Calculus for Economics, Commerce & Management
Prof. Inder K. Rana
Department of Mathematics
Indian Institute of Technology, Bombay

Lecture – 41
Some examples, constrained maxima and minima

Welcome back to our analysis of absolute maxima and absolute minima of functions of 2 variables. In the previous lecture, we had seen to analyze absolute maxima and absolute minima for a function of 2 variables we need to locate the critical points namely the points where interior points where the function of 2 variable both the partial derivatives exist and are equal to 0 or either of the partial derivative do not exist. So, those are the critical points and then look at the boundary points of the domain and find out the values of the function at all these points and compare and the largest value is the absolute maxima and the smallest value is the absolute minima with the function.

Let us apply these criteria to our example in our economics.

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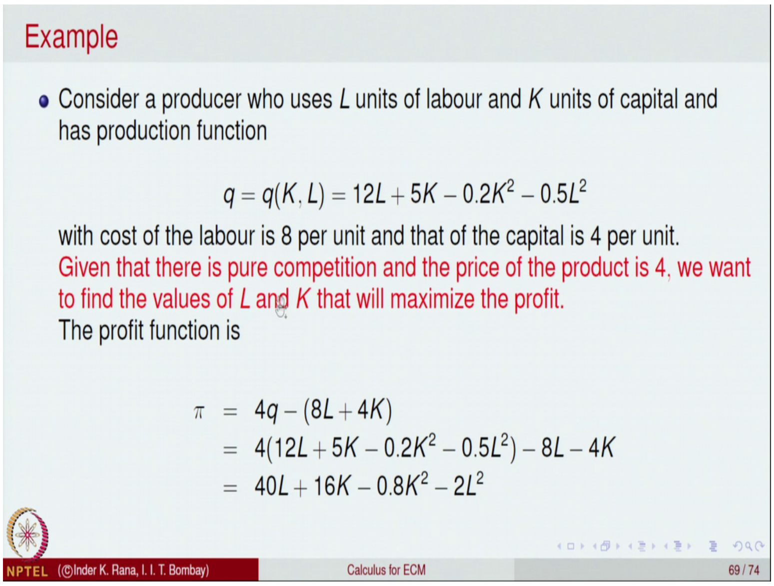

So, consider a producer which uses L units of labour and K units capital to produce a funct; to produce product and the product function is given by q which depends on the variables K and L and is given by 12 L plus 5 K minus 0.2 K square minus 0.5 L square. So, K is the capital and L is the labor and with the cost of the labour is 8 per unit each labour cost 8 and each capital investment is 4 per unit. So, that is what is given to us and what we want to do is given that there is a pure competition; that means, pure competition normally in economics means that the price of a product is fixed. So, let us say the price of the product is fixed at 4 we want to find out the values of L and K such that that will maximize the profit.

So, we want to maximize the profit this is the product function this is the cost input. So, we first we set up what is the profit function profit is cost minus profit is the revenue minus the cost. So, let us find out. So, pi the profit function which is a function depending on q and L both q is the product function. So, 4 times q because the price of the product is 4. So, the revenue will be 4 q minus the cost function. So, if L is the units of labour, 8 L plus K units of capital. So, 4 K.

So, that data is coming from. So, here that is 8 per unit for labour and 4 units per capital. So, this is the revenue and this is the cost. So, for q we put the value from here. So, this is 12 L plus 5 K minus point 2 0 K min us 5.5 L square. So, minus 8 L minus 4 k, once you put that you get the profit function as a function of 2 variables L and K to be equal to 40 L plus 16 K minus.

So, this is simplifying this is equal to 40 L plus 16 K minus 0.8 K square minus 2 L square. So, that is the profit function. So, what we want to do we want to maximize the profit function and then find out what is the input for the labour what is the input for the capital and what is the profit. So, this is the profit. So, differentiate this with respect to K you get partial derivative of pi with respect to K and similarly differentiate this with respect to L.

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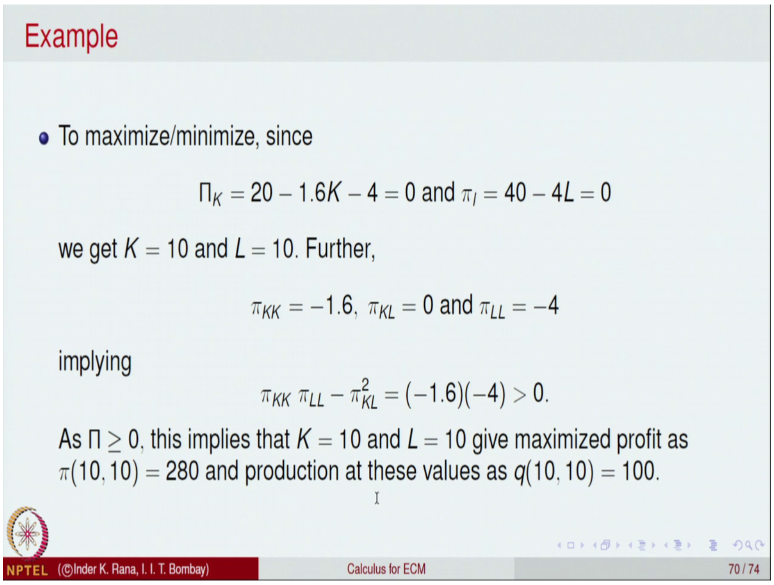

You get the partial derivative of the profit with respect to L and that gives us those 2 partial derivatives and we put them equal to 0. So, that gives you twenty minus 4 that is 16. So, that gives you K equal to tell 10 and L also equal to 10.

So, when input for the input K that is for the capital is 10 and input for the labor also is 10, then possibly the function profit function is maximum to check that is the case it is indeed a maximum will look at the second derivative test. So, second derivative of pi with respect to K from here profit with respect to K is point one minus one point six of this with respect to L. So, there is no L here. So, that is equal to 0 and partial derivative with respect to second derivative with respect L that is equal to minus four. So, and from this we look at the discriminant pi K K minus pi L L minus pi square K L and that is positive.

So, that the discriminant is positive and the first derivative is negative so; that means, the function has a maximum absolute maximum at the point K equal to 10 and L equal to 10. So, these are the inputs for the labour and the capital at which maximize the profit and the maximum profit is pi at 10; 10 that gives you 280 and the product function how many product; how many things should be produced. So, that is 100 products must be produced and then there will be a maximum profit of 280 K is equal to tell labour input for labor is 10 and you put for capital also is equal to 10.

So, this is how you apply the second derivative test to maximize or minimize a function let us look at one more scenario.

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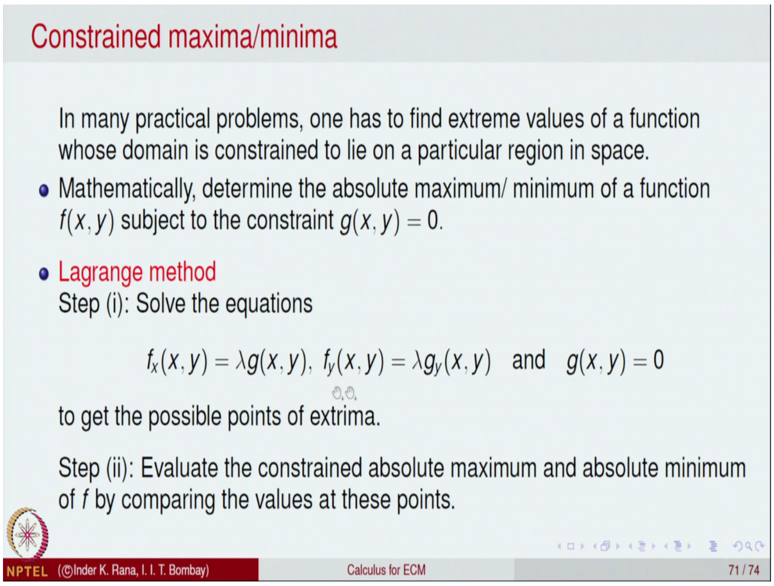

What is called the constrained maximum and minima sometime in many practical problems one has to find the extreme value of a function that is a maximum are the minimum what the domain, but the domain of the function is constrained to lie on a particular region in space. So, there are some constraints put on the function.

So, we are not looking at the absolute maximum and minimum of the function in the given domain of course, in the given domain, but we putting some restrictions on that absolute maximum and absolute minimum to illustrate this idea mathematically the idea is that given a function of 2 variables f xy, we want to maximize or minimize it subject to some relation between x and y. So, that that is called there is again a function of 2 variables mathematically and the; we write it as g xy equal to 0. So, fxy is to be maximized or minimized with a constraint gxy equal to 0, there is a algorithm for solving such kind of problems for 2 and 3 variables and so on.

So, let me just state that algorithm and we will apply in our scenario. So, the its called the LaGrange's method of constrained maxima and minima that says the following. So, look at the function f xy and look at the function g xy equal to 0. So, look at the relations step one solve the equations f xx a partial derivative of f with respect to x and there is a typo where it should be lambda times partial derivative of g with respect to x.

So, one equation is fx equal to lambda gx second equation is f y lambda gy partial derivative of f with respect to y equal to lambda times partial derivative of g with respect to y and the constraint g xy is equal to 0. So, we get 3 equations first you get equation is fx equal to lambda gx f y equal to lambda gy and the third is g x y equal to 0 in this 3 equations the variables to be determined are solutions to be found our xy and lambda.

So, these 3 can be determined from these equations once we have found those values of lambda x and y look at the points which you get from these solutions and find out the values of f about those points and compare and see which is the absolute maximum and absolute minimum.

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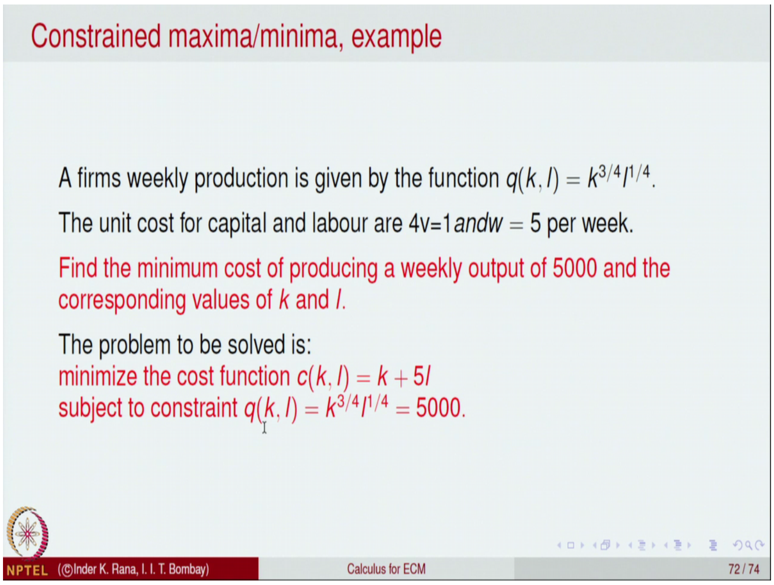

So, let me give you one example of this; a firms weekly production is given by the production product function is of capital and labor L and L and it is given by K raise to the power 3 by 4 L raise to the power 1 by 4 and what we want to do we were we are also given that the unit cost of capital and labour are v equal to one this is not 4 here this is a typo this is v equal to one and w equal to 5 per week.

So, the cost of capital is 4 capital is 1 and of the labor is 5. So, that the inputs. So, find the maximum minimum cost of producing a weekly output of 5000 units and the corresponding values of k and l. So, we write down what is want to minimize. So, we want to minimize the cost function. So, the cost function is for input for capital is 1 and the input for labor is 5. So, input is one times k plus 5 times l.

So, we get a function of 2 variables that is the cost function and we want to minimize the with respect to the constraint the product function is given to us. So, that is the constraint we have to obey the product function. So, the constraint is q k l equal to k raise to the power 3 by 4 and l raise to the power 1 by 4 equal to 5 and so, this is the constraint and this is the function which you want to minimize. So, as we said Lagrange method says find out the partial derivative of c with respect to k equal to lambda times partial derivative of q with respect to k 1 equation second equation will be partial derivative of c with respect to l equal to lambda times partial derivative q with respect to l and third equation is this constraint itself.

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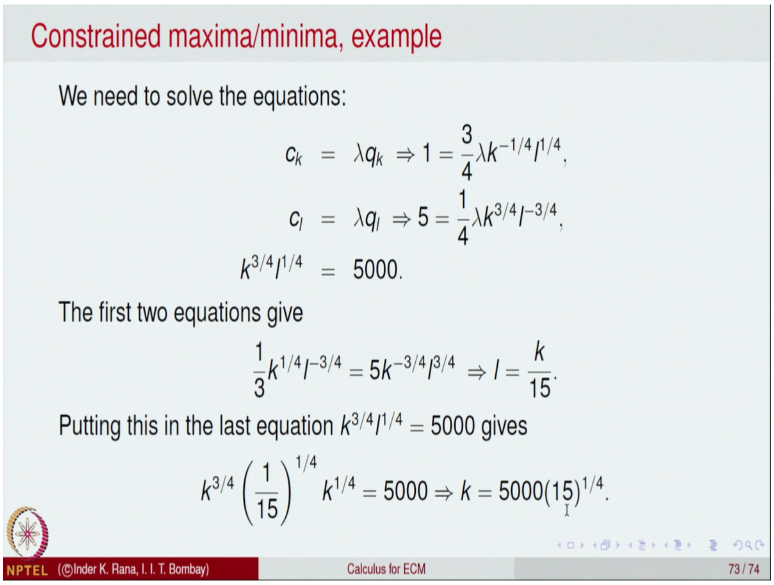

So, let us find out these 3 equations. So, this 3 equations are partial derivative c with respect to k equal to lambda k and that gives you one is equal to 3 by 4 if you look at the right hand side of the equation g and differentiate. So, power comes down 3 by 4 lambda k raise to the power minus 1 by 4 l raise to power 1 by 4.

Similarly, the partial derivative of c with respect to l is equal to lambda times partial derivative of q with respect to l. So, that is second equation that gives you 5 is equal to one by 4 lambda k to the power 3 by 4 l to the power minus 3 by 4 and the forth is the constraint itself that is k raise to the power 3 by 4 l raise to the power one by 4 equal to 5 thousand. So, these 3 equations are to be solved for k l and lambda which can be done quite easily. So, what we can do is from these 2 equations. So, we get that l is equal to k by 15. So, l is equal to ahs, we get that from these 2 equations you can solve them and get l is equal to k by 15.

So, that is solving equations and putting these values in the function constraint function give you K equal to 5000 to 15. So, the basic problem is Lagrange method says solve these 3 equations find the value of lambda K and l. So, in our case when we solve that the values come out to be equal to K is equal to this once you have obtained K we can put this value in this equation and get the value of l. So, get one lambda; if you want to find out lambda right or we have from these 2 equations, what we have done is we have found the value of lambda and equate it these 2. So, that gives you a relation between these first 2 equations give you the value lambda is equal to k by 15 and this when you put it in this equation you get the value of k.

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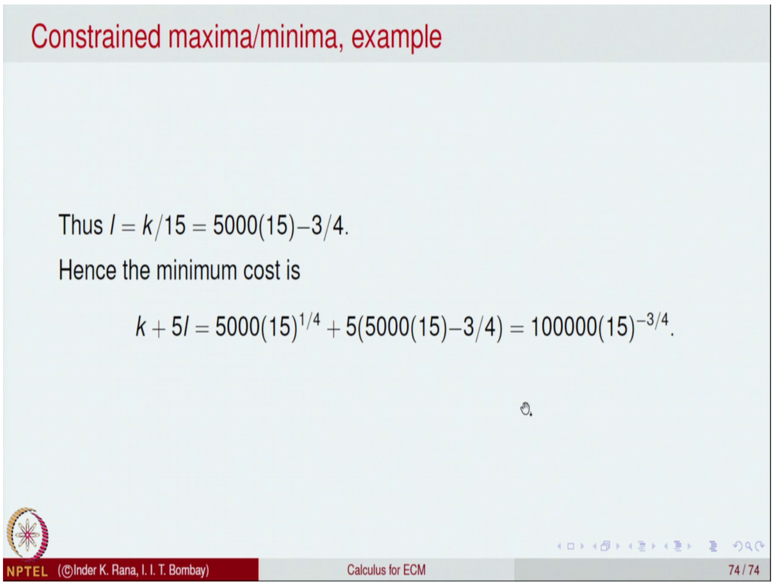

So, you get the value of k and from put back the value and you get the value of l. So, that gives you l is equal to k by 15 and that is equal to this value this power this is not minus this is a power minus 3 by 4. So, once you know what is k? What is l? We know k plus l is equal to that is the minimum cost function you can put the value of k and l and get the cost function. So, that is how you find the minimum value of the cost function.

So, this is what is called the Lagrange method of finding absolute maxima and absolute minima. So, with that we come to the conclusion of this course let me just revise this basically what we are trying to do in this course, we looked at to begin with we looked at what are the real number system and the quartile property there was the completeness property of real numbers. So, that said every monotonically increasing or decreasing sequence of real numbers which is if is increasing and it is bounded above it must converge which is bounded below decreasing and bounded below must converge.

So, those 2 that property completeness property we of real numbers is the one we assume that the real numbers have that property and then we looked at the concept of what is called a function. Function is a special type of relation. So, every point in the domain gets associated with the unique value one value does not get associated with 2 different values.

So, we looked at the concept of a function and then we analyzed an important concept in calculus namely what is called the limit of a function of one variable at a point. And we started with the idea that limit is the value of a function expected of it to be taken by looking at the values nearby for the limit the function need not be defined at that point. Once the concept of limit is established the value expected if it is equal to the actual value that is a continuity we defined the equation of continuity of functions of one variable.

So, limit continuity was defined and then we looked at the notion of differentiability and differentiability of a function of one variable came from various inputs one was rate of change of a function basically from physics it gives you the instantaneous speed at a point of a moving particle from mathematics geometry point of view. It helps you to define the notion of tangent to a curve at a point and basically in economics that gives you the marginal of a function.

So, marginal of a function is the derivative of that function at that point. So, that is how it is related with and similarly with these saw how the concept of differentiability gets related with a notion of coefficient of elasticity at a point and then we looked at the optimization problems of functions of one variable. We looked at what is increasing what is decreasing how what is the concept of local maximum what is the concept of local minimum what are the necessary conditions for local maximum local minimum and what are the sufficient conditions for local maximum and local minimum necessary conditions give you possible points where the local maximum or local minimum can occur and sufficient conditions give you a way of testing which are the points of local maximum or local minimum.

And then we looked at in particular we looked at we also looked at what is known as the convexity and concavity of a function of n variable and that was analyzing bending towards or bending away and economics that gave you the how the marginal is changing where as the marginal is increasing or marginal is decreasing and that also give you the notion points of inflection and finally, all of this put together gave you how the tips gave you the algorithm for sketching the graph of a function of one variable.

Once that was done we applied the concepts to various economic models and then we looked at functions of several variables mainly functions of 2 variables and we saw that we can analyze various things by looking at the partial derivatives administrating the as a function of 2 variables stating one variable as a constant and looking at the second variable as the varying. So, that gave us the notion of partial derivative and the notion of partial derivatives we mentioned they are not enough to imply continuity that is a stronger notion of differentiability for functions of 2 variables which we did not look at, but the notion of partial derivatives gave you conditions necessary conditions for analyzing points of local and maxima and minima.

And so, the points if is a point interior point the domain of the function and if it is a local maxima minima for a function of 2 variables then the necessary conditions at both the partial derivatives must be equal to 0. So, this along with the points where the function need to have partial derivatives and the boundary points or the possible points where the function can have local maxima or local minima and absolute maxima.

So, they helped us to analyze we also had tests of local maxima minima namely the second derivative test in terms of discriminant of the function and finally, we looked at absolute maxima and absolute minima and constrained maxima and minima and saw applications of this in various situations in economics commerce and management.

So, all the best revise the concepts.

Thank you very much.