Calculus for Economics, Commerce & Management
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Lecture – 39
Chain rules, higher order partial derivatives, local maxima and minima, critical points

In the previous lecture, we had started looking at the notion of partial derivatives and how do you compute partial derivatives using chain rule for composite functions. So, let us look at some examples of computing partial derivatives using chain rule. So, the first example we look at his the example f x y which is given by x square plus y square or x and y belonging to R 2.

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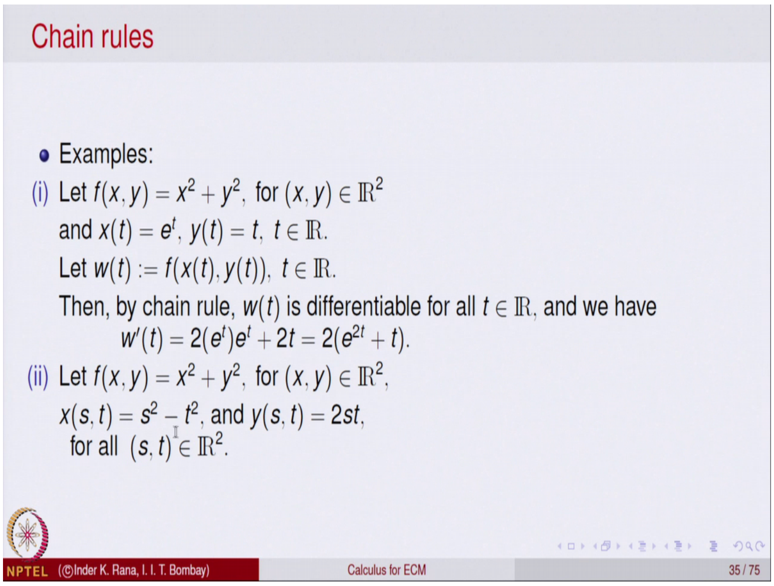

So, let us find out why it is where this is the function of 2 variables and let us assume x itself is a function of a variable t given by x t is equal to e raise to the power t. So, keep in mind it is exponential function e raise to the power t where e is a number, we had looked at this function in the first lecture or second lecture and y of t is equal to t for t belonging to R. So, x is a function of t and f is a function of x and y is also a function of t. So, what will be f as a function of t we can actually put the values of f values of x in this formula. So, there will be x square that is e raise to the power 2 plus t square. So, f actually becomes a function of one variable and we can compute the derivative as if it is a function of one variable.

But we want to illustrate this as a example of chain rule. So, let us look at wt as a function f of x t comma y t. So, this gives us the value f is a function of x and y and x is a function of t and y is a function of t. So, by chain rule what will be the derivative of w with respect to t it is a partial derivative of f with respect to x into derivative of x that is dx by dt plus derivative of y f with respect to the variable y in to dy by dt. So, partial derivative of f with respect to x that we get from this equation. So, that is 2 x y is kept constant. So, that is 2 x. So, and 2 x x is equal to e raise power t. So, you get the first term as 2 x 2 into e raise to power t into dx by dt, but xt is equal to e raise power t.

So, its derivative is dx by dt that e raise to the power t itself if you recall we had mentioned that for the exponential function the derivative is itself plus the contribution with respect to the second variable now. So, that will be partial derivative of f with respect to y into dy by dt. So, partial derivative of y with respect of f with respect to y is 2 y and y is equal to t. So, that is 2 t into dy by dt. So, dy by dt is equal to one. So, that is only 2 t. So, using chain rule the derivative of w with respect to t is this expression which you can simplify into 2 into 2 e raise power t plus t. So, that is let us look at the function same function fxy is equal to x square plus y square.

But here x; now is the function of 2 variables s and t x of s t is equal to s square minus t square and y of st is 2 times st. So, it is the same function of 2 variables is same as before x square plus y square, but now x is a function of 2 variables t and s. So, s square minus t square and y also is a function of 2 variables s and t. So, if like you can put these values in x square plus y square and will get f as a function of as a function composite function as a function of 2 variables, but the formula will look quite complicated. So, let us try to use chain rule to find the partial derivative of this. So, f is a function of 2 variables x and y x itself is a function of 2 variable s and t and y also is a function of 2 variables s and t. So, partial derivative of the composite function.

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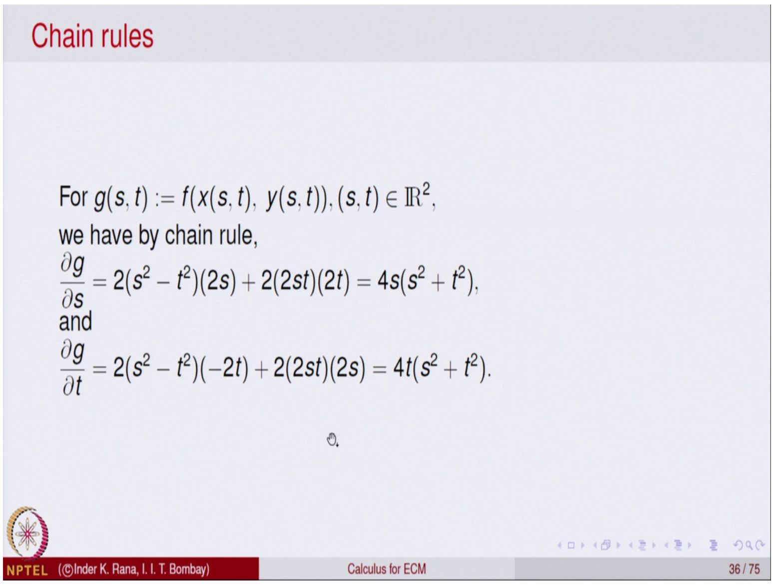

So, what will be the composite function f of x st comma y s t? So, we can compute the partial derivative of that function. So, the function is gst which is f of x t x st comma y s t s of t belonging to R.

So, g is a function of 2 variables and s right g is a function of 2 variables s and t over this comes as a composite function f of x st y st. So, g as a function of 2 variables will have partial derivative with respect to s and with respect to t. So, how do you find that? So, partial derivative of g with respect to s to compute that will look at. So, little bit 2 terms coming partial derivative of f with respect to x and if you are differentiating with respect to s then the partial derivative of x with respect to s.

So, that will be the first term plus partial derivative of f with respect to y multiplied by partial derivative of y with respect to s. So, that will be the partial derivative of g with respect to s. So, let us compute that the partial derivative of g with respect to s will be partial derivative of f with respect to partial derivative of f with respect to x into the de derivative of partial derivative of x with respect to s.

So, that it gives you this term plus the second derive second term that is a partial derivative of f with respect to y into the partial derivative of y with respect to s. So, that is a second term the similarly, we can find the partial derivative g with respect to t being first look at partial derivative of f with respect to t into the partial derivative of y with respect to t plus partial derivative of f with respect to y in terms of partial derivative of y with respect to t and all are put in terms of s and t. So, that your answer eventually is in terms of s and t. So, this is how you apply chain rule and compute the partial derivative of composite functions using what are called chain rules.

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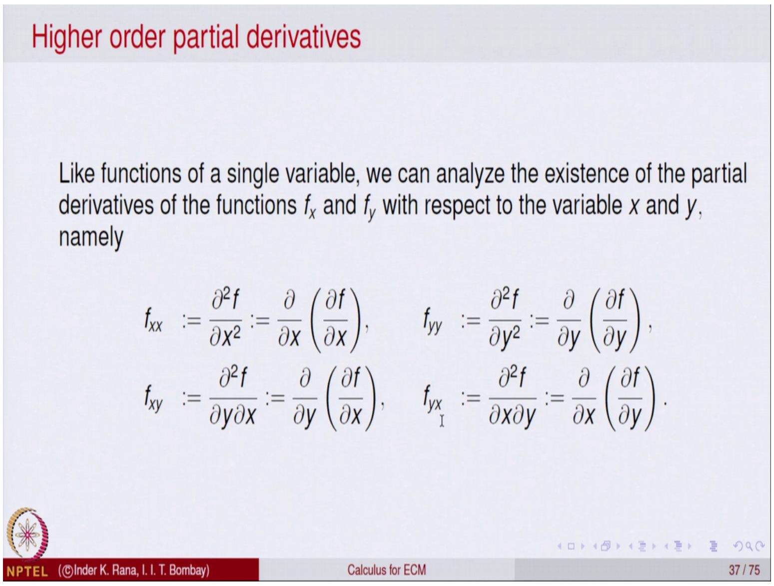

Let us go to slightly next concept see in functions of one variable we said that if the first derivative the derivative of a function exists at every point in the domain then we can ask whether the function is differentiable again and if it is. So, we will get the; what is called the second derivative and similarly the third order derivative and so on. Similarly the partial derivatives are functions of one variable only.

So, if the partial derivative of a function of 2 variables say fxy if the partial derivative of f with respect to x exist at all points then it is a function of 2 variables again and one can ask; what are its partial derivatives.

So, partial derivative f of x will have 2 partial derivative as again differentiating f of x with respect to x, and then differentiating this with respect to y or differentiating it with respect to x again itself.

So, each partial derivative will give you 2 more partial derivatives of higher order depending on the function. So, let us write these. So, that we are able to use them. So, the first one is fxx which is also written as partial derivative with a power 2 here. So, dell square f divided by dell x square. So, this is how you read this. So, this is dell square f dell x square or is the second order partial derivative of f with respect to x and x.

So, what is this you take the function find its partial derivative with respect to x and then again find its partial derivative respect to x. So, this is partial derivative of x twice you can call it. So, partial derivative of f with respect to x that gives you function find its partial derivatives with respect to x the other possibility is you look at similarly with respect to y. So, look at the partial derivative of f with respect to y. So, that gives you one partial derivative function right this is a partial derivative f with respect to y and that may be differentiable with respect to y again.

So, we can find the partial derivative of this function this is the function of 2 variables keep in mind the partial derivative f with respect to x is a function of 2 variables as well as the partial derivative of y f with respect to y is again a function of 2 variables if we assume it to be differentiable everywhere with respect to the variable.

So, then if the partial derivative of this function which is function of 2 variables with respect to y exists then this is called the second order partial derivative of f with respect to y or read as dell square f by dell y square or f y y or simply saying it is a second order partial derivative of f with respect to y other possibilities are this function exists. For example, partial derivative of f with respect to x exists, but we instead of asking partial derivative of that with respect to x again we can ask what is the partial derivative of this with respect to y that may exists. So, that gives you another possibility that the partial derivative of f with respect to x its partial derivative with respect to the variable y.

So, that is written as dell square f dell y dell x. So, note that here x is. So, as if the bracket is removed and this dell is multiplied and called dell square and dell y dell x right. So, there is only shorthand notation. So, you should read it as it is a second order 2 indicates the order f of the function f. So, partial derivative second order partial derivative of f what is the sec 2 orders first with respect to x and then with respect to y. So, you are going to read from right to left while as in if you can also write it as f of xy there you will read it is from left to right. So, you will be reading as x first and then with respect to y. So, keep in mind whether the; if this notation is used this is first find the partial derivative of f with respect to x and then with respect to y right. So, that is going this. So, this lower suffix is going from left to right where as here in the denominator if you think it is the denominator it is going from right to left.

So, this is a second order partial derivative of the function f where the first you differentiate with respect to x and then differentiate with respect to the variable y similarly there will be a partial derivative what is denoted by yx; that means, you first differentiate with respect to y and then with respect to x also called partial derive second order partial second order partial of f by dell square f divided by dell x dell y or second order partial derivative of f with respect to y and then with respect to x. So, that is order here that is left to right and it is right and left you take the function partial derivative of f with respect to y first and then take its partial derivative with respect to x.

So, there are four partial derivatives available second order partial derivatives for a function of 2 variables all of them may not exists some of them may exists, but these are the possibilities of. So, these are called four second order partial derivatives of a function of 2 variables these fxy and fyx they are called the mix second order partial derivatives because the variables x and y are mixed here. So, this is called fxx fyy; they are pure second order derivatives and these are called the mixed second order partial derivatives in general fxy is not equal to fy x.

So, caution should be taken that in general for a function of 2 variables the 2 mix of second order partial derivatives are not equal. So, there are sufficient conditions which ensure that when they will be equal will not go in to those conditions; however, will assume that for our purpose they are equal. So, those conditions are satisfied. So, these are called the higher order partial derivatives. So, they corresponds to something like second order derivatives of a function of one variable now we are defining them because they find applications in economics commerce and management.

So, let us look at one example let us recall the Cobb Douglas function which was defined as q is equal to QKL a into l raise to the power alpha once again this l is a typo it should be L raise to the power beta where a alpha and beta are positive constants K is the capital and L is the labor input for this function just.

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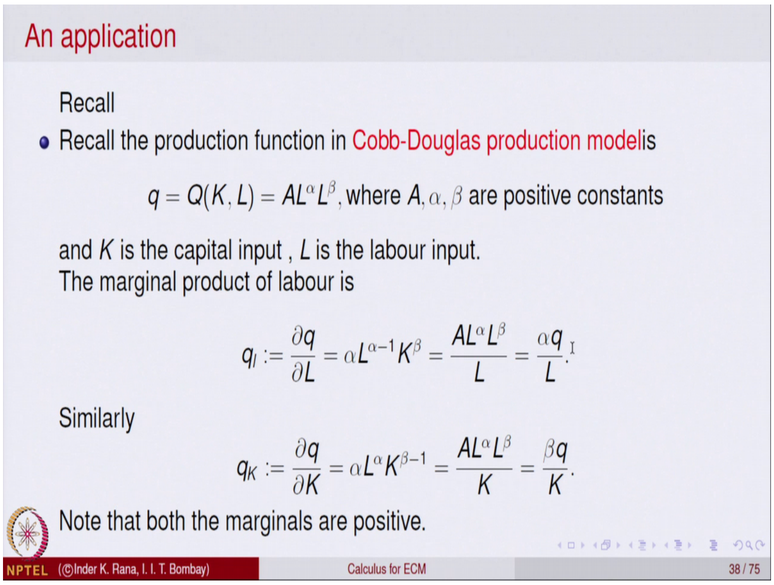

Now, we saw today that the marginal of the product of labor Q L is partial derivative of Q with respect to L was obtained as this and the partial marginal of Q with respect to K was today, we got as beta Q by L in the previous lecture. So, these are the marginals of the Cobb Douglas function which we obtained. So, let us look at the second order both the marginals are positive that we know. So, let us look at the second order for this.

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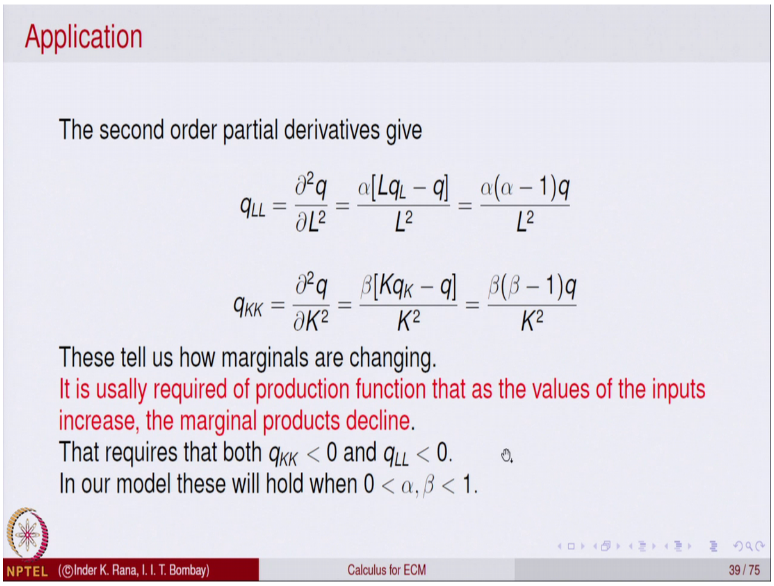

So, Q L L will be second order partial derivative of Q with respect to L. So, that we can differentiate by looking at this one alpha Q by L. So, it will be L square. So, by the quotient rule. So, this will be L square this will go up. So, alpha times L into derivative of Q with respect to L minus the Q time alpha Q into derivative of L that is one. So, let us put this values. So, that is equal to. So, L into derivative of Q with respect to L minus Q into derivative of L that is Q. So, will be simplify that is alpha into alpha minus one into Q divided by L square. So, that is a simple equation we get similarly Q; Q the second order partial derivative of Q with respect to K is partial derivative of.

So, take the first order partial derivative that is we obtained in the previous. So, that is the partial derivative. So, this function is the first order partial derivative second order partial derivative using the quotient rule. So, K square in the denominator numerator will be beta times K into derivative of Q with respect to K minus Q times derivative of K that is equal to one. So, that gives us the second expression namely the second order partial derivative of Q with respect to K is beta into beta minus one divided by K square into Q. So, these 2 expressions are the marginals they are the derivatives of marginals.

So, these will tell us how does the marginals change, right; what kind of functions the marginals are there. So, keep in mind that alpha and beta are positive and Q and L are positive. So, if alpha is between 0 and 1 and beta also is between 0 and 1. So, that will give us 1. So, normally in production in a production function we require that the values of inputs increase the marginal should decline if there is a increase in the values of the product function right then the; that means, if the inputs increase marginal should decline that is a normal principal guiding principal for economic models; that means, we want Q L L and Q K K to be a decreasing function for them to be decreasing functions we should have alpha to be less than one and beta should be less than L.

So, that is the reason one normally if you want this property to be true that the functions the derivatives are derivative of the marginals these are the second order partial derivatives if they are negative then that condition will be true when alpha and beta both are less than one they are non-negative quantities. So, they should be between alpha between 0 and one. So, in Cobb Douglas model normally one says alpha and beta are positive constants and they are between 0 and one. So, that is the basic assumption in Cobb Douglas model and the reason is as indicated here is basic principal of economics.

So, now, what we will be doing we will be will start looking at the notion of local maxima and local minima for functions of 2 variables if we recall for a functions of one variable how we looked at the notion of local maxima and minima we first obtained a necessary condition namely that if at a point a function has local maxima or a local minima then the derivative and the derivative at that point exists, then the derivative should be equal to 0. So, that gave us necessary conditions gave us the possible points where the where the local maxima or the minima can occur and then we had various tests the continuity tests the first derivative test and the second derivative test to analyze whether those possible candidates for local maxima or minima are actually local maxima or minima for the function or not.

So, they were the sufficient conditions. So, we had necessary conditions for local maxima minima and sufficient conditions for local maxima minima the same process we are going to follow for functions of 2 variables. So, first we will define what is called as the local maxima and local minima for a function of 2 variables and then give necessary conditions for a function of 2 variables if it has a local maxima and or a local minima at a point then what are the conditions and then will look at the sufficient conditions to analyze those possible candidates where the function can have local maxima or local minima. So, let us look at the definitions.

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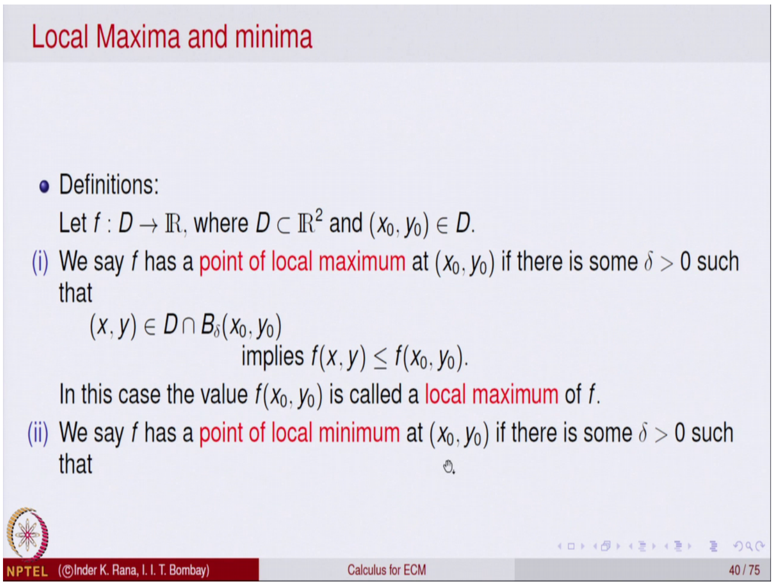

So, the first definition says for a function of 2 variables with a domain D. D is a subset of r 2 a point x 0 y 0 is called a point of local maximum. So, if there is a ball b delta x 0 y 0 is nothing, but a ball centered at x 0 y 0 of radius delta; that means, this is a set of all points at a distance maximum of delta. So, let me just explain because we are not really explained this very well in the beginning, for a in the plane.

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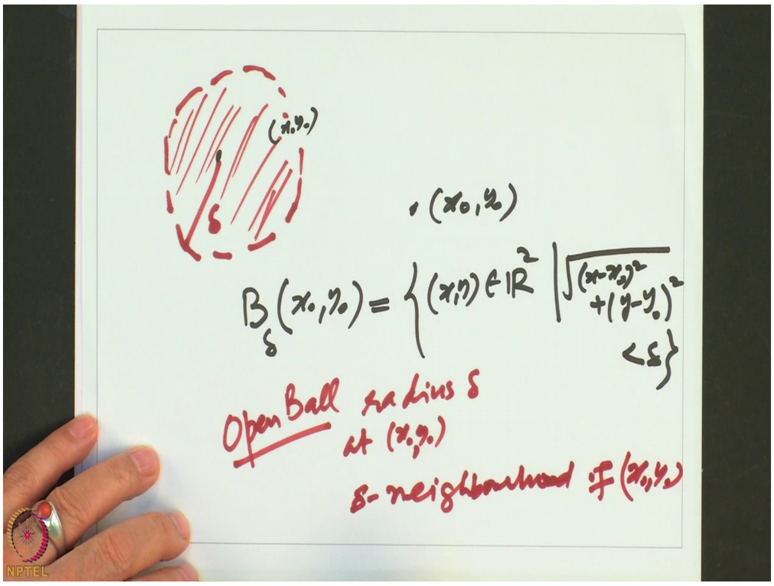

So, if it take a point x 0 y 0, then what is ball centered at x 0 y 0 of radius delta. So, this is nothing, but this is a set of all points x and y belonging to r 2 such that the distance between x 0 y 0 and xy is less than delta. So, what is the distance? So, the distance is x minus x 0 square plus y minus y 0 square this is less than delta. So, this a set of all points and in the picture.

So, if this is the point x 0 y 0 then these points are nothing, but the points which are at a distance of delta so; that means, at a distance of delta you are looking at, but strictly less than delta. So, you will be looking at all points in the disc, but minus the boundary the boundary is not going to be included. So, all points inside that is the ball that is a ball. So, we say it is a ball radius ball of radius delta at. So, at the point x 0 y 0 because we are not including the boundary one normally actually qualifies open ball of radius delta this is also sometimes called the delta neighborhood b neighborhood of the point x 0 y 0.

So, recall in a function of one variable we defined the neighborhoods by the open intervals centered at that point here a b of x 0 y 0 is an open ball in the plane centered at x 0 and y 0. So, what we are saying is a point of local maxima at the point x 0 y 0 if there is a neighborhood of that point in the domain of course, such that at all these points xy the value of the function is the largest.

So, f of x 0 y 0 is bigger than f of x y for all points in the delta neighborhood of the point x 0 y 0. So, in this case we say it is a point of local maximum. So, locally, what does locally means in a open neighborhood of that point x 0 y 0 and the domain the value is the largest. So, locally the value is the largest and similarly we say it is a point of local in this case we say that this value of the function the point is called the point of local maximum and the value of the function at that point is called the local maximum of the function.

One can define similarly the point of local minimum at x 0 y 0 to be there the points.

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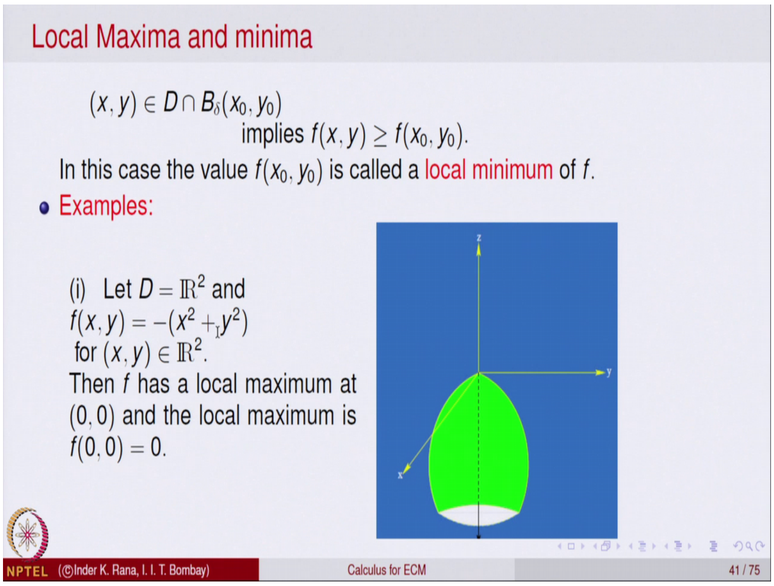

Such that there is a neighborhood delta bigger than 0 with the property that the value at the point x 0 y 0 is smallest for all points in that neighborhood open neighborhood open ball around the point x 0 y 0 then you say it is a local minima and the value of the function at that point is called the local minimum that point is called the local minimum.

So, you can have look at some examples if I look at the function f x y is equal to minus of x square y square then we had tried to visualize the graphs of this functions their surfaces this will look like a adverted cup and the negative; that means, the surface is going to lie below the xy plane. So, the point x 0 y 0 the point 0 0 is going to be a point of local maximum the value will be 0 0.

So, the point 0 0 is going to be point of local min maximum and the value 0 is the local maximum for such a function.

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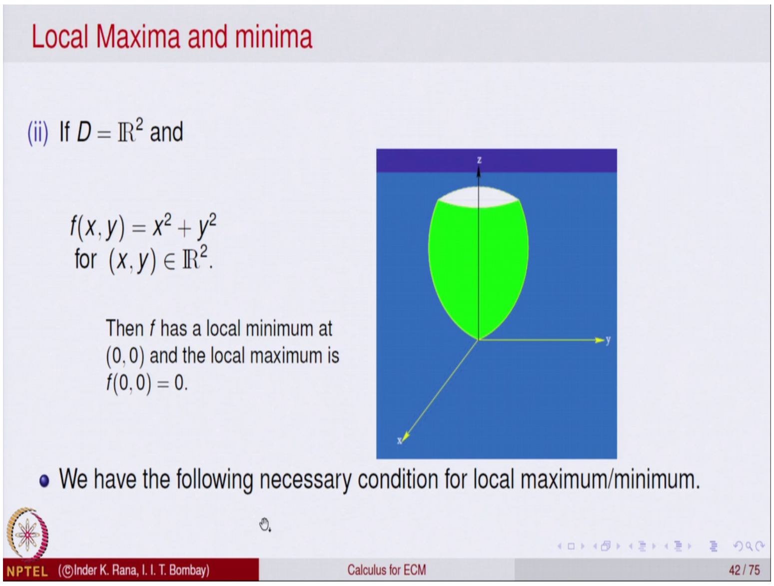

If I just invert it. So, the function becomes fxy is equal to x square plus y square for a xy belonging to this then we get that the point 0 0 is a point of local minimum and the value at 0; 0 is the local minima for the function. So, this is a simple example of local maximum and minimum for the functions like functions of one variables we have the following necessary condition for local maximum and minimum.

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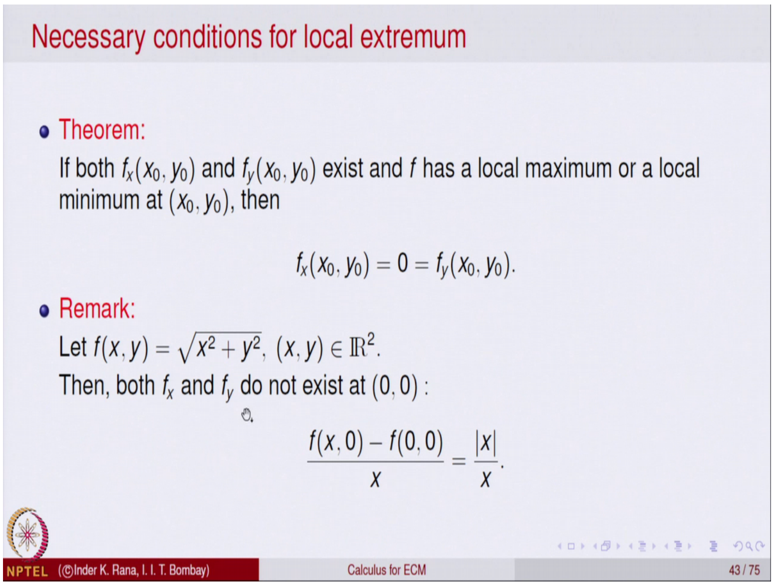

So, let us write this as a theorem that if f is a function which is defined in some open domain in r 2 a function of 2 variables if both the partial derivatives exists and f has a local maximum or a local minimum at the point x 0 y 0 either of it, then the partial derivatives of with respect to x and partial derivative of y with respect to f with respect to y at that point must be 0.

So, what we are saying is if a point x 0 y 0 is a point of local maximum or local minimum for the function and the partial derivative is exist at that point then they must be equal to 0 which is perfectly similar to the function of one variable if local maximum or a local minimum for a function of one variable exists and if the function is differentiable then the derivative must be equal to 0.

So, that is the necessary condition that we get for a function of 2 variables to be point of local maximum or a local minimum, but keep in mind that if we look at the for examples the functions square root of x square plus y square this function does not have local maximum does not have partial derivatives at the point 0 0 that you can easily check because this ratio f of x 0 f of y 0 divided by x they will be equal to mode x by x which is that limit does not exist at the point 0 0. So, for this function the partial derivatives do not exists, but the function has got a local minimum at the point 0 0.

So, keep in mind the function can have a local maximum or a local minimum without partial; of partial derivatives is the point of local minimum without.

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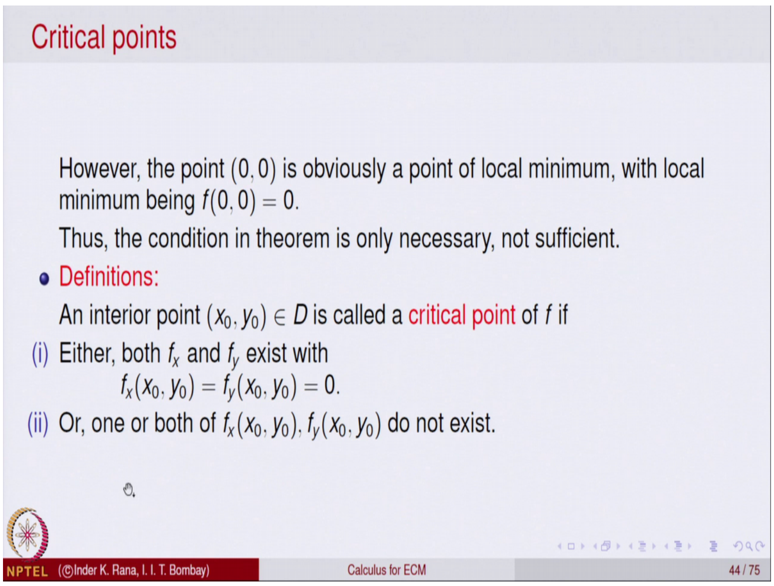

So, that is the condition in the theorem is only a necessary, but not sufficient condition. So, let us define what is called a critical point. So, like in one variable an interior point in the domain of the function will be called a critical point if either both f of x and fy exists and are equal to 0 and second possibility is one or both of them do not exists like in one variable derivative may not exist or the points where there could be the boundary points the third one we are not listed here we should list that also. So, those also could be the points of. So, these are the interior points.

So, for interior points these are the only 2 possibilities partial derivatives exists and are equal to 0 or either or both of the partial derivatives do not exists. So, keep in mind this is for the interior points ok.

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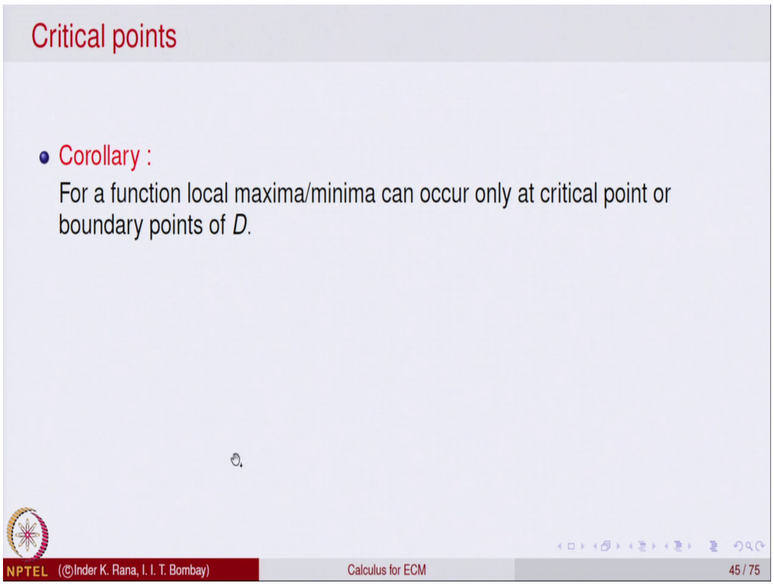

So, as a corollary of this we will get that a function can have a local maxima minima can occur only at the critical points or possibly at the boundary points of the domain like in the function of one variable the critical points and the boundary points together give the possible candidates for the function to have a local maxima or minima. So, like in one variable we give sufficient conditions for the possible points to be local maxima or minimum same we will do it for functions of 2 variables in the next lecture, will continue our study of points of local maxima minima for functions of 2 variables in the next lectures.

Thank you.