Calculus for Economics, Commerce & Management
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Lecture – 138
Marginals in Cobb-Douglas model, partial derivatives and elasticity, chain rules

In the previous lecture, we had started looking at the notion of partial derivative of functions of several variables we looked at some examples how partional partial derivatives are applied in marginals of a model. Let us start by looking at the Cobb Douglas model which we have introduced in the previous lecture and looked at the marginals for that. So, let us just recall that the production function.

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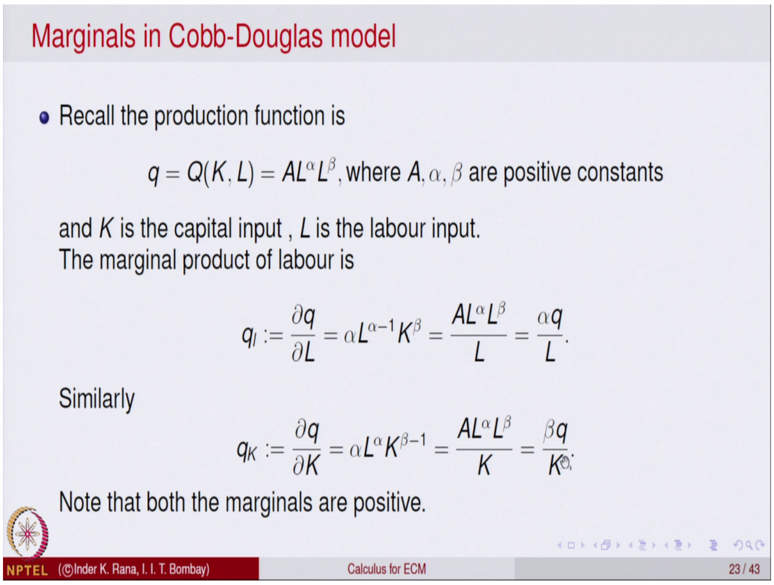

In Cobb Douglas model is given by q is equal to it is a function of 2 variables K and L where K is the capital and L is the labor input and it is given by A into L raise to the power alpha and K raise to power beta where this should be k. So, where A alpha and beta are positive constants and K is the input capital input L is the labor input. So, model of a change here as this is be it is a function of 2 variables a in to L raise to the power alpha K raise to the power beta. So, this is a typo keep in mind.

So, once that is given once you differentiate this with respect to L. So, q with respect to L is partial derivative of q with respect to L. So, here A is a constant and L raise to the power alpha. So, this alpha comes out and. So, it is A into alpha raise to the power one minus one and this because is K it does not depend on b alpha labor that is capital. So, that remains as it is so that is equal to alpha into L raise to the power alpha minus one into K raise to the power beta we can put the values of we can multiply its numerator and denominator by L. So, that will give you alpha L raise to the power alpha K raise to power beta divided by L so, but now if that numerator is alpha times this.

So, numerator is alpha q by L. So, the marginal with respect to L can be written as alpha q divided by L similarly the marginal with respect to K is the partial derivative of q with respect to K will be A L raise to the power into beta K raise to the power beta minus one and once you put this well use multiply and divide it by K and use that equation you get beta in to q raise beta q by k.

So, we can represent the marginals in terms of the original product function as it is. So, let us see an interpretation of this. So, both the marginals are first of all positive because alpha is a positive constant q is positive L is positive beta is positive. So, q marginals are both positive. So, keep in mind what is marginal marginal is the partial derivative. So, there being positive indicate the function is going to be increasing. So, let us analyze this further from the previous equations let us calculate from this equation, let us calculate alpha; alpha is L times q L divided by q and similarly beta is K times q K divided by q.

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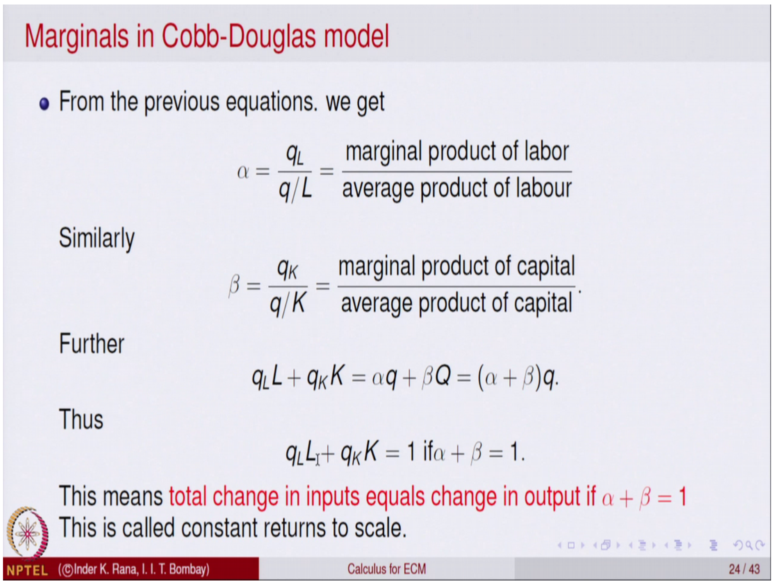

So, let us use these values to find alpha and beta. So, it is q L divided by q by L and now let us interpret this was the partial derivative of q with respect to L.

So, there is a marginal product of labor and q by L is the average product of labour. So, alpha the constant is equal to the marginal product of labour divided by the average cost of the labour similarly from second equation you will get that beta is q partial derivative of q with respect to K divided by q and K q by k. So, the numerator it is marginal with respect to the marginal product of capital with respect to the capital marginal and the denominator is nothing, but q by K. So, that is a average product of the capital. So, alpha and beta are equal to this. So, if we add them these 2 equations. So, q L multiplied by L plus q K multiplied by K. So, this taking it up is alpha q plus beta q. So, that is equal to alpha plus beta into q some small typo here small q is same as the capital Q.

So, that is the only; so, keep in mind small q is same as capital Q. So, you get this q L marginal multiplied by L is marginal of capital multiplied by the capital is equal to alpha plus beta times Q. So, that says that if you want this quantity left hand side to be equal to one then you should have alpha plus beta equal to one because when alpha is equal to one beta is equal to one you will get this is equal to one.

So, that; that means, that means that the total change in input. So, q L marginal into L that is a total change in input plus the total change in output that is q K multiplied by q is 1 if alpha plus beta is equal to 1. So, this scenario m a economics is called as that constant returns to scale. So, whatever is the total input that equals to the total change in the output?

So, that is how marginals are used in economics for the functions of several variables.

Let us continue our study of partial derivatives of functions of several variables 2 variables.

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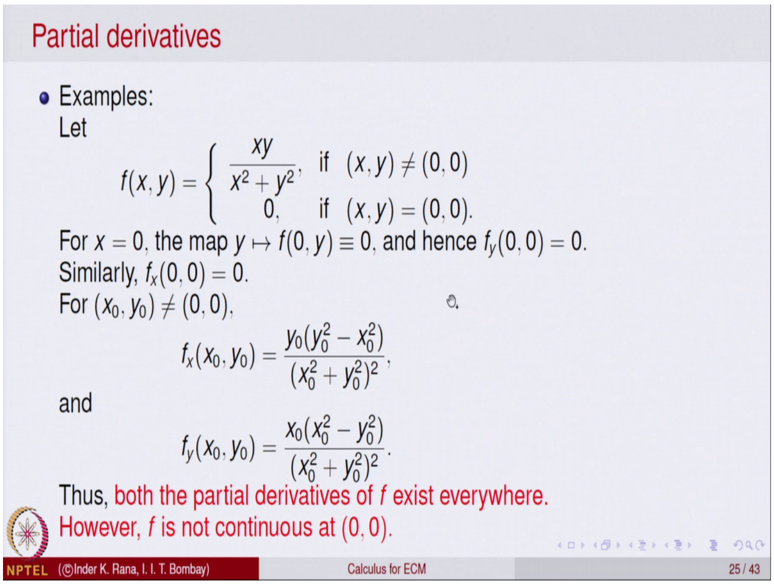

So, let us look at the function fx y is equal to xy divided by x square plus y square if xy is not equal to 0 and it is x y at the point 0 the value is 0. So, this is the function which is defined by this formula and the denominator is not equal to 0. So, that is equal to 0 when x is equal to 0 equal to y. So, at that point we define the value to be equal to 0. So, for this function, let us fix the x variable as 0 and cursor the map y going to f of 0 y. So, when x is fixed as 0. So, the function is x y that is 0 divided by y square. So, that is equal to 0. So, as a function of the variable y when x is fixed at 0 this is the constant function y going to 0. So, naturally for this the partial derivative at this as a function of one variable is differentiable and the derivative is equal to 0.

Similarly, for when you put y equal to 0 again this function is identically equal to 0. So, the derivative at 0 0 is again equal to 0. So, for this function both the partial derivative at 0 exists and are equal and equal to 0; however, if you put if you take the point not equal to 0; 0 and look at the derivative. So, then the value then the function is given by this formula. So, to find out the partial derivative for with respect to x; when x is fixed as x 0 with respect to x when y is fixed at y 0 will have to apply the quotient rule. So, applying the quotient rule, when we want to find the partial derivative of f with respect to x, at the point x 0 y 0, partial derivative will be the denominator raise to the power 2 at the point x 0 y 0. So, that is x square y square raise to the power 2 into the derivative of that into derivative of the den this denominator as it is.

So, that will be x square plus y square into the derivative of the numerator with respect to x. So, that will be y. So, the first term will be y into x square minus y square minus the second term will be x y into the derivative of x square plus y square that is 2 x. So, that will give you 2 x square, so that you can combine together. So, the partial derivative at a point not equal to 0; 0 comes out to be with respect to x is equal to this. So, and similarly the partial derivative with respect to y at a point y 0 x 0 is given by x 0. So, using quotient rules with the point y 0 of x here and the point x 0 of x here you can calculate the partial derivatives they come out to be equal to this.

Now, let lets analyze both the partial derivatives of f exists everywhere because they existed at 0 0 and they existed at this point. So, partial derivative for this function exists at every point; now one would like to know what about this function as a function of 2 variables. So, that we will not claim that this as a function of 2 variables is not continuous at the point 0; 0 to see why that is. So, one method could be if you take new approach seek to find out continuity at the point 0 0 we have to approach fxy we have to led fxy go to find the limit of this as x y goes to 0 0 to. So, let us find it this limit along a particular path.

So, let us say for example, you take y equal to x. So, you approach the point 0 0 along the line y equal to x then the function along y equal to x will be fx x. So, that will be x square divided by x square plus x square. So, numerator and denominator will cancel out x square will cancel out, but instead of taking just a line y equal to x you could have taken also a line y equal to m of x. So, line with a slope m. So, if you take y equal to mx and find out the limit of this along the line y equal to mx that will be x comma y will be mx. So, will mx square and denominator will be x square plus m square x square.

So, x square will cancel out and the ratio will turn out to be one over one plus m square so; that means, along the line y equal to mx the function is constant and is equal to one over one plus m square; that means, along different lines as you approach the point 0 0 the function values approach different values; that means, the limit of a function does not exist and hence the function is not continuous at 0 0. So, this is how one analyses continuity of functions of 2 variables at a point will not have much opportunity to use continuity, but concept of continuity of functions of 2 variables can be defined as in terms of as for function the one variable.

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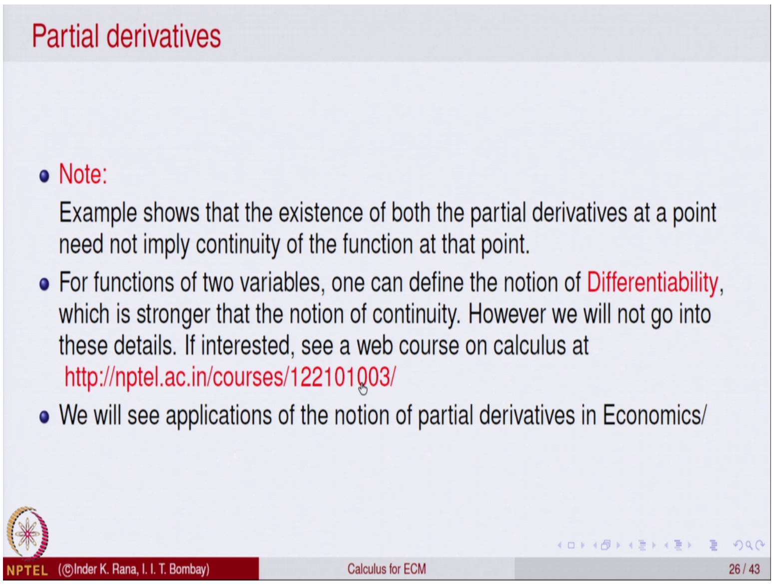

So, let us go a step further. So, note that. So, this example shows that the instance of both partial derivatives at a point need not imply continuity of the function at that point. So, existence of both the partial derivative is not good enough a condition to ensure continuity like in one variable which at differentiability implies continuity that is not the case in function of 2 variables just a instance of partial derivatives is not good enough; one can define the notion of differentiability for a function of 2 variables which is stronger than the notion of continuity and also stronger than the notion of resistance of the 2 partial derivatives, but will not go into that in this course.

So, those of you who are interested in knowing more about this probably should look at the web course which is on mathematics one which is available at this site; this is a NPTEL web course on calculus. So, if you are interested in knowing more about the differentiability of functions of 2 variables go to this site and you will find the complete course on calculus of one and several variables which you can read for your more knowledge completeness if you like.

So, with that reference we move on and look at the applications or notion of partial derivatives in economics. So, recall for a function of one variable we are defined the notion of elasticity of demand for function of one variable that was proportionate percentage change in the output with respect to the input.

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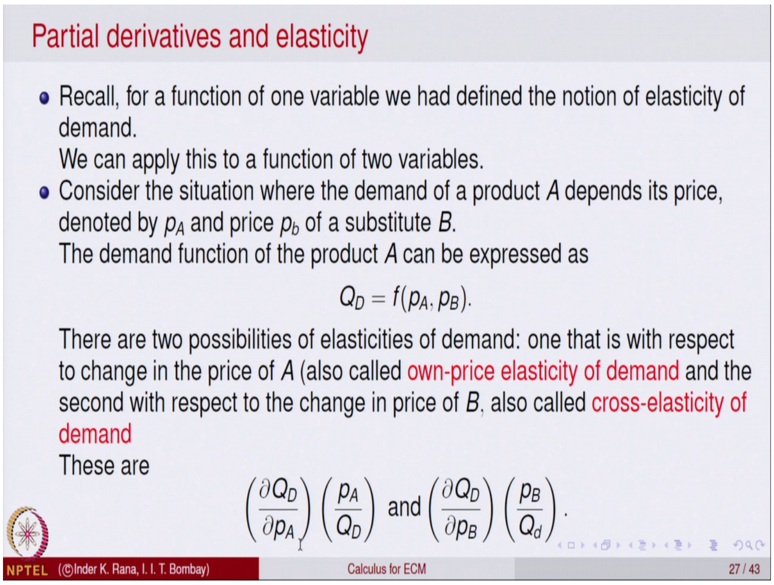

So, we can apply that concept to a function of 2 variables. So, let us look at that. So, consider a situation where the demand of a product A; a product is called A is being produced and the demand for this A depends on its price. The price of the commodity A is P of A and the price P of B of A substitute B which is also available on the market. So, you can think of A and B; 2 products available on the market with the same use and for A; the price is P A and for B the price is P of B. So, let us look at the demand function.

So, demand function Q of D will be a function of P of A and P of B, right. So, price of A and price of B together both of them will decide; what is the demand of that product, right. So, this is a function of 2 variables P of A is the demand price for A and P of B is the price for the substitute which is B. So, there are 2 possibilities of elasticity of demand for such a function of 2 variables one is that with respect to the change in price of A; the com commodity itself. So, this is called own price elasticity of demand.

So, the elasticity of demand with respect to the product, A is called own price commodity ela on price elasticity of demand one can also look at how the things are changing with respect to the product B. So, the second with respect to the change in price of b is called cross elasticity of demand. So, elasticity of demand which is with respect to the price b price of b with a fixed price of a fixed will be called as the cross elasticity.

So, it is going over to the substitute. So, here is A on price. So, let us try to compute both of this. So, the own price elasticity is the derivative of the demand with respect to a. So, that is because of the function of 2 variables. So, it will be partial derivative of the demand function Q of D with respect to the price P of A into the price ratio of price of a into Q of D. So, that is formula for function of one variable if you treat Q as a function of the one variable P A itself then this is the elasticity of demand by that formula and similarly if you treat a fixed and look at as a function of the variable B, then the elasticity of demand Q of D will be partial derivative of Q demand Q D with respect to P B the price of b into the price of B divided by the demand Q of D.

So, these are the 2 coefficients of elasticity of demand. So, this one is what is called own price elasticity of demand this is called the cross elasticity of demand let us compute these 2 quantities.

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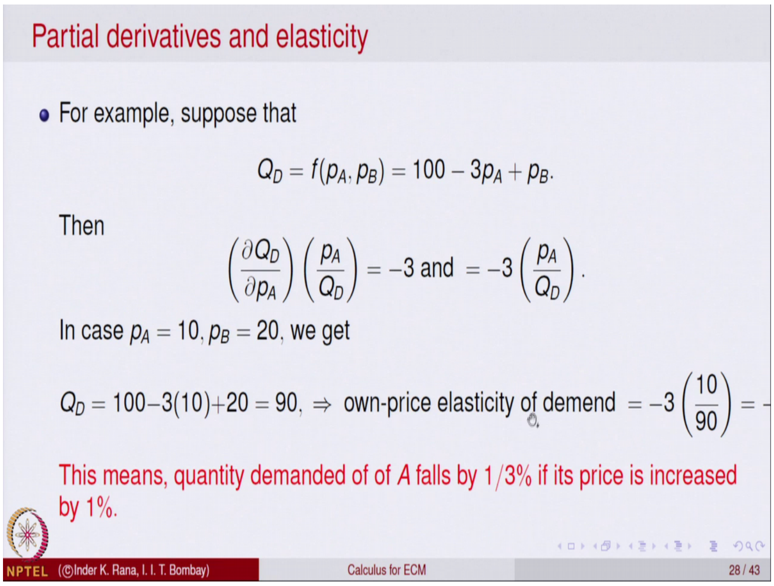

For a particular example, let us say Q of D which is the function of the 2 variables P A and P B is given by the following relation, it is 100 minus 3 P A plus P of B. This is just a illustration of that we are trying to do.

So, if Q of D is this, then we can compute what is elasticity of demand with respect to a. So, its partial derivative of this function with respect to a once you differentiate this with respect to the price of A. So, only variable P A is coming here. So, it will be minus 3. So, price elasticity of demand with respect to the price of A is given by minus 3 because this quantity is partial derivative is equal to minus 3 and in to P A divided by QD. So, that as it is P A divided by QD. So, this is the function of P A and P B similarly the second one if we look at the particular example particular case when P A is equal to say 10 and P B is equal to 20.

So, when P A B P A is equal to 10 P B is equal to 20. So, we put these values here you get q d is equal to hundred minus 3 into 10 plus 20 is 90. So, that gives you that the value of the own price elasticity of demand which was equal to minus 9 times P A by Q D is equal to minus 3 times P A is 10 and Q D is 90. So, 10 by 90, so that is the price own price elasticity of demand. So, that minus 1 by 3 you have to simplify this so; that means, what; that means, the quantity of demand of a falls; that means, the quantity demanded of the product a falls by 1 by 3 percent if the price is increased by one percent. So, if P A is equal to right.

So, that is a change in proportionate change is one by 3 percent. So, the quantity demanded. So, this own price on price elasticity of demand be negative; that means, it is going to a decrease when the pri when there is a increase in the price. So, quantity demanded of a falls by 1 by 3 percent because this is minus 1 by 3; if the price is increased by one percent. So, that is the interpretation of this.

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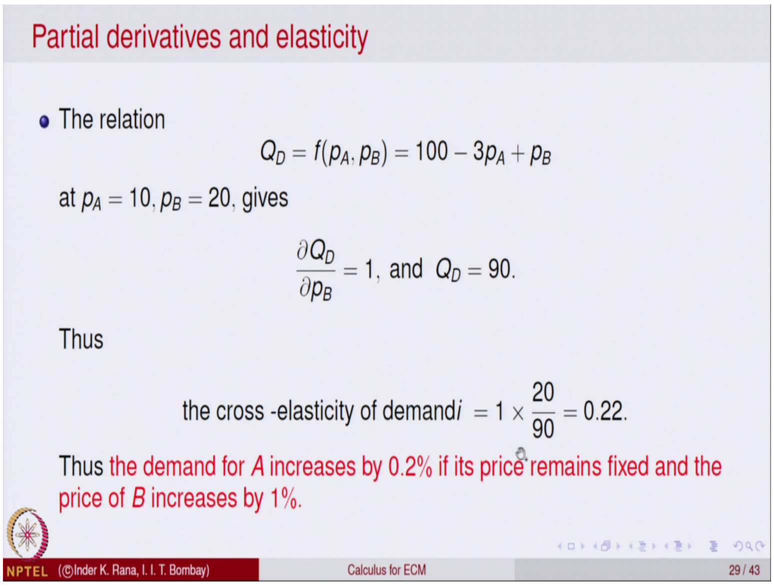

So, let us look at the second case when Q of D as before is same function 100 minus 3 P A plus P B; P A is equal to 10 and P B equal to 20. So, we saw that that gave us partial derivative of Q D with respect to PB. So, we are keeping P A fixed and only differentiating with respect to PB. So, that gives you this all becomes 0 and get the value equal to 1 and at that point Q D is equal to 90. So, we get the cross elasticity of demand being equal to that partial derivative of Q D with respect to P B into P B by QD. So, that is 20 by 90. So, that is point 2 2.

So; that means, what; that means that the demand for A increases. Now this is positive. So, this is increasing function. So, the demand for, this means that if the demand for a will increase by 0.2 percent. So, this is you can call it as 0.2 percent, if the price remains if its own price remains fixed and the price of b is increased by one percent because this is cross elasticity of demand. So, one percent change in the price; B will result in 0.22 percentage; change in the price of a price of A is kept fixed only the price of B is changing.

So, if the company B who is producing product B increases its price the demand for a increases. So, by this much percentage, this is the interpretation of marginals coefficient of elasticity in the scenario of functions of several variables. So, we give a simple example to illustrate that. So, next we will look at the methods of computing partial derivatives of composite functions partial derivatives or functions of 2 variables is keeping one variable fixed looking at the derivative of the function with respect to the other variables.

So, rules of partial derivatives for algebra partial derivatives is same as for one variable namely partial derivative of the sum is equal to sum of the partial derivative as long as the same variable is get fixed and. So, on; however, the chain rules change a bit because the second variable contribution is also will be there functions of 2 or more variables. So, we want to look at the scenario of chain rules. So, they are same as looking at the derivatives partial derivatives of composite functions or functions of several variables. So, let us look at that. So, let us there are many possibilities. So, will let us look at one by one some of them what is called chain rule one says the following.

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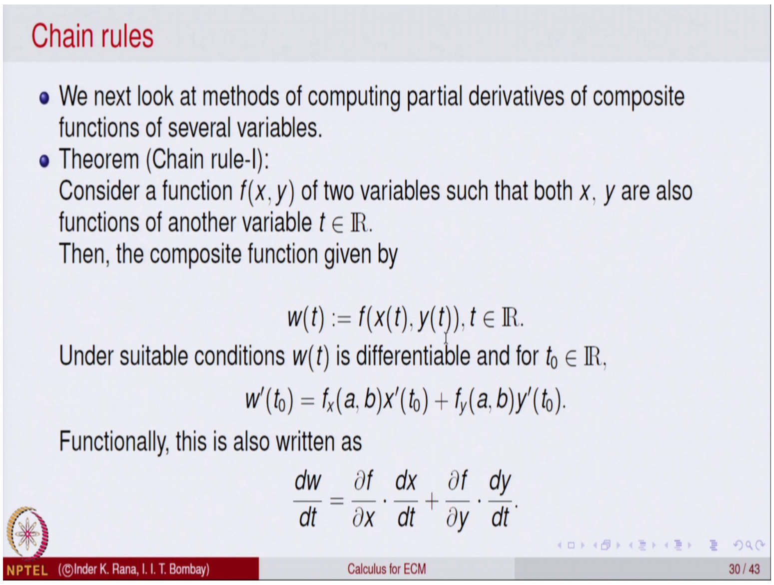

Consider a function f x y of 2 variables such that both x and y variables themselves are functions of another variable t say in belonging to r. So, f depends on x and y, but x and y both depend on another variable t so; that means, as t changes x will change and y will change and correspondingly f will change. So, we will have a composite function. So, the compounded function is f x depending on t.

So, write it as f of xt comma yt. So, let us denote this by w. So, w of t is equal to f of xt comma yt where t belongs to r. So, eventually what is happening is the variable t gives raise to appear xt comma yt appear in R 2, right. So, we have got 2 functions t going to xt R to R t going to yt that is again a function R to r, but put together as a pair we get a function t going to x t comma y t. So, that this inner thing is a function from real line to the plain.

So, that is the function of 2 variables. So, while you get 2 variables, and f is a function of 2 variables; f can be evaluated at xt comma yt. So, the resultant thing is a number again. So, this is a composite of 2 functions t going to xt comma y t and then f taking at point 2 f of xt comma yt that is again in R. So, wt is a real number, but in between it becomes a function of 2 variables. So, if xt and yt are differentiable functions and if f is also a nice function which we call as differentiability of the function f then this composite function is differentiable.

So, one says that under suitable conditions on w 2 is differentiable this function of one variable is differentiable and what is a derivative of this the derivative is obtained by looking at the derivative of f because f is a function of 2 variables. So, it will have 2 contributions whether you are moving looking at the rate of change according to the variable x or according to the variable y. So, the formula says that the rate of change of w which is a composite function at a point t naught is same as the partial derivative of f at a point a, b. So, what is a?

A is x nou xt naught and b is y of t naught; so, that value. So, fx at xt naught y t naught into d derivative of x so; that means, there is a contribution in the direction of x axis. So, that is a contribution of the partial derivative of f and the f is evaluated at that partial derivative is evaluated at the point x t naught y t naught. So, here x t naught is taken as a and y t naught is taken as b into the derivative. So, here in this contribution is as if you are moving along only one variable x and looking at the contribution of the chain rule. So, that, but there are 2 variable. So, the other variable also contributes equally. So, plus f y at the same point a b into y dash of t naught.

So, this is what is called the chain rule when t goes to a function of 2 variable and then it to function of 2 variables goes into real line. So, this is written as dw by dt. So, w is a function of 2 variable. So, f of x y. So, partial derivative of f with respect to x into dx by dt and partial derivative of a with respect to y into dy by dt. So, this is what is called as a chain rule under these composite functions. So, keep in mind. So, it is something like one variable, but there are 2 terms coming.

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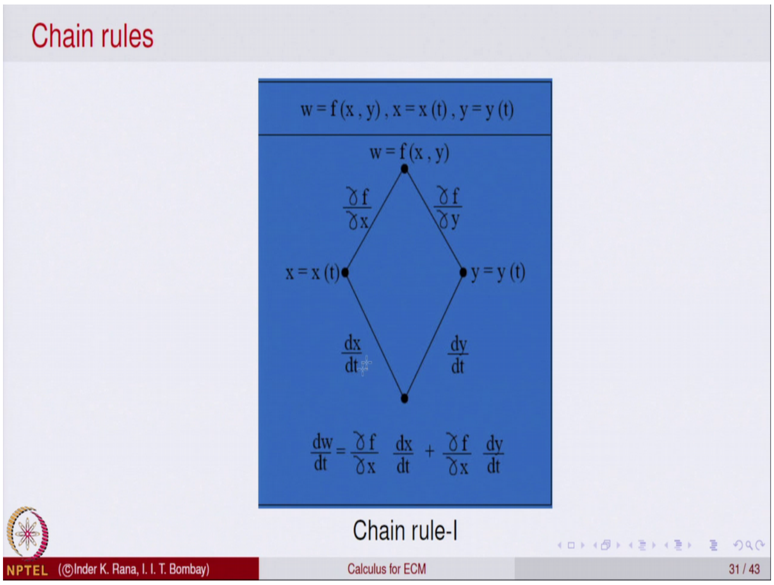

So, similarly we can write it in a pictorial form w is a function of 2 variable x and y x is a function of variable t y also is a function of t. So, when you want to find out the derivative d w by d t. It is a contribution along the variable x contribution along the variable y. So, along the x it is partial derivative of f with respect to x into dx by dt plus partial derivative of y f with respect to y into dy by dt. So, that is a chain rule example illustration for the previous consideration we can have a something similar slightly more complicated one where w is a function of 2 variables x and y. Now x itself is a function of 2 variables r and s and y also is a function of 2 variables r and s.

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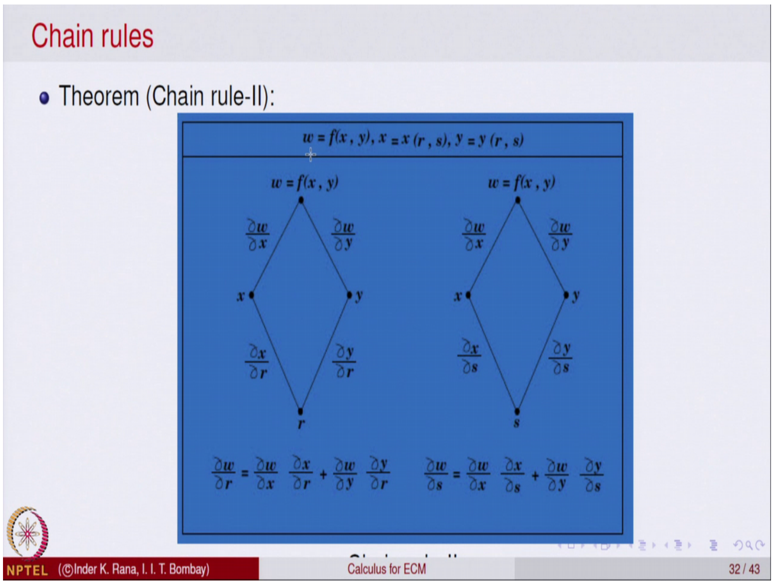

So; that means, r comma s goes to a point x r s comma y r s. So, it is a r 2 to r 2 and then there is a function f from r 2 to r. So, w becomes a function of 2 variables r and s. So, it is a composite of 2 functions of and both of them are functions of 2 variables. So, you can ask; what is the partial derivative of w with respect to the variable r what is the partial derivative of w with respect to the variable s. So, each one is computed as per the earlier conditions as per the earlier methodology algorithm w is a function of 2 variables x and y. So, contribution with respect to x plus x itself is a function of 2 variables.

So, its contribution with respect to the variable r; so that gives you the contribution of the variable x the first term partial derivative of w with respect to x into partial derivative of x with respect to r plus the second branch will give you the partial derivative of w with respect to y into partial derivative of y with respect to r. So, that is the second. So, this gives you the partial derivative of w with respect to r and similarly you can calculate the partial derivative of w with respect to x s that is partial derivative w with respect to s is partial derivative of w with respect to x into partial derivative of x with respect to x plus partial derivative of w with respect to y plus partial derivative of y with respect to x s.

So, this is you get 2 partial derivative again by chain rule. So, similarly depending on the situation how much which variables are composite; how many variables are involved partial derivatives are obtained, and these are all called chain rule formulas. So, basically the idea is wherever a function of more than one variable is coming each variable it will give its contribution in the partial derivatives. So, we will look at more illustrations of partial derivatives in the next lecture.

Thank you.