Calculus for Economics, Commerce and Management
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Lecture – 04
Operations on sets, cardinal number, real numbers

So, welcome to today's lecture. Let us just recall what we are done last time. We had defined the notion of A set. We had defined various ways of representing a set. We had looked at various operations on sets some of them we mention again, union of sets intersection of sets, compliment of a set and the notion of disjoint sets.

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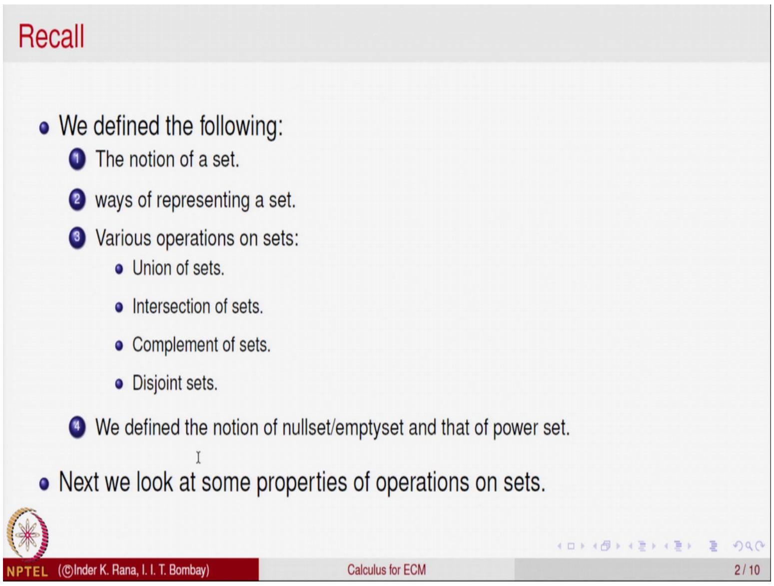

Let us just some of the things that we dint do last time. These are called operations on sets. Given sets you can generate more sets out of them by various operations. So, let us look at some of them.

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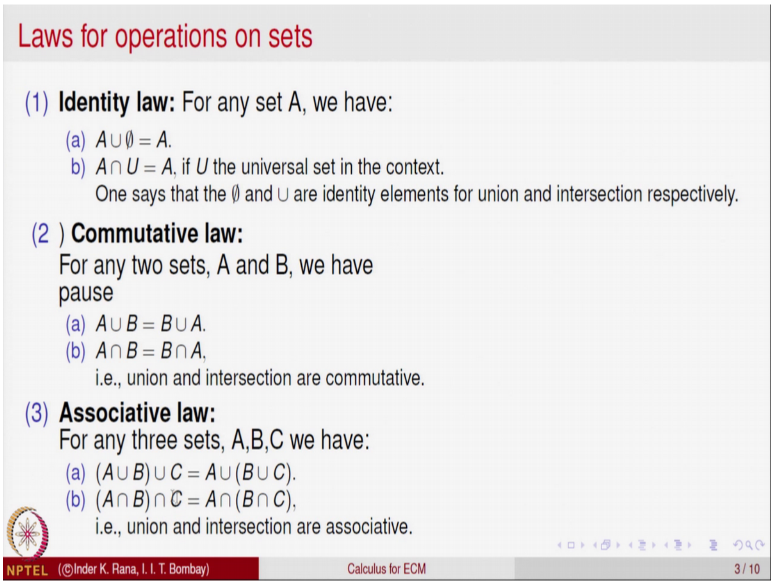

These are also called laws if you want to call them. So, first is called the identity law; that means, if you take any set A and take it is union with the empty set, then it is A itself. That is quite clear, because empty set has no elements.

So, you are not adding anything to A, it is just A itself. The next one says if you take U a universal set; that means, that A is a subset of U, then A intersection U is A. Because A is a subset of U right. So, you are going to pickup those elements in U which are also in A. So, that is A itself. So, A intersection U is A itself when U is the universal set. Or even when you can say that A is the subset of b, then also it is true. So, you can say that phi this is null set or empty set is also called phi, this is the Greek letter phi. So, phi is an identity and identity elements for the union and the intersection. If the commutative laws for set operations are as follows the first one says for any 2 sets A and b, A union B is same as B union A.

So, essentially to form the set A union b, what you are going to do is pick up the elements of a pick up the elements of B and put them together in one set. So, whether you taking elements of a first and then elements of B or elements of B first. And then elements of a does not matter you are going to put all of the elements of A and of B together in one set. So, this seems an property. And similarly, A intersection B is same as B intersection A. So, what is common between A and B is same as common between B and A. So, that is right. So, basically what we saying is the logical terminology A and B is same as B and A. And A or B is same as B or A. So, these are the 2 laws which gives us commutatively. Associative law says that you can perform operations of union or intersection in any order you will get the same result.

That means if you given 3 sets A B and C, then you take A and B and take their union. So, we will get a new set and take it is union with C, then it is same as first take B and C take their union. So, we will get B union C, and then take it is reunion with A. So, both will give you the same result. So, the set is same. So, A union B union C is same as A union B union C. And similarly, for intersection, A intersection B intersection C is same as A intersection B intersection C. So, these are all simple observations which one can right down proves analytically also, but does not bother too much about these proves. We understand that associative means the order in which you do the operation of union for 3 sets does not matter. And similarly, the order or intersection of 3 sets does not matter you will get the same outcome.

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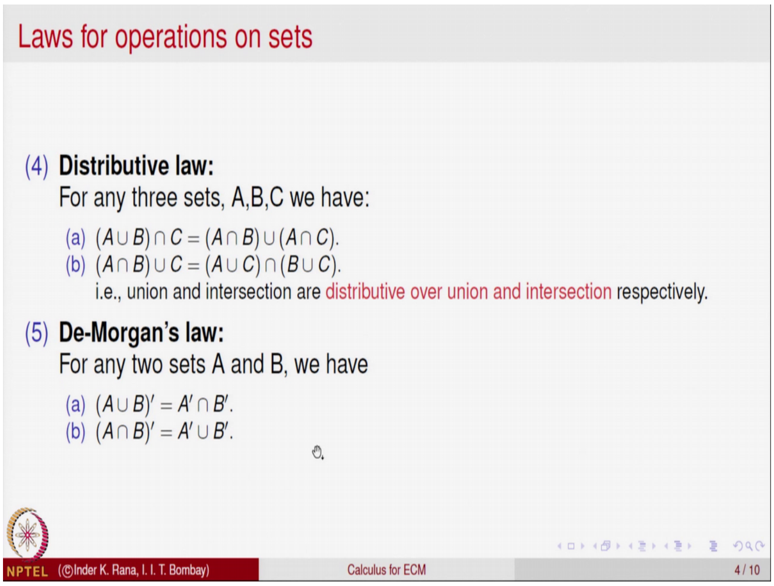

Here is a distributive property which is something similar to distribution of product over some and in the numbers. So, let us look at that for any 3 sets A B and C. If you take union of the sets A and B say I take intersection of that with C, then it is same as first taking intersection of A with C. And then taking intersection of B and C and putting them together in union. So, it says this operation of intersection right. So, A union B it distributes over both of them. So, A union B intersection C, is same as A intersection C union B intersection C. And similarly, the distributive property of union over intersection. So, A intersection B and if you take union of this with set C is same as A union C intersection with B union C.

So, these are called the distributive properties, how these operations distribute over each other. So, one says union and intersection are distributive over union and intersection respectively. There are de Morgan laws, which relates to compliments of a set. So, it is says for any 2 sets A and B. If you take the union and then take the compliment. It is same as taking the compliments first and then taking their intersection. So, A union B compliment is equal to A compliment intersection B compliment right. So, this says if you want to interchange the order of union and compliment, then you have to bring in intersection in between. And similarly, A intersection B take first the intersection of 2 sets and then take their compliment it is same as taking the compliments of each one of them and taking their union.

So, this is these are called de Morgan laws, and they are useful in operating with sets. So, basically what we are saying is given the collection of sets, you can generate new sets out of various operations and these are governed by these laws. So, let us just go back and say something again about this laws, one is identity that A union phi is A and A intersection the universal set is A. So, universal set is identity for intersection and empty set is the identity for union. Similarly, commutative law says A union B is B union A and A intersection B is B intersection A. These laws are basically something similar to keep in mind a similar thing that happens in addition and multiplication of numbers.

So, for example, you if you treat union as the plus, then any number plus 0 is equal to that number. And similarly, this is multiplication a multiplied by U, U is if you take it as 1, then it is A. So, this is basically going parallel to that. And so, addition is commutative multiplication is commutative. So, here the union is commutative intersection is commutative. Similarly, the associatively of intersection, and the union and then the distributive properties of addition of intersection over union, and distributive property of union of over intersection. So, these are the basic laws and then de Morgan laws, our compliments namely A union B compliment is same as A compliment intersection B compliment. And similarly, A intersection B compliment is same as A union B. A compliment union B compliment right.

So, this operations on sets we are not giving any formal proves of these things. So, what we will use them as an when required, right. There is also notion of the cardinal of a number. Cardinal number of a set that essentially is say how many elements are there in a set.

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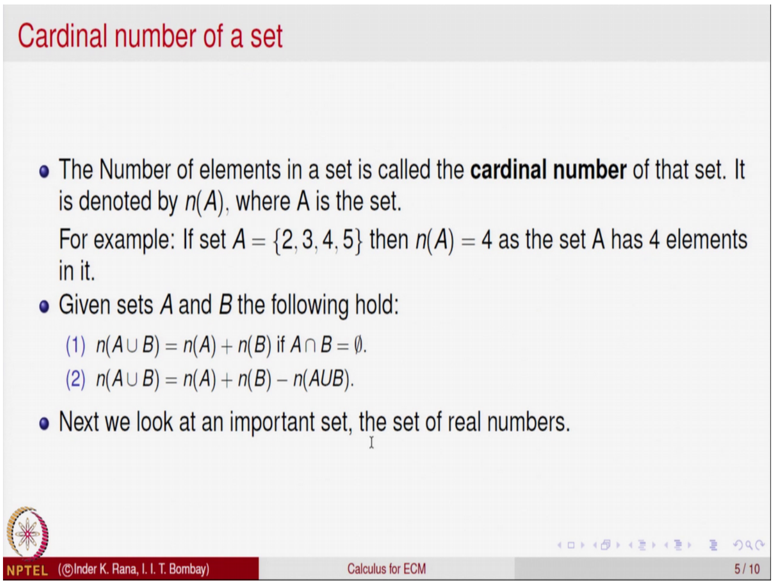

So, with the cardinal number of a set is denoted by n of A where A is a set keep in mind this makes a sense only when the set is a finite set. So, we are not going into detail what is called a finite set and what is a infinite set. So, intuitively if you can count the number of elements in a set. Then you associate a number called n of A with it, right. That is called the cardinality of the set or the cardinal number of the set. For example, for the set A 2 4 2 3 4 5 it has 4 elements 1 2 3 and 4.

So, n of A is equal to 4. So, given 2 sets A and b, here is some properties of the number of elements. It says the number of elements in A union B is equal to number of elements in a plus number of elements in b, provided there is nothing common between them. Because if there is something common then you will be counting it twice. So, for that you will have to modify in general and number of elements in A union and B is number of elements in a plus number of elements in b, and here there is a problem this is a typo it should be number of elements of A intersection B.

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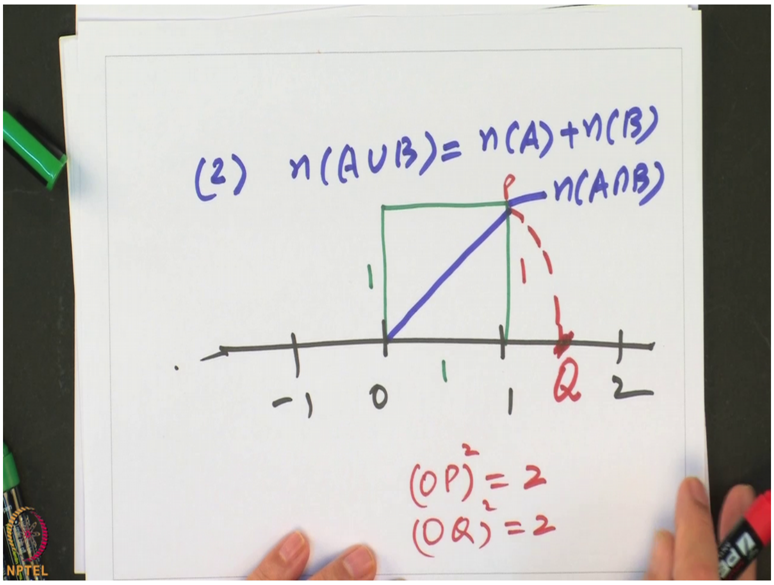

So, let me just write out the correct statement, the second statement should be written as the number of elements in A union B is equal to number of elements in a plus number of elements in B minus number of elements in A intersection B. So, this is the correct statement which should replace here. So, essentially this is because when you are looking at A union B. That is the collection of All elements of A and B together. So, you will be counting elements of a there you will be counting elements of b, but if there are some elements which are in both; that is, A intersection B, then you will be counting them twice. So, you have to replace them. So, you have to subtract once those elements in the number.

So, these are basic set theory and basic properties of sets their operations and their loss. So, you can pick up any book, you I have of standard 11th tenth or 11th where set theory is discussed. All these are discuss very nicely in that in in books. For example, NCRT 11th standard book. I think should we dealing with these kind of things. And so, revise basics set theory, because we will have opportunity to use them later stages. So, next what we are going to do? Yes, we are going to describe a very important set, which is going to play a fundamental role in our subject. That is a set of real numbers. If I ask if you think you know real numbers, then there is a slight probably discrepancy or slight gap in your knowledge. Because, normally what are what is a real number is not a that is easier job to describe.

So, we will also not be describing completely, what is a real number? How these are obtained and so on, but we will treat real numbers as a set with certain properties. So, we will describe now what are the real numbers as I said and what are it is properties. So, let us look at that.

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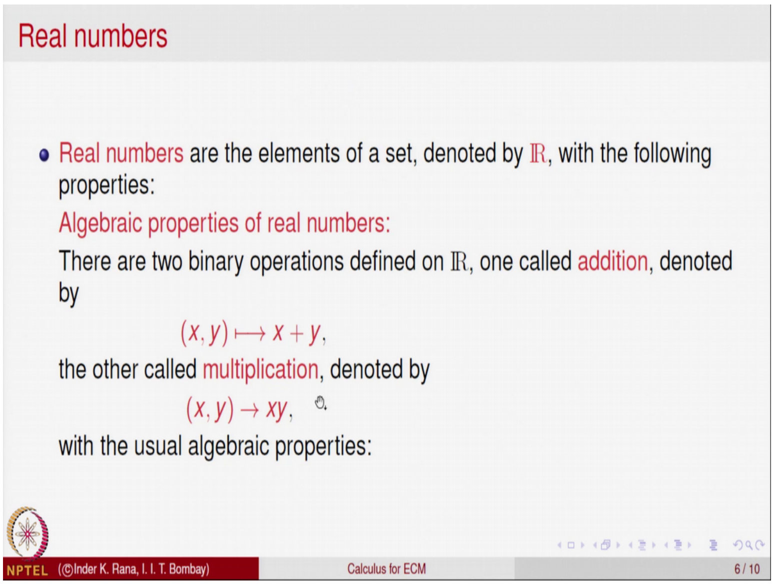

So, real numbers are the elements of a set, and this set is denoted by script R. This is a English alphabet R, with a one, one additional line on the side to indicate that this is a special set this is a set of all real numbers. It has the following properties first of all there are some algebraic properties associated with these set; that means what? That on this set R there are 2 binary operations defined. One is called addition. So, that means, what given 2 elements 2 elements section y of this set you can generate a new element called x plus y.

So, this is operation of addition. So, x plus y is again an element of the set R. So, binary operation means this is for a pair of elements right x comma y, you can find you can manufacture a new element called x plus y. This is a association for a pair x comma y you can associate a number x plus y n R itself. So, this is called the sum. So, and the other one is for the pair x comma y, you can associate a number which will denoted by x y. So, that is called the product of the 2 elements x and y. So, there is a operation of addition there is a operation of multiplication.

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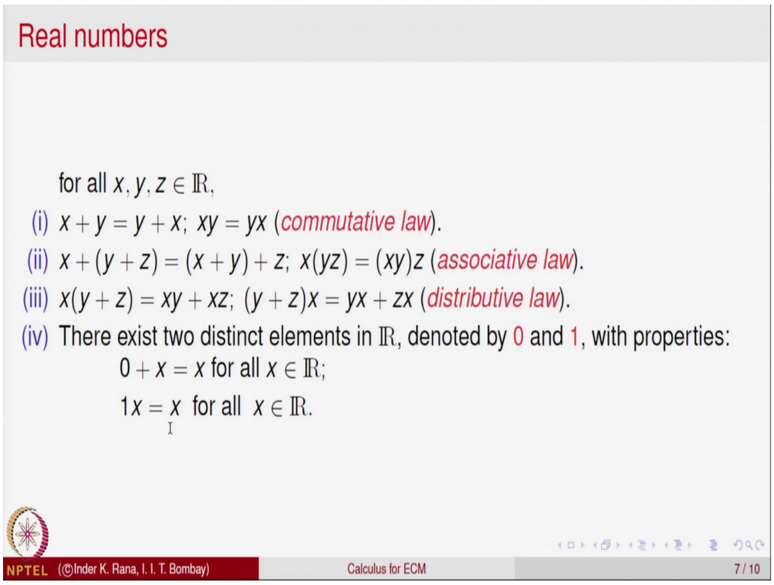

These are 2 binary operations called binary operations on the set R with the following properties. The properties are as follows for all any 3 elements x y and z in R x plus y is same as x y plus x. And x y is same as y into x; that means, if you take the sum of x and y it is same as the sum of y and x. And similarly, the product of x and y is same as the product y x. This is called the commutative law. And similarly, and next is what is called the associative law; that means, given 3 elements x y and z in R, you can take their sum in any way you like.

So, first take the sum of y and z, and then add to it x or you can first take the sum of x and y, and then add to it z you will get the same number. And similarly, if you take the product of y and z first. And then take it is product to be x that is same as taking the product of x and y first, and then taking it is product with z. So, these 2 are commutative properties, sorry, associative properties of addition and multiplication. The next is what is called the distributive properties, namely multiplication and addition distribute over each other. So, the namely x multiplied by y plus z if you multiply the sum of y plus z with x.

That is same as multiplying x with y first and multiplying x with z and adding those 2 products to get x plus x y plus x z. So, this multiplication here distributes over radiation. Similarly, you can think of multiplication from the left or addition from the right or from the left some from the left. So, y plus z you first take the sum, and then multiply by x that is same as y x plus z x. So, this is called the distributive property. So, what we saying is that operations of addition and multiplication are commutative associative and distributive over each other. There are 2 unique elements in the set R, they are denoted by 0 and 1 with the following properties namely. If you add 0 to any element x that is x itself. And similarly, if you multiply any element x with 1, that is this x itself.

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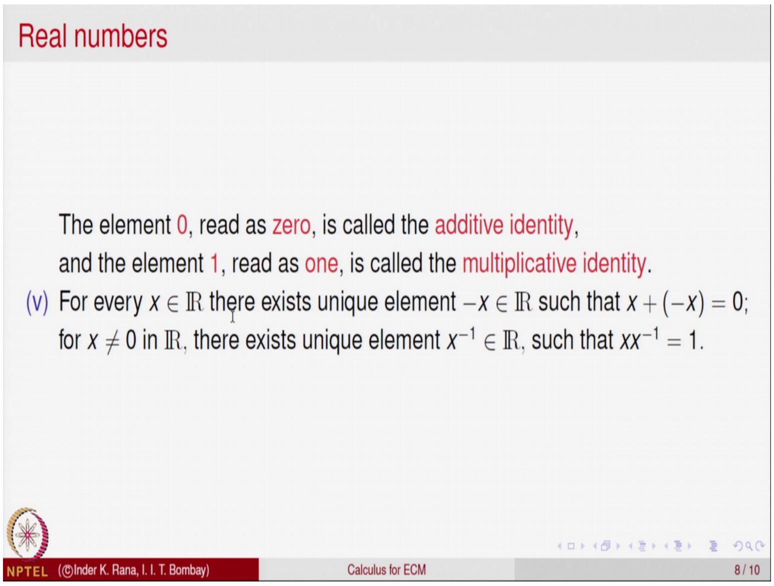

So, 0 1 says 0 is identity for additive identity and one is the multiplicative identity; that means, x plus 0 is x, and one into x is a x for every real number x. And next what you want to say is that for every x belonging to R for every number for every element in R, there is a unique element denoted as minus x, this is red as minus x which is again an element of R such that x plus minus x is equal to 0. And similarly, for every non-0 element x which is not equal to 0, 0 with additive identity. There is a unique element called x inverse. So, this is x upper minus this is called x inverse or it is also sometimes called 1 over x. So, such that x multiplied by x inverse is equal to 1.

So, this number minus x is called the additive inverse, and the number x upper minus 1 is called the multiplicative inverse of the number x. So, every real every element in R has a additive inverse, and every non-0 element in R has a multiplicative inverse. So, these are the algebraic properties. Next, we discuss what are called the order properties of these numbers of this elements in R.

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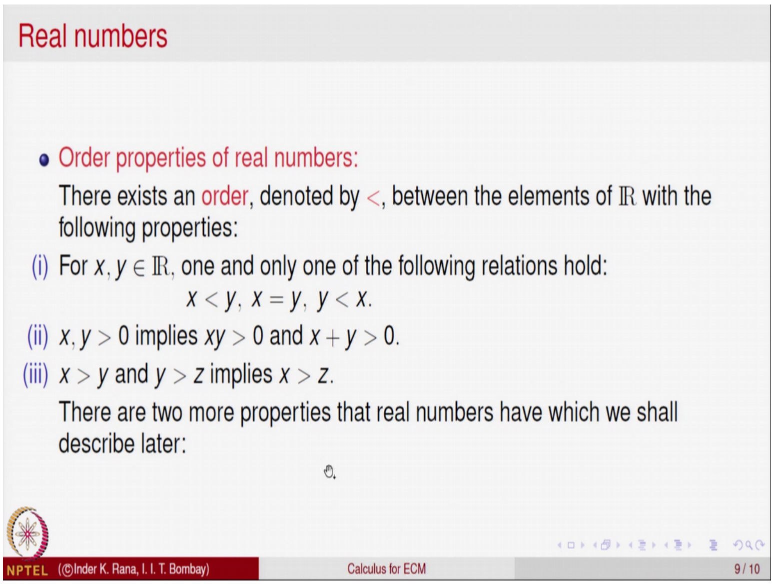

It says that there is a order among the elements of R; that means, given 2 elements x and y, you can say either x is less than y, or x is equal to y or y is equal to x. So, in order means any 2 elements in the set R can be compared, but the property says that only one of this properties will be true either x is same as y. Or you can say x is less than y or y is less than x. And this order how does it interact with the addition and multiplication, say you say if that x and y are bigger than 0, you remember 0 was the additive entity.

So, if x and y are elements which are bigger than 0, right there is an order. So, bigger than 0 than x product and sum is also bigger than 0. So, normally in such elements are called positive elements. So, if x is bigger than 0 y is bigger than 0, then these elements are called positive elements. And so, the properties says that the sum of positive elements is positive, the product of positive elements is also positive. And finally, there is a law of transitivity. That says if x is bigger than y, and y is bigger than z, then x is bigger than z. There are 2 more properties of the real numbers which we shall describe later. But before I do that let me just mention that what we are done is we have formalized the properties of numbers that we have been using since our education in school as in terms of a set theory and binary operations. There are 2 binary operations of addition and multiplication there is a order with certain properties.

So, what are those 2 extra properties special to it one is called the Archimedean property. And the other is called the completeness property. These 2 properties we shall describe soon we need say bit of more machinery a bit of more mathematics to describe these 2 properties. So, we will do it soon. So, keep in mind real numbers the set is A is a set of R with 2 operations addition and multiplication with those properties, and there is an order on it with problems some properties. And there are a properties called Archimedean and completeness that will describe.

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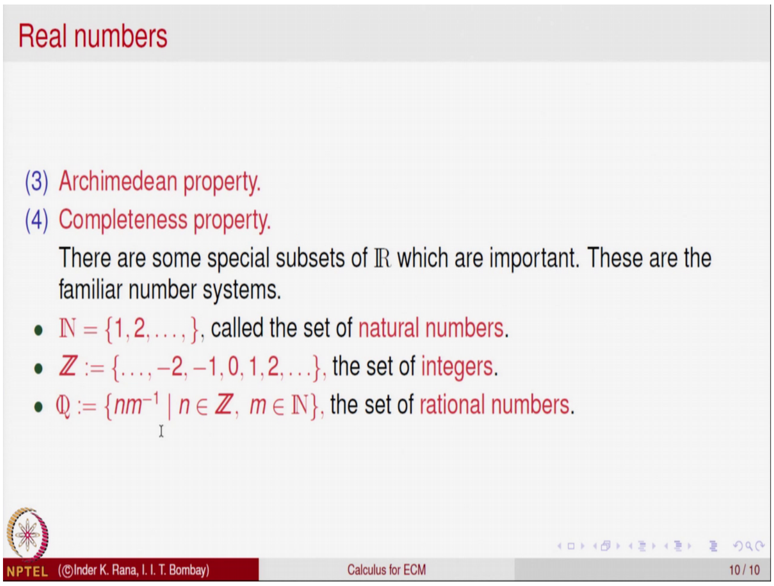

So, let us describe some special subset of this set R, which will play a role in our study. We in fact, we start with natural numbers in our school education. Here we are doing the other way around we already have a set R is with this properties.

So, in this set R we already have the multiplicative identity 1, right. If you add one to itself you will get a new number that we denoted by 2, and to the number 2 if you add one itself again you will get some new number called 3. So, this is the set of natural number. So, among this set of real numbers R we are identifying our natural numbers. So, these are the numbers starting with the multiplicative identity, and inductively adding one to it the result. So, 1 2 3 these are the notation for the natural numbers. Once you have the natural numbers we have the multiplicative we have the additive identity is 0. So, 0 1 2 3 if you take those things these are normally called whole numbers. And for one there is a additive inverse minus 1, for 2 there is a additive inverse minus 2. So, we put together all this things this is set of integers.

So, set of integers is the set of natural numbers along with 0, and along with the additive inverses of another natural numbers. So, this is denoted by z, it is a English alphabet z with a script. So, it is called z n natural number z integers, and then we have the fractions. So, how does a fractions appear in our number system. So, given a number n which is a integer, which could be positive and negative. And look at a number m which is a natural number. So, because m is a natural number it is not equal to 0. So, it will have a multiplicative inverse m minus 1. So, take the product of this this is you get a new number. So, this numbers collection of All such numbers is denoted by Q script Q and call the set of rational numbers. In our familiar notation if m universe is written as 1 over m, then this is just n over m, right the fractions that we are familiar with.

So, this is a set of rational numbers.

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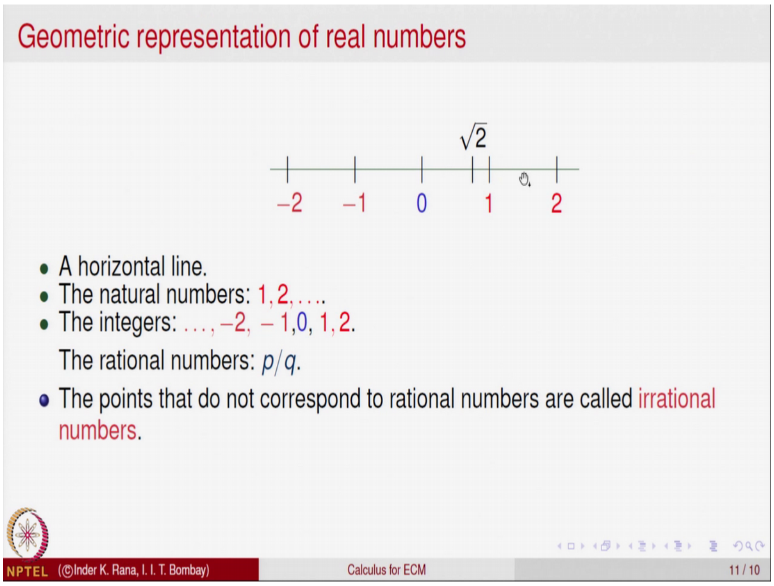

Let us try to give a geometrical representation of our numbers. So, we draw a horizontal line. So, this is a horizontal line, on this line you make cuts take any point on this line and call it as 1, take some unit and make mark a point and call this is as 2. So, you called this unit of distance, same distance you go to the next step on the right and on the left mark points. So, this you call as 0 on the left side the next point you call as minus 1 the next as minus 2 and so on. So, what we are done is, we have put all the integers on a horizontal line we are given them a position on the horizontal line, right. You could have started with 0 also. May I have kept mark a point and call it 0 equidistant points call on the right-side call as 1 2 3 so on. On the left equidistance points call minus 1 minus 2 minus 3 and so on.

So, integers have been given some position on the number line. Next let us look at fractions which are rational numbers. So, if you think of a rational number as p by q. So, what we are doing is exposing this number is between 0 and 1 p by q is a number exposing it is between 0 is bigger than 0 and less than 1. Then it will we want to put a give a position to it somewhere here. So, that means, what we do is 0 to 1 is divided into q equal parts. So, divided into smaller parts q equal parts and in the p th part will denote p by q. So, that is the fraction that is a part of 1 over q the one of the unit divided into q partition the p th part. So, this way every fraction gets represented by a point on the number line, but there are some points on the number line.

So, what we have done is we have put fractions rationales on the number line. The question is does this fill up the whole number line; that means, can I say conversely every point on the line gets represented by a rational. That claim is not true, and that was the beginning of discovery of real numbers. So, what we are saying is there are points on the line, which do not represent a fraction. And discovery of this things these points are normally called irrational numbers.

So, for example, square root 2 is one such number how do you get square root 2 is this this position is shown wrong, it should be on the right side of 1 by this unit. So, basically if you take a square root of unit length and take the diagonal that diagonal will have length square equal to 2. So, if you draw it here this point, you will get a point here whose square is equal to 2, right. The and one can prove that this number this point which you get is A is not a fraction.

So, basically let me just draw a picture and show it to you. So, namely what we are saying is on this line, you have made point 0 1 2 minus 1 and so on. So, if you draw a square of unit length that this is 1, this is 1. And if you take the; if you take this diagonal the length of the diagonal will be equal to. So, by Pythagoras theorem this is one this is 1. So, OP square will be equal to 2. So, if I draw this point. So, we will get a point here Q such that OQ square is equal to 2.

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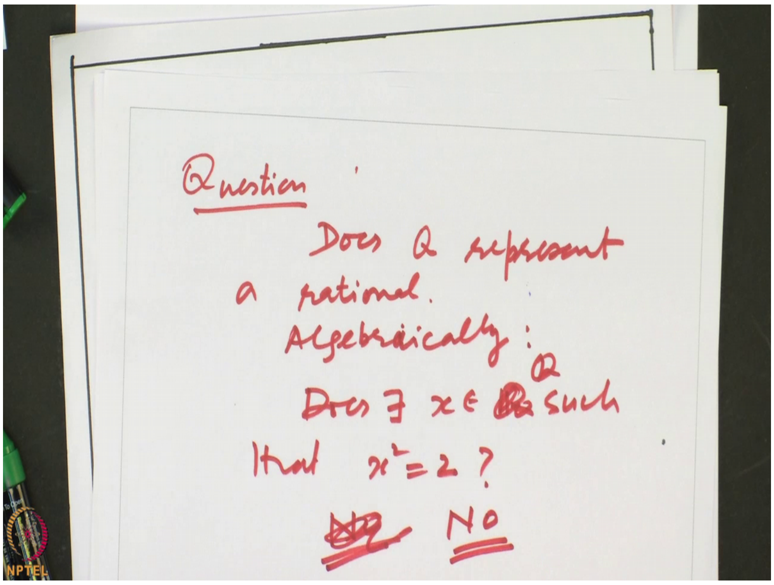

That means OQ, what is this point Q? Does it represent a rational? So, question, does Q represent a rational? Geometrically this means analytically, algebraically. This means does there exist a number x belonging to R such that x square is equal to 2. No, no, sorry, the rational number in Q such that the answer is no. So, that is why one have to invent. So, there is no rational number whose square is equal to 2. So, one at to invent some numbers beyond rational numbers and they are called the irrational numbers.

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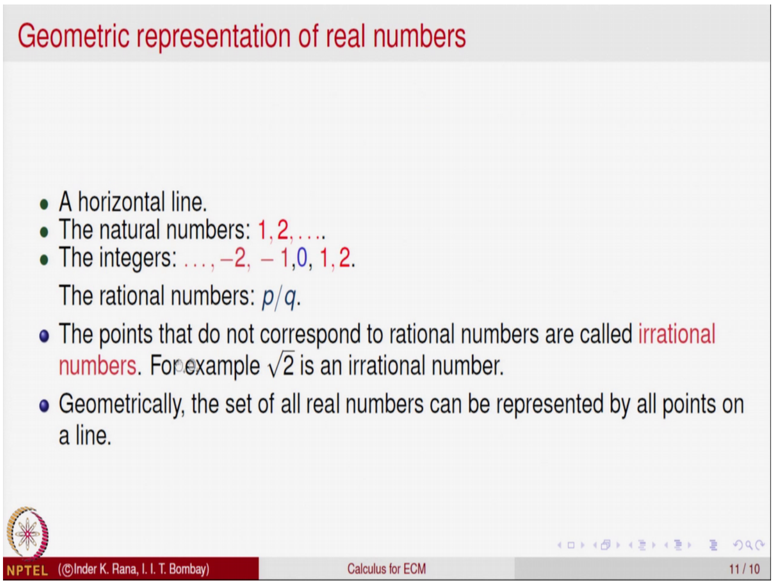

So, horizontal geometrically the set of all real numbers can be represented by all points on the number line. So, we continue our discussion of A number line further in the next lecture.