Calculus for Economics, Commerce & Management
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Lecture - 40
Saddle points, derivative tests, absolute maxima and minima

Welcome back to our discussion of points of local maxima and local minima for a function of 2 variables. in the previous lecture we had seen a necessary condition for a function of 2 variables to have a local maxima and minima at a point. So, we said if a function of 2 variables has a has partial derivatives at a point, and if the point is a local point of local maxima or a minima, then the partial both the partial derivatives with respect to the variables x and y should be equal to 0.

So, this gave us the possible candidates like in one variable where the function can have local maxima or minima. So, the points or the interior points in the domain, where the partial derivative is exist and are equal to 0, or either of the partial derivatives does not exist, and the third possibility is the boundary points for the domain. So, we looked at some examples, and we saw there were some points which were neither local maxima or local minima, but the partial derivatives were 0 at that point.

So, let us that motivated us to look at what is called the saddle points for a function of 2 variables. So, we will define today what are the saddle point whatever what are called the saddle points for a function of 2 variables, and analyze them. So, a saddle point.

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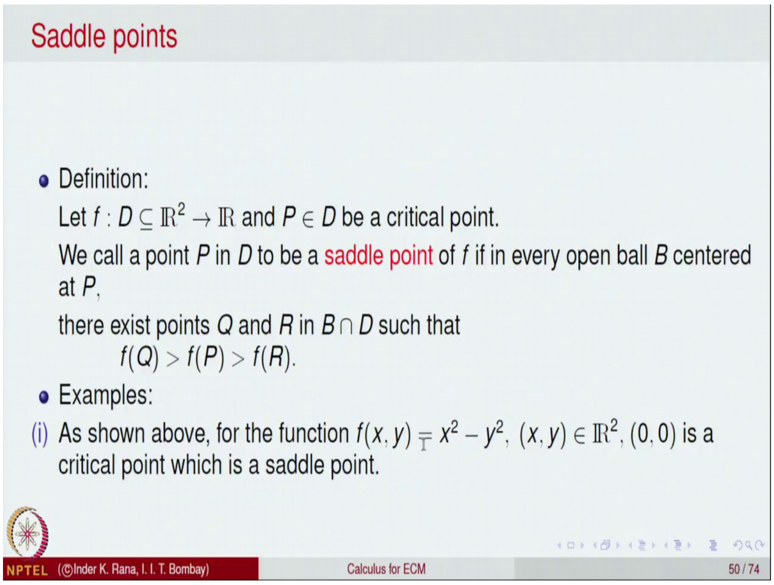

Is a; suppose f is a function of 2 variables from R 2 to R. And P is a point in the domain, and let P be a critical point. So, the critical point is the interior point where the derivatives are equal to 0. So, we call P to be a saddle point if for every open ball centered at P, there is a point Q in the and there is a point R in the ball of course, in the domains is that, the value at the point Q is bigger than the value at the point P, and is bigger than the value at the point P.

That means every neighborhood of that point P, we has a point where the value is bigger than the value at P, and the point R where the value is smaller than the value at the point P. Such points are called saddle points. For example, let us look at the function fxy equal to x square minus y square, where x and y belong to R 2. And for this if we look at the partial derivatives, they turn out to be 2 x and 2 y, and when they put them equal to 0, you will get 0 0 is a critical point. And our claim is that this is a saddle point. To show that, that is a obvious actually, because if I take a point close to the region and where y is 0. So, that is the point on the x axis, then the value will be positive. And if I take a point on the y axis then x will be 0, the value will be minus y square.

So, close to 0 we can have points where the value is positive and also, the points close to the origin where the value is negative. And the value at 0 0 the value is 0. So, 0 0 is a saddle point by the above definition.

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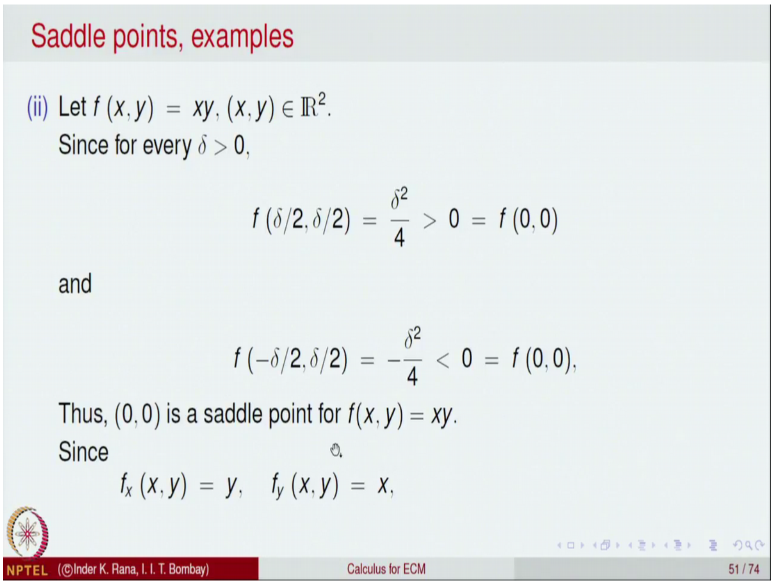

Let us look at to the curve fx or function f x y is equal to X Y. The domain of the function is the whole of R 2. And if I look at for every delta, look at the value of the function at the point delta by 2 delta by 2, then f of x y will be equal to delta square by 4, which is bigger than 0. At the same time, if I look at the value of the function at a point say minus delta by 2 and delta by 2, then the value of the function will be minus delta square by 4 which is less than 0.

So, close by choosing close delta sufficiently close to 0, we will see that we can find points closed in the domain of the function, close to the origin where the values are positive and another point where the value is negative. So, this means 0, 0 is also a point saddle point for this function of 2 variables f x y equal to x times y. note that 0 0 is a saddle point, and when if you look at the partial derivatives for this function, the partial derivative with respect to x is equal to y, and the partial derivative with respect to y is equal to x. And so, that means, 0 0 is in fact, a critical point. So, 0 0 is a critical point, but it is neither a maxima nor a minimum for the function. So, that means, 0 0 is a critical point and a saddle point.

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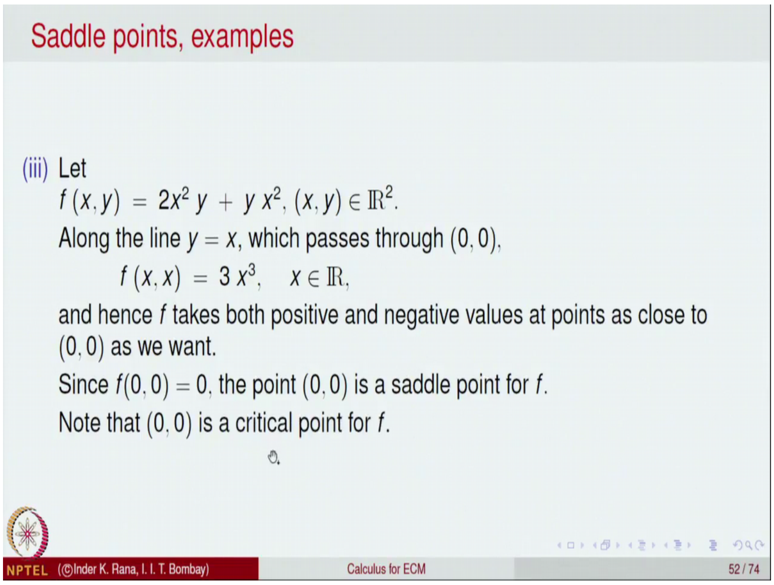

We can also look at some more examples like 2 x square y plus yx square. For this if I look at the line passing through the origin, the line y equal to x passing through the origin, then the function looks like x is equal to y. So, that is x cube 2 x cube and x cube. So, that is 3 x cube. So, that means, close to 0 0, if it is on the positive side it will take the positive value. And on the negative x negative close to 0 it will take the negative value. So, 0 0 is a point of is the saddle point for this function. So, this is the way we can analyze saddle points by definition. Namely these are the points where, which are critical points for the function, but the function is neither taking the local maxima or nor is a local minima at that point.

In fact, every point for every neighborhood there is a point where the value is bigger and the neighborhood where the value is smaller.

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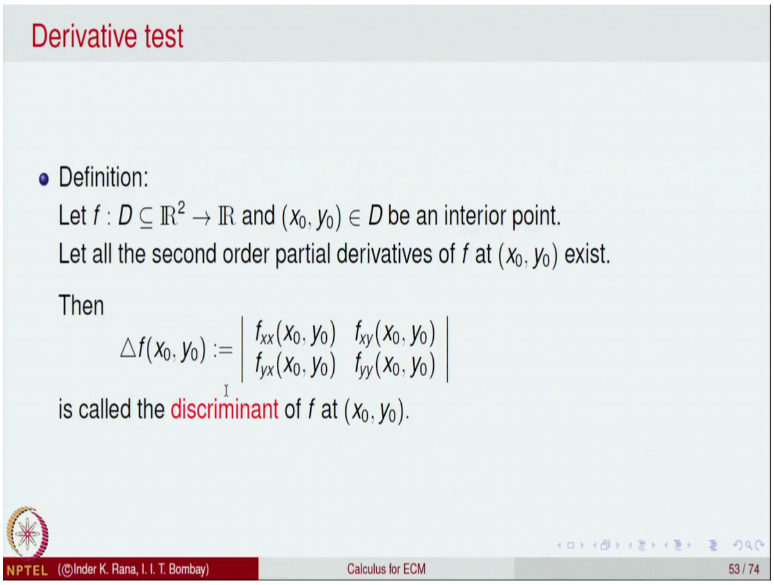

So, let us now look at what is called the derivative test for a function of 2 variables to be having a local maxima or local minima at a point. So, let us take a function of 2 variables f x y, and let us x naught y naught be a interior point. So, such that the both the second order partial derivatives of f at x 0 y 0 exist. Then we define a quantity called the discriminant it is a 2 by 2 determinant. So, that is defined as the this is a notation used for discriminant, and defined as 2 by 2 determinant, f where the first row is fxx the second order partial derivative at x 0 y 0 f x y the partial derivative at the point x 0 y 0 and here a cross is fy x the partial derivative at x 0, y 0, and fy y at x 0, y 0.

In case we have not seen this kind of object earlier, this is just you cross multiply the and you write. So, this is can also be written as fx x into f yy minus fxy f yy. So, that is the expanded form of this 2 by 2 determinant, a discriminant. And whether it is called the discriminant is a 2 by 2 determinant.

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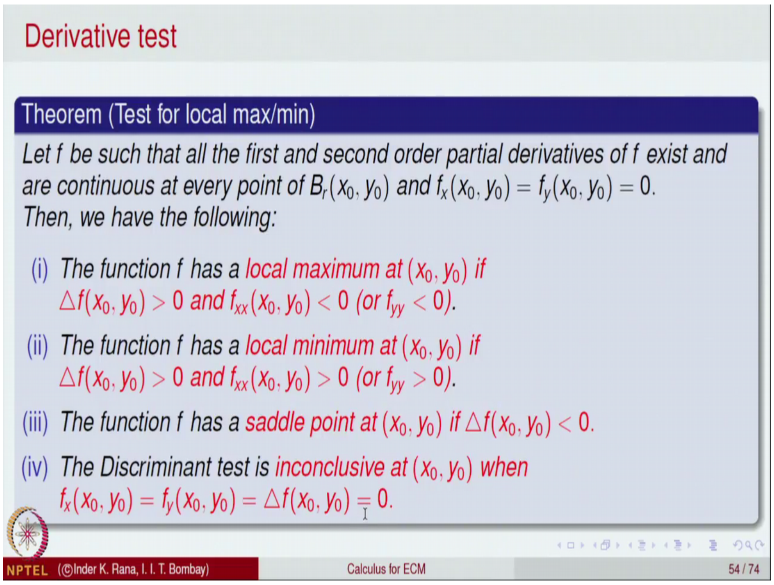

So, the derivative test is in terms of this discriminant, which is something similar to the second order derivative test for a function of one variable. So, let us state this test for the local maxima and local minima for a function of 2 variables. So, let f be a function of 2 variables such that all it is first and second order partial derivative they exist and are continuous at the point x 0 y 0, in a neighborhood of the point x 0 y 0.

Now, this condition basically is put to ensure that the mix second order partial derivatives are equal. So, that is a sufficient condition. So, we want that the second order partial derivative is exist, and are continuous at every point in a in a neighborhood of the point x 0 y 0. And the first order derivatives are 0. So, that means, this is a critical point. So, it is a critical point the second order partial derivatives exist, and our mixed partial derivatives are equal. So, we say that the test says that the function will have a local maxima at the point x 0 y 0, if so, this is a sufficient condition, if the discriminant at that point is bigger than 0, and the partial derivative fxx at that point is less than 0, or equivalently fy y is less than 0, either of it is enough to ensure.

So, these 2 conditions that the discriminant is bigger than 0, and the partial derivative xxx second order partial derivative with respect to x is less than 0. Or the partial derivative with respect to y is less than 0. So, this 2 conditions ensure that the function has a local maxima at the point x 0 y 0. if the discriminant is still bigger than 0, but the partial derivative at the point xxx a partial derivative second order partial derivative at the point x 0 y 0 is bigger than 0 or equivalently the second order partial derivative with respect to y is bigger than 0. Then this will ensure that the function has a local minimum at the point x 0 y 0.

So, this condition is for sufficient condition for local maxima, and this is a sufficient condition for local minimum. And in case if the discriminant is less than 0. So, this was bigger than 0 this is bigger than 0 this is in case the discriminant is less than 0, we can straight away conclude that the function will have a saddle point at the point x 0 y 0. And in case discriminant is equal to 0, no conclusion can be drawn, from the conditions of xxx a double derivatives or anything the function can have a local minimum function, can have a local minimum and may not have either of it if this condition is satisfied.

So, for a sufficient condition sufficiency condition for local maxima or local minima, first the discriminant should be positive. And if the second order derivatives fxx is less than 0, then it is a local minimum. If it is bigger than 0, it is a a local minimum. So, these are this is what is called a second order derivative test for local minimum and local minimum. We will see some applications of this how this is applied. So, let us look at an example f xy is equal to 4 xy minus x 4 minus y 4.

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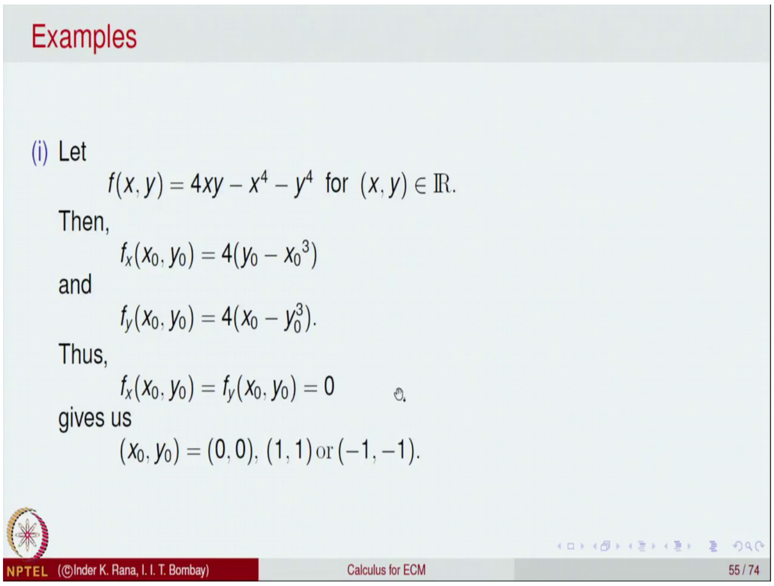

So, this function is the has partial derivatives of with respect to x and y at every point. So, let us find out the partial derivatives. And you find the partial derivatives with respect to x. So, it will be 4 x, right with respect to x. So, that will be 4 y minus 4 with respect to x. So, it will be 4 y minus 4 x cube.

So, this will be the derivative partial derivative at the point x 0 y 0. Similarly, when you take the partial derivative with respect to y with respect to y, it will be 4 x minus this will be 0, and this will be minus 4 x 4 y cube. So, that will be the partial derivative at the point x 0 y 0. So, to find the critical points we put them equal to 0, and once you put them equal to 0 and solve these equations from these 2 equations, it is easy to see that the solutions are 0 0 1 1 and minus 1 minus 1. We have not given the details here, but these 2 equal to 0, and we analyze because once it is equal to 0; that means, y 0 is equal to x cube. y 0 is equal to x cube and put the value here and solve and you will get the required values.

So, there are 3 critical points for this function 0 0 1 1 and minus 1 minus 1. So, let us look at the second order derivatives. The second order partial derivative with respect to x x. So, we will look at this point. So, with respect to x we want to differentiate. So, this will give you 0, and this will give you 3. So, minus 3 into 4 that is minus 12 x square and similarly the other partial derivative f yy at x 0 y 0.

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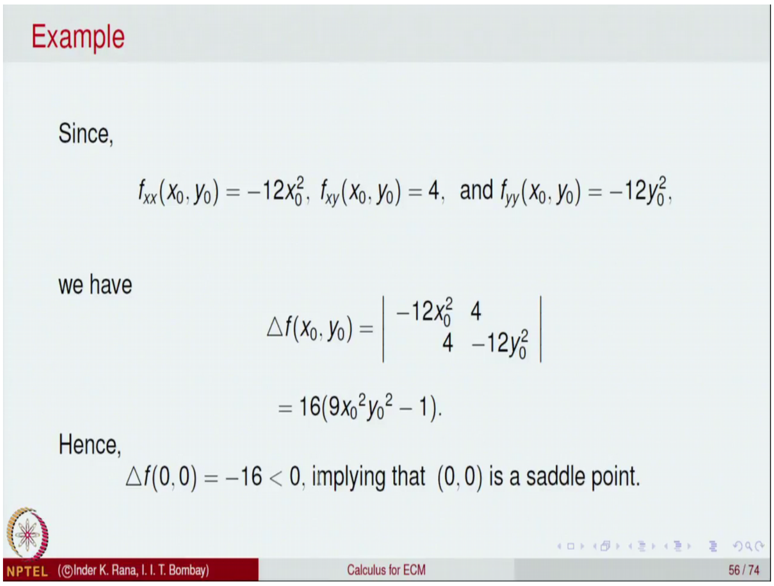

We can calculate and the partial derivative of f x y is equal to 4. So, all the 3 partial derivatives are available, under that because we had put those conditions that mixed partial derivatives will be equal once the partial derivatives exist and are continuous.

So, this is going to be the case. So, the nearly this is required. So, in the discriminant the first term will be the partial derivative with respect to x x. So, that is 12 x square the second term is the partial derivative x second order partial derivative that is equal to 4. So, the first term in the second row is the mixed partial derivative that is 4 again. And fy y that that is minus 12 y square. So, once you open that. So, and cross multiply; so, when you cross multiply 4 from where is common for so, 3 is common. So, that is 16 on the first-row or first column either one 4 is a common second also 4 is common so 16.

So, when you multiply this, this will give you 12 144 x square y square minus 16. So, 16 out is common. So, mine x square y square minus 1. So, that is a discriminant, and we have to analyze this discriminant and those critical points which were 0 0 1 1 and minus 1 minus 1. So, at 0 0 the value is minus 16 that is less than 0. So, we can safely conclude that the point 0 0 is a saddle point. Because discriminant less than 0 gave you conditions sufficient a condition was that there should be a saddle point. So now, let us look at the discriminant for the value 1 1, when it is 1 1 it is going to be positive.

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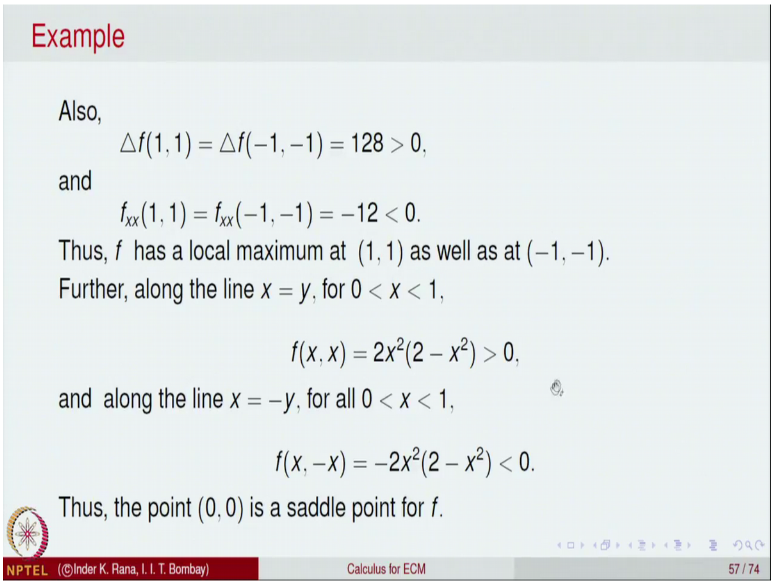

So, it is 128. So, the same value at the point minus 1 minus 1 because it is x naught square y naught square. So, it does not matter whether it is positive or negative. So, at 1 1 or minus 1 1 both the points the value of the discriminant is 128 which is bigger than 0.

So now let us analyze what is the value of f xx at the point 1 1, and similarly at the point minus 1 minus 1, which you when you put the value in this at when you put is one that is minus 12, when you put it one that is minus 1. So, you square so, either way it does not matter. So, whether plus 1 or minus 1 the value of the second order derivative is at 1 1 and minus 1 minus 1 are negative less than 0 so; that means what? That means, the points 1 1, and minus 1 minus 1 are points of local maxima. Because the discriminant at these points is bigger than 0 and the second order partial derivatives are less than 0.

So, these points are local maximum. So, both the points 1 1 and minus 1 minus 1 points of local maximum and for the function, we have already concluded 0 0 is a saddle point, but you can also analyze it by definition along the line y equal to x, fxx is it always bigger than 0. And along the line x is equal to minus y, the function f yy fx minus y is less than 0. So, as close as you want near 0 0 you can have a positive value at a point and a negative value at 0 0 the value being 0. So, the function has a saddle point at the point 0 0.

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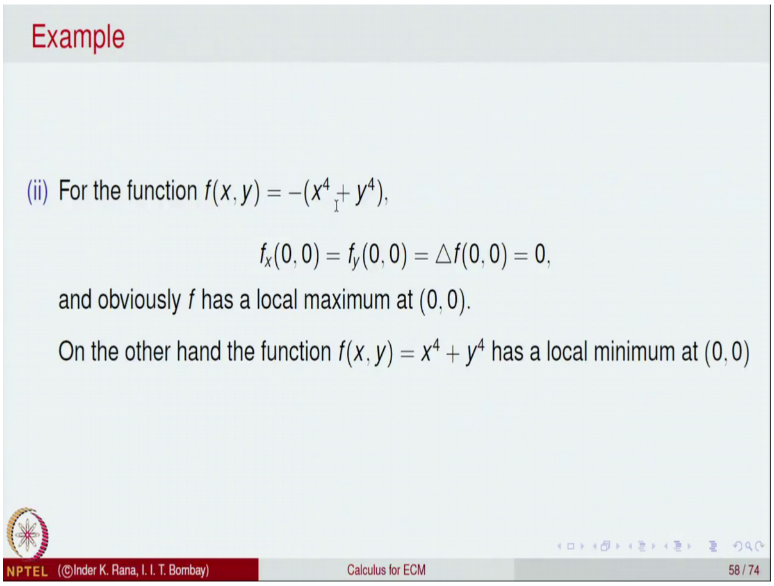

So, one can also analyze more examples like the function fx y equal to minus x 4 y 4. First order partial derivatives are equal to 0. And for this because it will give first order partial derivatives it will give you for x cube this will give you for y cube 4 2 equal to 0 0 0 is a critical point. And you calculate the second order partial derivatives, they are all 0. So, discriminant is also equal to 0. So, discriminant is 0. So, second derivative test is inconclusive for this case we cannot apply it; however, if we look at the function at 0 0 the value is 0, at every other point the value is negative. So, 0 0 is going to a point of local maximum for this function though the test fails.

Similarly, if I invert the function, change the sign here x square x 4 plus y for the all this will remain, the same the function will have a local maximum at the point 0 0. Because in that case f at 0 0 is 0, but f at every other point will have a positive value. So, the idea of this example is that when the discriminant is 0, the and even if the point is a critical point. It may be a local maxima it may be a local minima as just now I exhibited. So, finally, let us look at another example of this function, for x cube y minus 4 x y cube.

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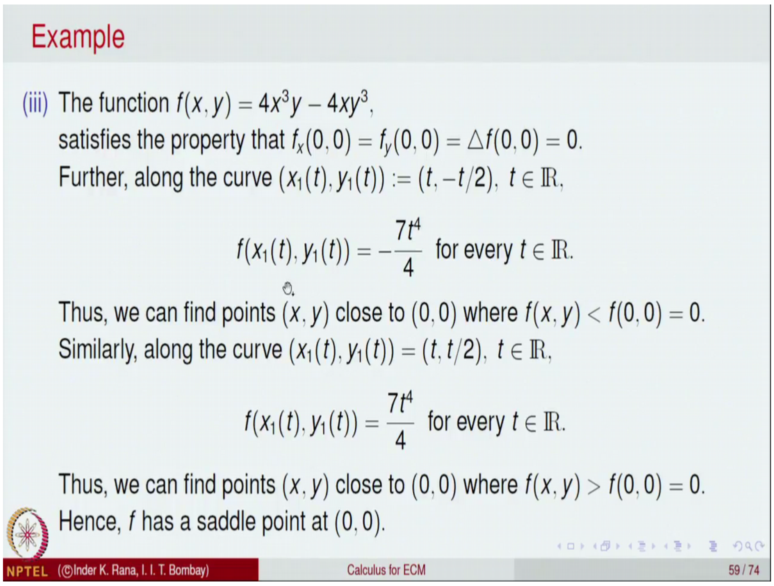

And calculate the partial derivatives, and they come out to be 0 discriminant comes out to be equal to 0. But at this the point 0 0, is a point of is a saddle point, which we can verify as follows, look at the values of the function along the curve t and comma t minus t by 2.

If I look, so, that means, x is equal to t and y is equal to minus t by so, this is the curve going through 0 0, and along this the value of the function is minus 74 by 4 which will be negative, for all points close to 0 0. And similarly, if we change t to t by 2, the curve is x one t is equal to t, and y 1 t equal to t by 2. So, along this curve which is passing through 0 0, the value of the function is 70 by 4 by 4. So, it will be positive. So, very close to 0 0 the function takes the both the values positive and negative values. So, 0 0 is a saddle point for this function, and the test fails.

So, these examples illustrate that the test may fail, but still the function may have a local maxima may have a local minima or a saddle point at the point where the test is failing.

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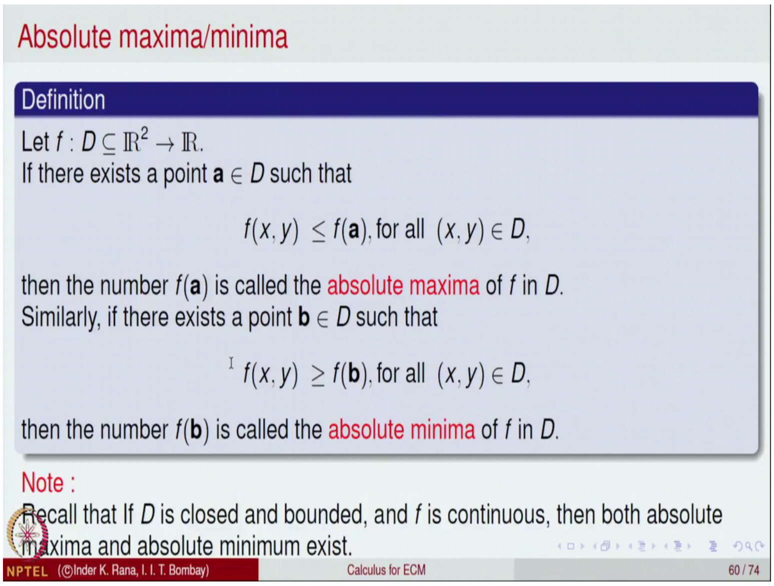

So, these examples illustrate those only conditions are sufficient, only when the discriminant is bigger than 0. Like one variable we can also look at the functions of 2 variables and define what is called an absolute maxima or a absolute minima of the function in the domain. So, f is a function defined in a domain R 2 to R, a point in the domain a vector a point a in the domain d is called a point of absolute maxima, if f of a is bigger than or equal to f of x y for all x y in the domain.

So, it is a global property. For all values in the domain f of a is the value which is the largest. So, this is called absolute maxima. And the point is called the point of absolute maxima. And similarly, you can define absolute minima to be the point be in the domain say that f of x y the value of the function at every other point is bigger than or equal to the value of the function at that point.

So, that point is called the absolute point of absolute minima, and the value is called the absolute minima for the function. Going parallel 2 functions of one variable and again, going parallel to the function of one variable, recall that if a function is defined in a closed bounded interval for one variable the absolute maxima and the absolute we need my exists. And same property is true for functions of 2 variables.

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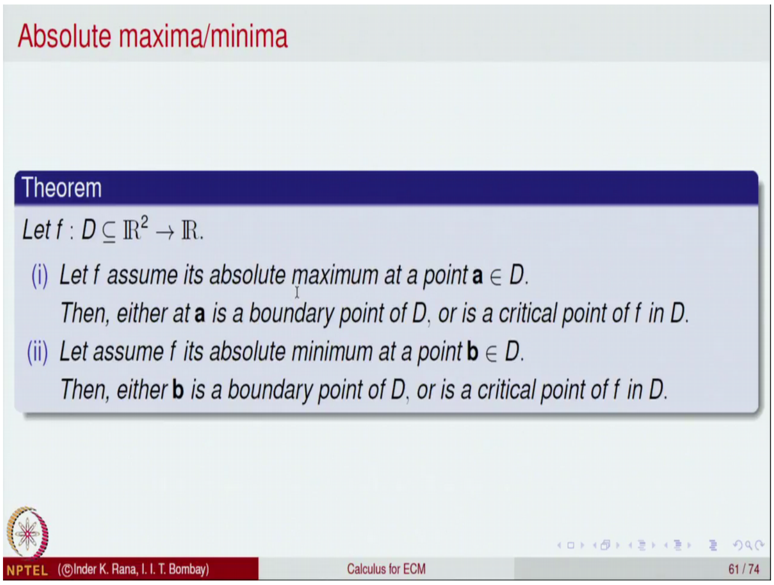

ao, what are the points where absolute maxima can occur? So, clearly a function is defined in a domain d in R 2. So, if a function has the absolute maxima at a point a. So, then the point a either is a boundary point or is a critical point for the function. So, that means, what either a should be a critical point or a is a boundary point it s a critical point meant, that either it is an interior point where the both partial order derivative for first order partial derivatives are equal to 0. Or the points where either of the partial derivatives does not at least one of the partial derivatives does not exist. Similarly, the for absolute minima.

So, that means, to find out the possible candidates where a function of 2 variables can have absolute maxima or absolute minima are the points, the boundary points or the critical points. So, to locate analyze a function and find it is a point of episode maxima and absolute minima, what one does is; look at the function find out it is critical points, find out the boundary points, and analyze the values of function at these points perfectly similar to one variable situation.

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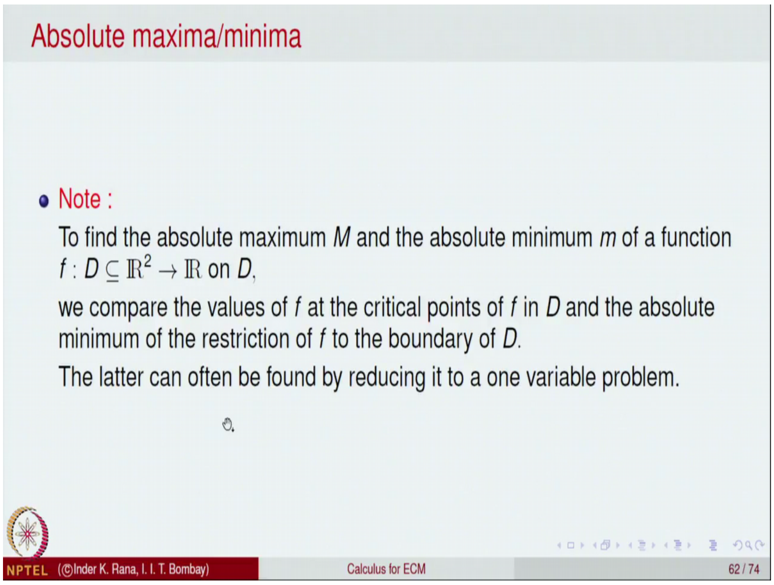

Some examples let us observe as I said we find out the critical points. We found out boundary points find out the values and compare ok.

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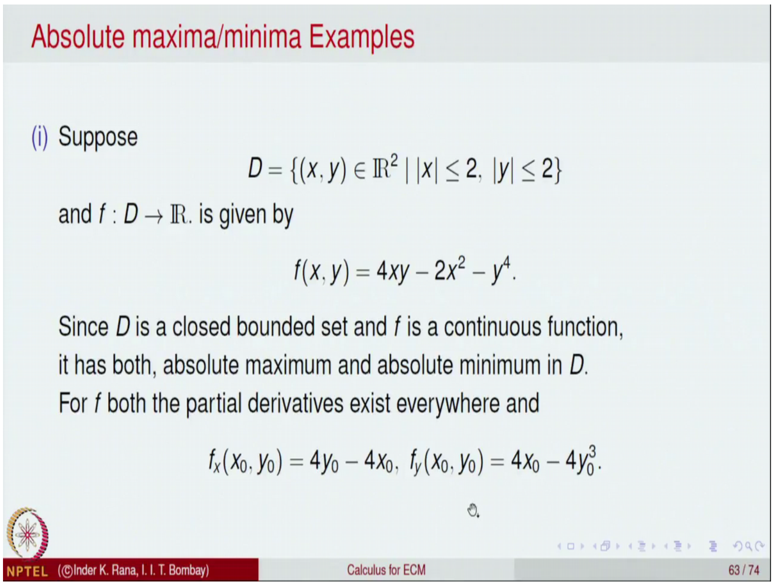

So, let us analyze there one example. So, domain of the function is given there is x y in R 2. Mod x less than 2 mod y less than 2; that means, it is a square in the plane centered at 0 0 with length x going up to 2 and on the side minus 2.

So, the 4 corners are given by 2 0 minus 2 0, and similarly at the top it will be right. So, centered at the origin going up 2 units going down 2 units so, that is the square and here is the origin. So, that is that so, the function given to us is 4 xy minus 2 y square minus y 4. So, we want to find out the absolute maximum of this function in this domain. So, this domain has a interior and as well as a boundary the interior is when mod x is strictly less than 2, and mod y is strictly less than 2. So, that is a inside of the square, and the boundary of the square is the normal boundary of the square, where mod x is equal to mod y is equal to 2.

So, let us analyze that. So, it is a continuous function. So, it will have absolute maximum and absolute maximum. And to find out that, let us look at the partial derivatives. So, these are the partial derivatives of the function of 2 variables.

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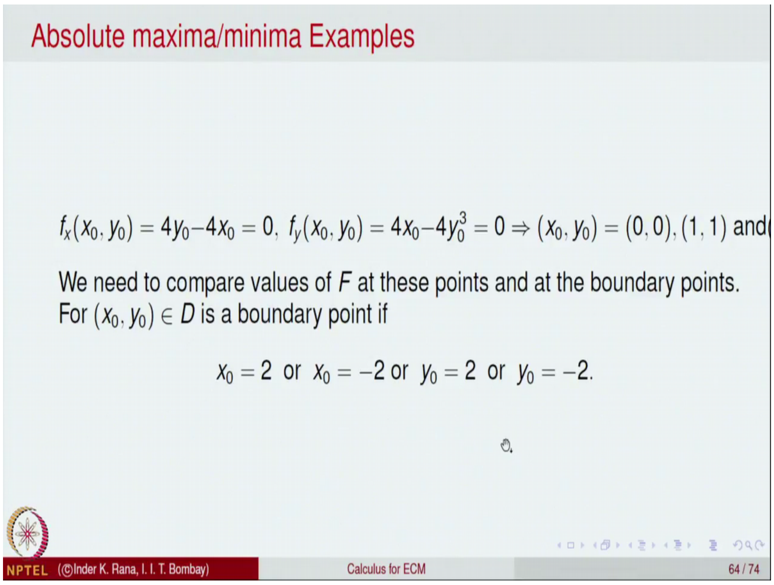

And when you put them equal to 0, that gives you 0 0 1 0 1 1 and minus 1 minus 1. So, therefore, lo 4th point is not visible here. So, that is so, we need to compare the values at these points. So, and the boundary points on a boundary point x is equal to 2, or x is equal to minus 2, y equal to 2 and y equal to minus 2.

So, when we put that values in the function.

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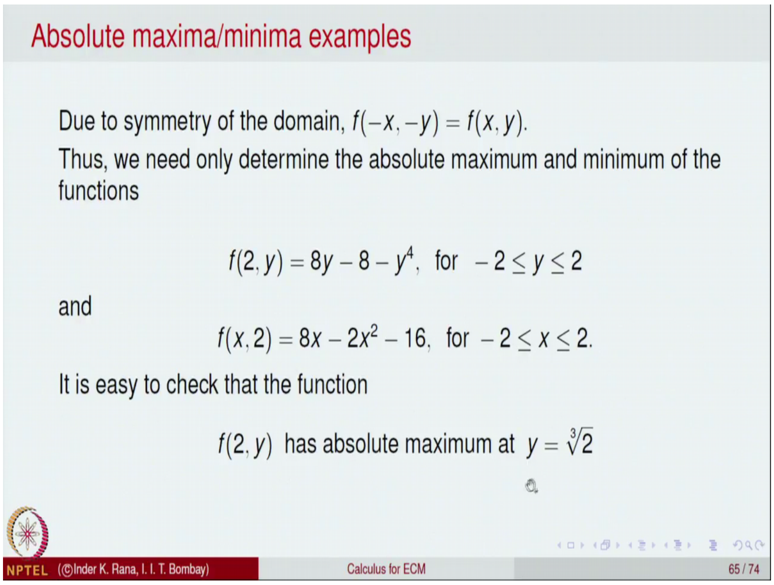

So, and function is symmetric with respect to x and y. So, we need to determine the absolute maximum and absolute maximum of the function, we have to only find out f of 2 y. So, there is a boundary point when x is equal to 2 that vertical line is given by this, and y goes between minus 2 to 2. So, as a function of one variable and similarly for x 2 when y is equal to 2 so, that is the vertical line at the top, and x is varying between minus 2 to 2. So far as for a function of one variable, we can analyze the conditions for this to be equal to 0, and find out epsilon. So, one can check we leave it for you to check that as a function of one variable in y, it has a absolute maximum at the point y equal to cube root of 2.

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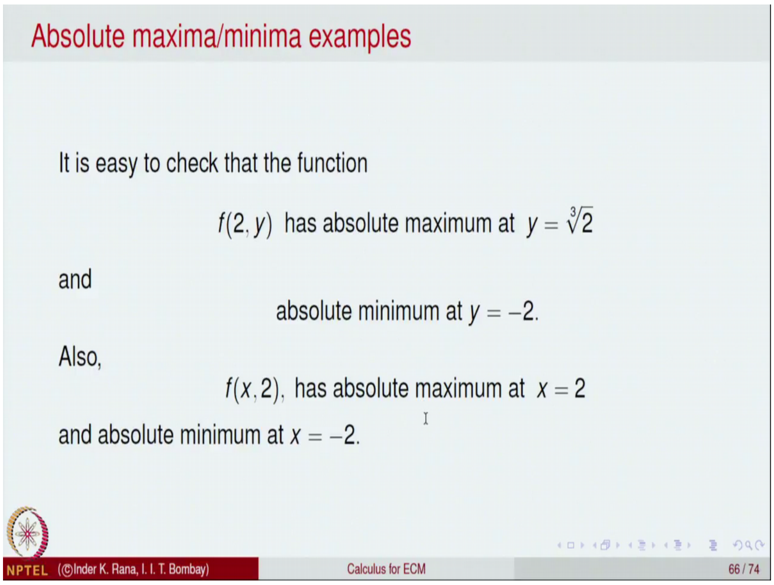

And similarly f 2 y has a absolute maximum at the point y is equal to cube root of 2 and the maximum at the point y equal to minus 2.

So, this is a function of one variable analysis will give you these values. So now, we can compare all the values of the function at all the points f 0 0 is 0 f 1 1 is 1, f 2 cube root of 2 is equal to minus 40 f 2 2 it is equal to.

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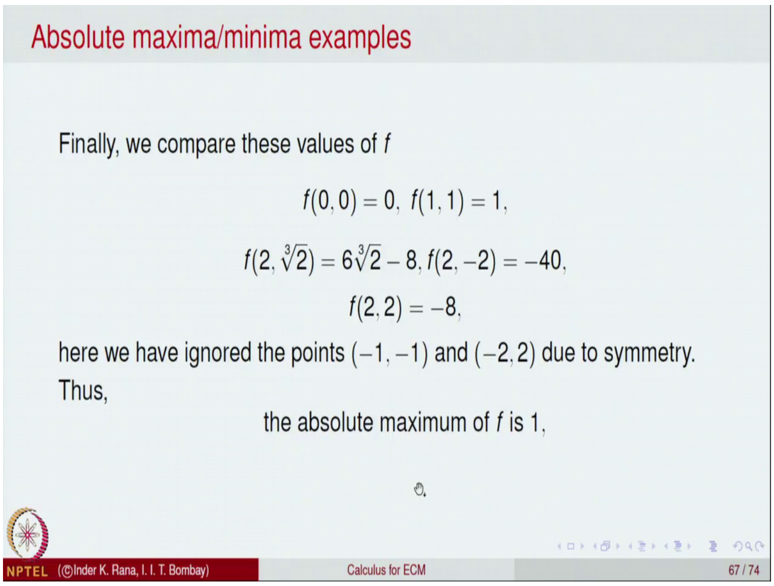

So, at all the points boundary points and the critical points find the values and compare. So, we see that f of and also, we had we have ignored the points minus 1 and minus 2 2 because of symmetry. So, we can include that also. So, that means, epsilon maximum f is 1, and at the point 1 1 and minus 1 minus 1.

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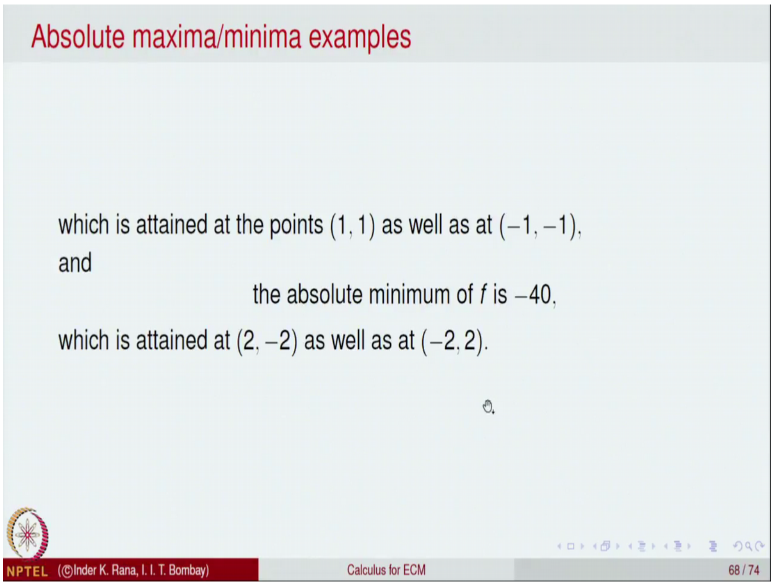

And similarly, it has a absolute maximum is minus 40 at the point 2 minus 2, minus 2, minus 2. So, this is how you analyze absolute maximum and absolute maximum of functions of 2 variables. will see some example of this in our economics model also.

Thank you.