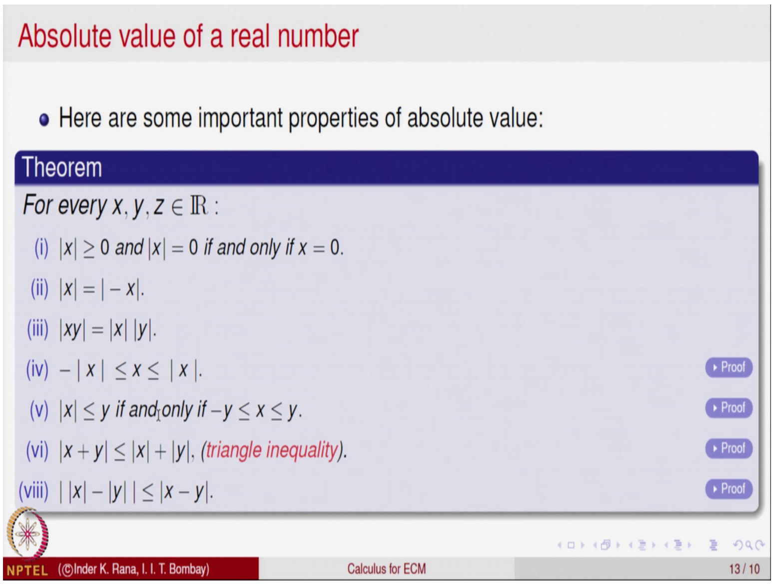

This notion of absolute value; so, this is what we have indicated here. It has some important properties, so let us look at those properties; we will not prove those properties but they are quite reasonably obvious. Namely absolute value of a real number is always bigger than or equal to 0; that is quite clear by definition and it is equal to 0 if and only if x is equal to 0; that means, if x is 0; it is distance from 0 is 0.

And if the distance of a point from x is 0, then the point must be 0 itself. So, this is the put as x is bigger than or equal to 0 and always and it is 0 if and only if; only in the case only and only in the case that x is equal to 0. The second property says absolute value of x is same as absolute value of minus x; that is quite clear from our definition because mod of x is a non negative number, so mod of x is same as mod of minus x.

How absolute value behave with respect to multiplication? It says, if you take product up to numbers x and y and take its absolute value; that is same as the product of the absolute values. So, how we are describing here in this property; how does absolute value behave with respect to multiplication? It says the absolute value of the product is same as product of the absolute values.

And here is an obvious third property namely for a number x; is always between minus of mod x and plus of mod x; mod x is a non negative quantity and x is equal to mod x, if it is positive and if it is negative this number is minus, so it is minus minus that is x.

So, it is always between minus x and x; one of them will be true. And here is if mod x is less than y; it says this property mod x is less than distance of y is bigger than mod x, if and only if either the point y is on the right side of x or minus y is on the left side of it; we do not know. So, x may be positive; x may be negative, so it says if y is bigger than mod x then this can happen if and only if; x is bigger than minus y and x is less than or equal to y.

And here is the last property which is quite important, which relates absolute value with addition. It says that the absolute value of the sum of two numbers is less than or equal to the sum of the absolute values. In multiplication, it was equal; the absolute value of the product was equal to product of the absolute value, this is not in general true for addition.

What is true is we can only say that the x plus y absolute value is less than and equal to absolute value of x plus absolute value of y. This is normally called triangle inequality, it relates the property of distance is in a triangle the sum of two sides is bigger than the third side. But at present, we will only look at because points are on the real line; we will say that the absolute value of the sum is less than or equal to sum of the absolute values that is called triangle inequality.

These properties of absolute value; one can prove but we will not be discussing the proves here. Some of you who are interested in knowing the proves of these things; can pick up a book on advance calculus, if you like. Or if you like under the NPTEL; I had developed a web course on calculus, it is available online on the NPTEL site; look for web course on calculus and in the first chapter these properties are proved.

So, if you like to read more about calculus; you can always refer to the web course on calculus on the NPTEL web site. So, use internet fruitfully and look for that course and read it if you find it interesting. So, these are the properties of absolute value that will be using in adieu in our course. So, let me repeat once again for every real number x; we can associate a quantity called mod x that is the distance geometrically it is a distance of the point x from the number line.

This distance is always bigger than or equal to 0, it is 0 if and only if x is 0. So, distance of x is same as distance of minus x and the absolute value of the product is product of the absolute values. And here are the inequalities that the number x is always bigger than or equal to minus mod x and less than or equal to mod x.

Because it depending on whether x is positive or negative, it will be equal. And a mod x is less than y, if and only of y is bigger than x and x is bigger than minus y. Here is the property, which relates addition with triangle equality property; which relates addition with this property, namely the sum absolute value of the sum is less than or equal to sum of the absolute values.

Here is a straightly technical stating inequality which you may or may not able to visualize, but anyway this is quite simple to prove. And we may not be requiring much use of it; it says if you take a real number x and take a real number y, look at their absolute values. So, absolute value of x and absolute value of y; these are again points on the number line, the distance between mod x and mod y is less than or equal to the distance between x and y.

So, this another property of the absolute value; see one thing I should point out at some point in higher mathematics, the geometric intuition is not possible and that is why one has to at some point leave the geometric intuition to do the abstract mathematics. So, here probably you may not be able to visualize; what is this property geometrically. So, these are the properties of absolute value that will be using in our course right, so let us go over a bit further.

So, till now we looked at what is a real number; we looked at representing numbers on the line and we defined the notion of absolute value of real numbers.

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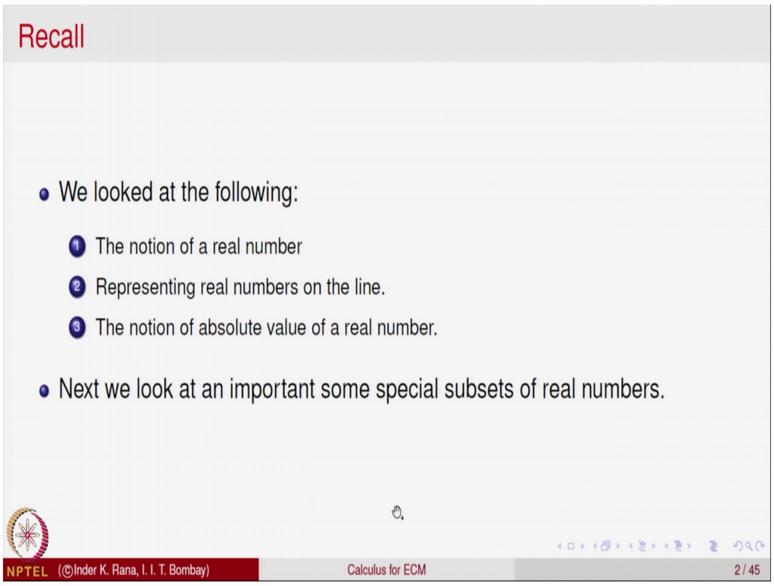

Next, what we are going to do is; we are going to look at some important special subset of real numbers which will play a role in our future part of the course.

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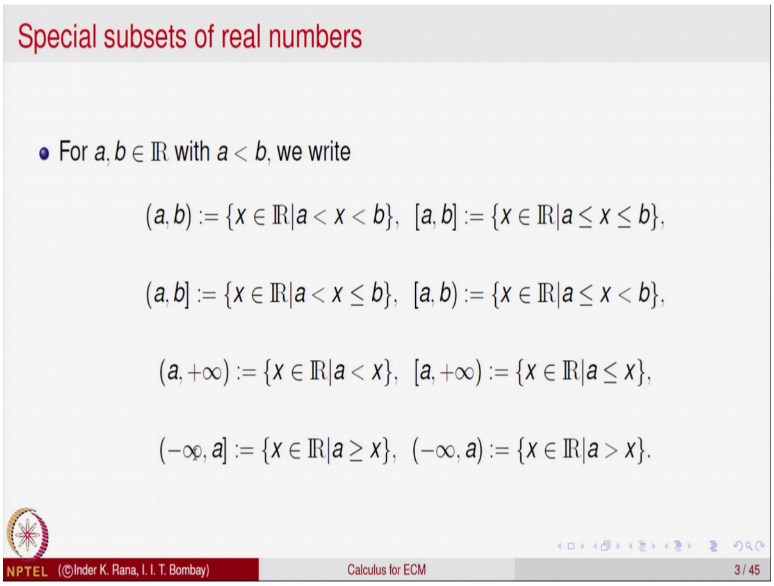

So, let us start describing this special subsets; given real numbers a and b, let us assume a is less than b. We write with a; this bracket a comma b, so we write bracket a comma b to be the set. So, this is the notation for a set; what is this set? It is all real numbers which are between a and b. So, all number x which are strictly bigger than a as strictly less than a.

So, this set we will denote it by round bracket a comma b; round bracket close. Similarly, the square bracket a comma b will denote all real numbers x belonging to R; such that x is bigger than or equal to a, but less than or equal to b. So, here in this x was strictly bigger than a and strictly less than b; here x can be is actually equal to a also and equal to b also. So, here you can think of that; a and b are not part of this set the first one, while a and b are the part of this set.

So, you can quite easily compare the two; this first set is a subset of a b. In fact, a proper subset because the point a and b belong to this set but they do not belong to this one. So, similarly there are some other sets; let us describe them. I am taking a circular bracket a comma b with a square bracket on the right side; this means I am looking at that set of all points x real number, which are bigger than a and less than or equal to b. And when I put square bracket on the left and circular bracket on the right; that means, I am looking at points x bigger than or equal to a and x less than b.

So, it should be clear some pattern is emerging when it is a circular one; circular bracket means there is a strict inequality, when it is a square bracket that is less than or equal to. So, here is a next one which is slightly different from these two; we write a comma this. This is a symbol which is read as infinity, so a plus infinity; circular bracket a plus infinity means I am looking at all the real numbers, which are bigger than a; x bigger than a.

And if I put as square bracket on the left; that means, I am also including the point a in the set. So, I am looking at the set x bigger than or equal to a, so that is square a comma plus infinity. So, this is a symbol plus infinity; so, keep in mind this symbol it is like a flat 8; 8 is normally written as this, if you make it horizontal this looks like 8.

So, this is plus infinity read as; so, a comma plus infinity circular bracket; circular bracket means I am looking at all numbers x bigger than a. And if I put a square thing here; that means I am looking at all numbers bigger than or equal to. So, keep that pattern in mind when bigger than or equal to; that means, a square bracket. And similarly minus infinity to a, so if I put minus sign before the symbol infinity; this is read as minus infinity and square a; that means, I am looking at all numbers which are less than or equal to a. So, from a onwards; including a and if I put minus infinity comma a circular bracket, that is all numbers x which are strictly less than a.

So, these are special type of sets and these are called intervals; all these are subset of real line and they are called intervals. Just keep in mind whenever is a circular thing coming, there is a strict inequality. Whenever there is a less than or equal to; there is a square thing coming and also keep in mind this plus infinity and minus infinity are just symbols; they are not numbers.

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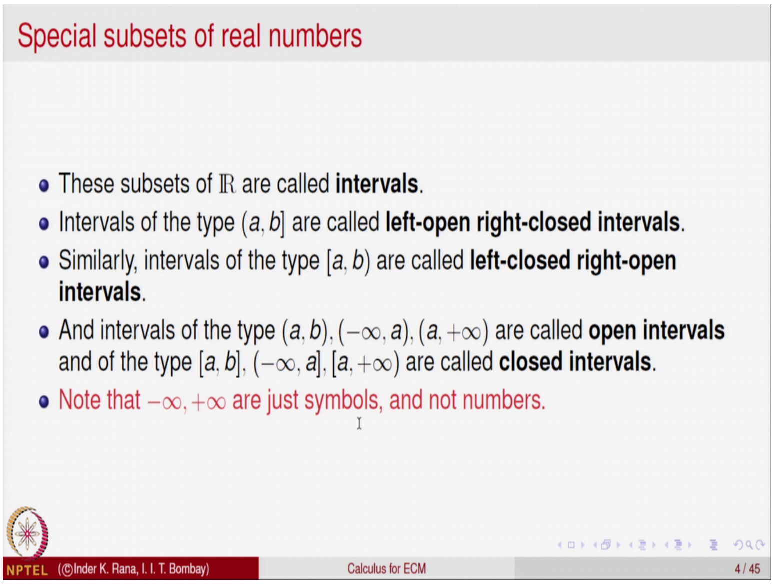

So, all these are subsets of the real line; so, when I am not including a. So, this is a circular one and including b these are called left open right closed intervals. So, intervals of this type are called left open the left point a is not included and b is included, so left open right closed. So similarly this type of intervals; a square bracket on the left, so this is left closed, right open intervals.

These intervals a comma b with circular brackets and minus infinity comma a; a comma plus infinity, these type of intervals are called open intervals. And when there is a square bracket with infinity is coming, these are called closed intervals.

So, these are names given to special type of subsets of the real line but all along please keep in mind that minus infinity and plus infinity are just symbols, they are not numbers; do not confuse them that plus infinity is a number which is bigger than or equal to any real number a. It is not a number; it is just a symbol indicating; is a shorthand to describe the intervals.

So, let me just once again go back; so these are the type of special subsets. So, this is the open interval a b, this is a closed interval a b, this is a left open, right closed interval; this is left closed; right open interval, this is you can call this as a unbounded interval or a infinite interval, it is an open interval; open on the left of course and this is a interval which is closed on the right, this is a interval which is closed on the left.

So, this is a closed interval, this is a closed interval, this is an open interval and this also is an open interval. So, these are special type of intervals; a special subset of real line which we will be using and keeping in mind plus infinity minus infinity are not numbers; they are just symbols to signify.

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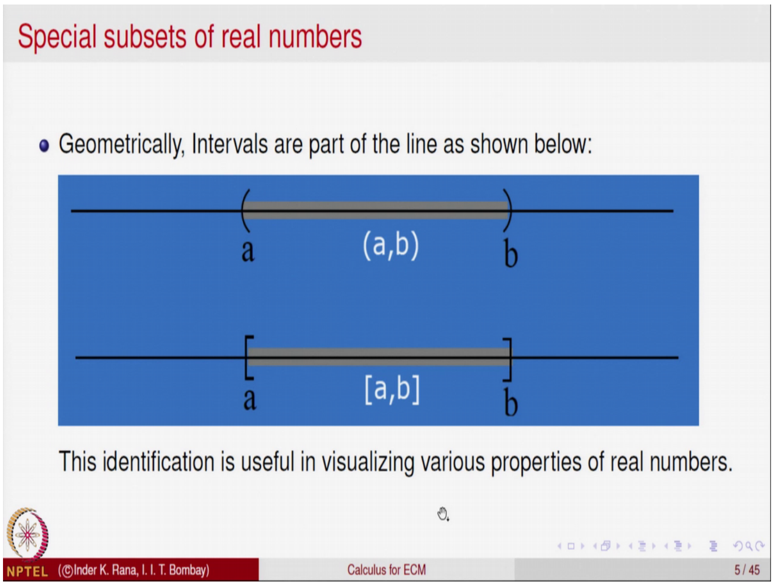

So, if you look at geometrically the open interval a; b, we can represent as circular bracket here, circular bracket here. So, all the points in between; this represents the interval a comma b. And similarly if I put a square bracket here and a square bracket here then the points inside; included a and included b are this is geometric representation of the closed interval a b.

This is a open interval a b; the all points on this segment excluding a, excluding b and this is all points on the segment with a and b included; that represents the closed interval.

These representation will be useful in proving some properties of real numbers, so they guide us to write the analytical proves later on.

So, keep this in mind; there are some special type of intervals which will be requiring.

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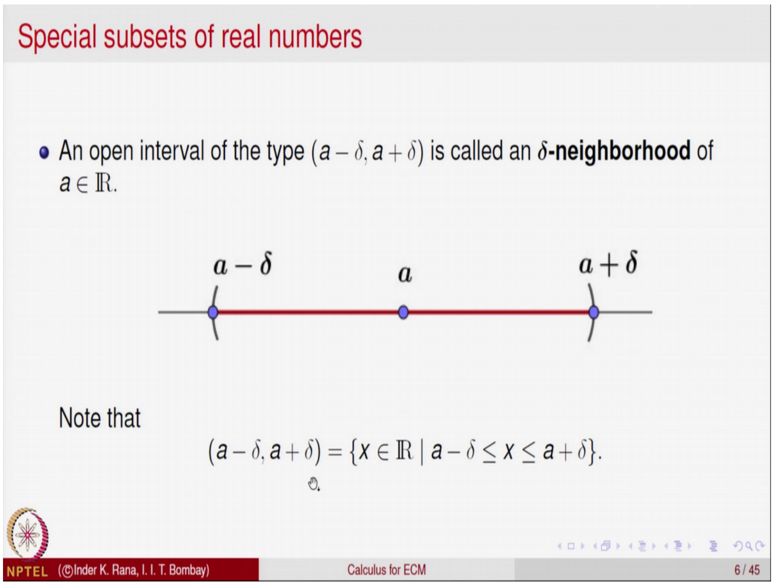

So, consider an open interval of the following type a; a is a real number, delta is a real number because n 0. So, look at a minus delta and a plus delta; so, geometrically you are looking at this is the point a and you are going a distance delta on the left, you will get the point a minus delta. Go to the right; we will get the point a plus delta, so look at the open interval with; this is called the left end point.

So, left end point is a minus delta; the right end point is a plus delta. So, all points in this segment excluding a minus delta and a plus delta, this is a interval; this is an open interval; its length is 2 delta; on delta on this side and delta on this side this is called a delta neighborhood of the point a. So, such type of intervals are normally called delta neighborhood of the point a.

So, we will be using this in our future discussion, so keep with this in mind. So, geometrically we are saying this geometric thing is written algebraically as and set theory. So, interval a minus delta to a plus delta excluding a and excluding b; here is a typo error, this should be strictly bigger, this should be strictly bigger. So, because we are taking open, so this is a typo error, so please keep in mind that this should be strictly less and this should be strictly less.

So, now we have described real numbers; we have described the notion of absolute value. So, keep in mind the absolute value allows you to discuss the notion of closeness of points on the line because it measures the distance. So, if you want to say two points are close; you will say that the absolute, if a two points are x and y are close to each other, you will say that their distance between them is small; that means, you will say the absolute value of x minus y is small.

So, absolute value allows you to describe closeness of points on the line that is a important thing. So, now we are going to look at a new concept called sequences of real numbers.

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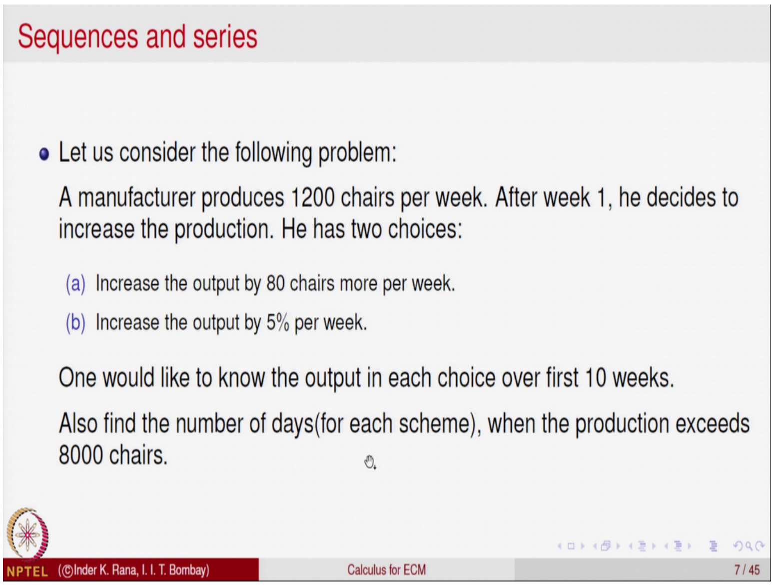

So, let us look at a problem; a manufacturer produces 1200 chairs per week. After week 1, he decides to increase the production and he has now two choices; one he can increase the output by 80 chairs more per week; that is one possibility. And the second is he can increase the output by 5 percent per week.

So, he has two options available; one he can increase the output by 80 chairs per week; every week he can go on increasing 80 chairs or he can also have the option, probably he has two factories; in another factory he can increase the output by 5 percent per week.

So, these are the two possible choices he has; so what is want to discuss? One would like to know the output in each choice for the first 10 weeks; in the coming 10 weeks what will be the output of the number of chairs being produced? Also probably one would like to find out the number of days for each option, when the production exceeds 800 chairs; so let us try to understand each option one by one.

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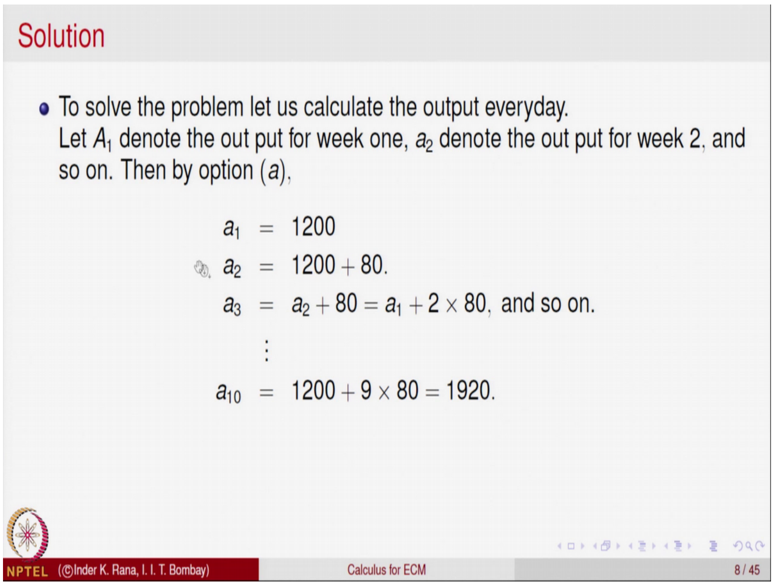

To solve the problem let us calculate the output everyday; let a 1 denote the output for week 1, let a 2; denote the output for week 2 and so on. So, a 1 is the output; so a 1 will be equal to 1200 that is given to us. Now if he goes for the option a; that means, what he is able to increase the production by 80 units more; that means, a 2 will be 1200 plus 80.

And what will be a 3? A 3 will be equal to from the second week of 1200 plus 80; another 80 will be added on. So, after the third week it will be; the first week plus two times 80 will be added. So, if you go on doing it; at the tenth week 1200 is the first one, 9 times 80 more chair would be added; production will go by that many units. So, total units will become 1920, so what do I done is; we have generated a observation, this is the first one; production on the first week. This is a production on the second week, if the chairs is 80 per week added, third week and so on.

So, this data that we have generated is that of the production at every week. So, we have generated a sequence of numbers which describes the production of first week, second week, third week and so on; so, the point is we have generated a sequence of observations. So, we will continue in the next part of the lecture; this a bit more.