Calculus for Economics, Commerce & Management

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Lecture - 37

Partial derivatives and application to marginal analysis

So, let us continue our study of functions of 2 variables. In fact, the study of calculus of functions of 2 variables goes parallel to the study of calculus tools for functions of one variable. If you recall for a function of one variable, we defined what is a function of one variable, we looked at the domain, what is the range, and then we looked at the graph of function of one variable. And then we started looking at the concept of limit continuity, and differentiability and it is applications in economics.

One can develop various rigorous tools for analyzing what is the meaning of limit of a function of 2 variables, what is the notion of continuity of functions of 2 variables and so on. The only thing is once you have defined notion of limit everything else follows. But this being not a course in calculus of several variables, we will not to go into all those details. But we will try to utilize maximum per possibly the notion of calculus of one variable, how does it give results in analyzing functions of 2 or more variables, and the applications in economics commerce and management.

So, at some points I will just give you a reference of what time assuming and what we are going ahead. So, the basic idea is that for a function of 2 or more variables, one would like to utilize what is the what are the results that we have already developed for function of one variable; that means, given a function of 2 or more variables we should try to bring it down to function of one variable. And that is not difficult as we can do it as follows.

So, what we are saying is some of the properties of the function of 2 variables can be analyzed by assigning some fix numerical value to all, but one variables. So, given of a function of say 2 variables you fix one of the variables assign some particular value, and analyze then it becomes a function of one variable only. So, all the tools available for one variable calculus will be applicable. This does not give you everything, but most of the things required for applications this is good enough. So, let us start looking at f, as a function of 2 variables f xy. So, let us fix the variable x as x is equal to x naught.

So, the value of the variable x is fixed with x is equal to x naught.

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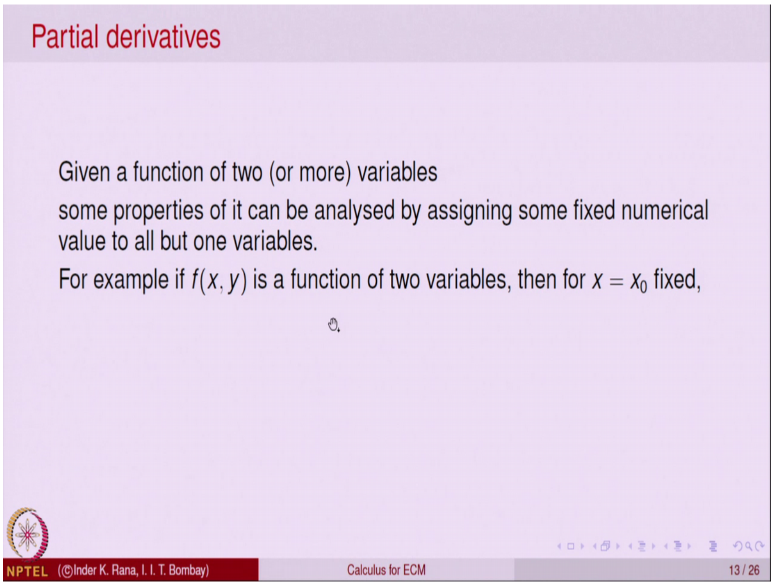

So, we are going to calculate what is f of x naught comma y for all y. So, x naught is fixed. So, only y is changing. So, this becomes a function of one variable y going to f naught of xy. So, this is becomes the function of a single variable. Similarly, you can fix up y equal to y naught. And then look at the values of fx comma y naught not as x varies in the domain. So, of course, you have to look at x comma y naught in the domain here also you have to look at all these points x naught comma y in the domain of the function. So, fix up a point x naught, you get a function of the variable y. And you fix up a value of y naught you fi get a function, function which depends only on x of a single variable. So, these are the 2 function, they do not give complete information about the function, but they give a reasonable amount of information about the function.

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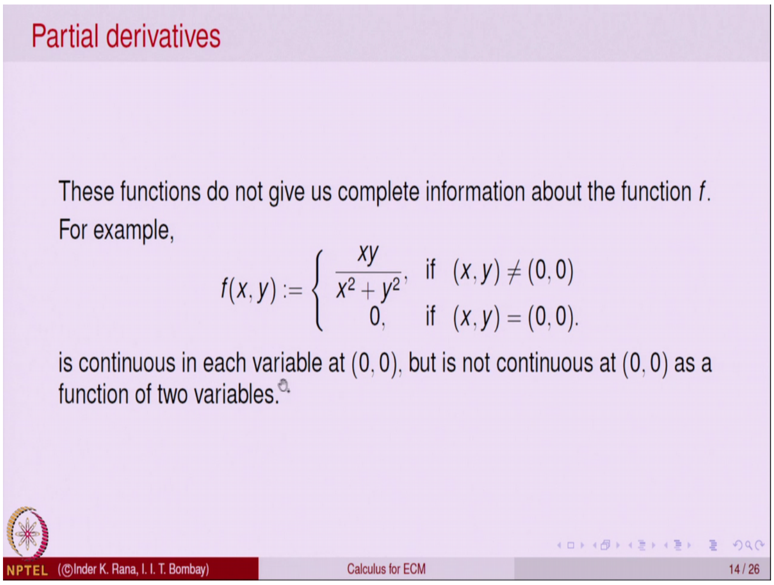

So, these are normally these functions, normally are called the coordinate curves for the function. In some sense you are fixing one of the coordinate x 0, and let the other coordinate y vary. And similarly, one coordinate y is fix at y 0, and x is allowed to vary. So, these are called the coordinate curves, we will not be having much opportunity to use them. But anyway so, let us look at function of 2 variables f of xy equal to xy divided by x square plus y square of course, to make it. So, that this is defined everywhere this formula does not give you the value at the point 0 0. At every other point when xy is not 0 0 this formula makes sense. So, the function is given by this formula when xy is not 0 0, and it is given equal to 0 if xy is equal to 0 0. So, let us try to find out if you fix up say the x coordinate or the y coordinate at 0, let us say. So, let us say fix. So, here is the statement which is a continuous in each variable at x 0 y 0, but it is not just continuous at the point the function of 2 variables.

So, let us just have a look at it what we are trying to say. So, let us look at the function. So, f of xy is equal to xy divided by x square plus y square.

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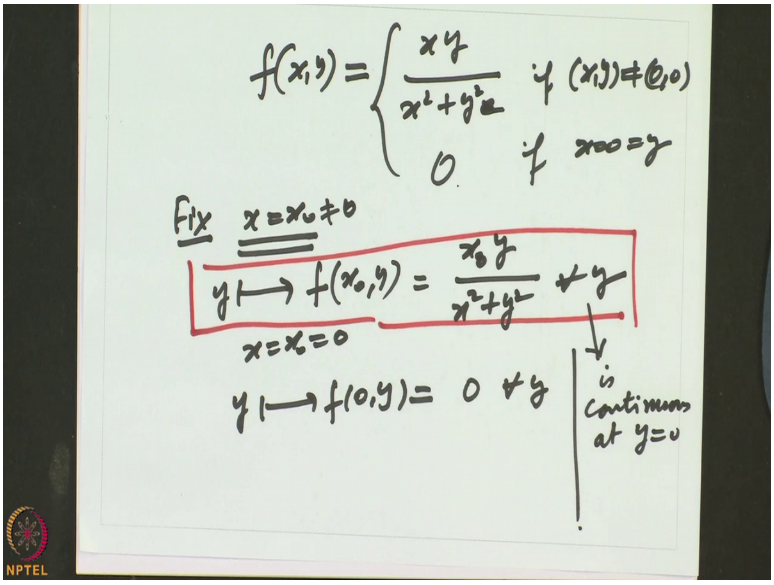

So, this is y squared if xy not equal to 0 0. And it is 0 if x is equal to 0 is equal to y, right. Now let us fix x is equal to x naught. Then what is the function f of x naught y is a function y going to this. So, what is that equal to if x naught is not equal to 0, then the function will be x naught y divided by x square plus y square, right. So, that is a function of the variable y. And if x is equal to x naught is equal to 0, then what is the function y going to f of 0 y. So, what is f; that means, x is equal to 0. So, it is 0 into y is 0 whatever be the denominator it is 0 for every y. So, this is for every y. So, for the value x is equal to x naught fix if it is not 0 this is the value. And this is the value if it is equal to x naught is equal to 0.

Now so, we are looking at this. So, this as a function of y. So, look at this function, this as a function of y we can ask the question this as a function of y is it continuous at the point y equal to 0. So, that mean what? If we let y go to 0 what will happen by the limit rules whatever b x naught y goes to 0. So, numerator goes to 0. So, this goes to 0 and the value at 0 is 0. So, this as a function of y is continuous at y equal to 0. And similarly, we can analyze what is the function when other point is fix right.

So, what we are saying is when you fix the point. So, this is continuous at 0 0 in each variable; that means, if you fix one of the variables as x naught, then becomes a function of y, and that is continuous at the point 0. Similarly, as a function of the other variable with y naught fix it is a function of x that also is continuous as a function of one variable, but one proves we are have defined what is the continuity of a function of 2 variables, but one proves that this function is not continuous as a function of 2 variables.

So, what I am trying to give you is that, when you restrict a variable in a function of 2 variables, it gives you some idea about the properties of a function, but it does not give you complete idea about that function. Example this example tells you that one once each variable is fix it is a continuous function, but when you jointly it is not a continuous function. So, something is missed when you restrict your attention to a particular variable. But still fixing a variable and analyzing the it gives a lot of information. So, how are the so, these are the coordinate maps x going to f of xy naught, when y naught is not fixed.

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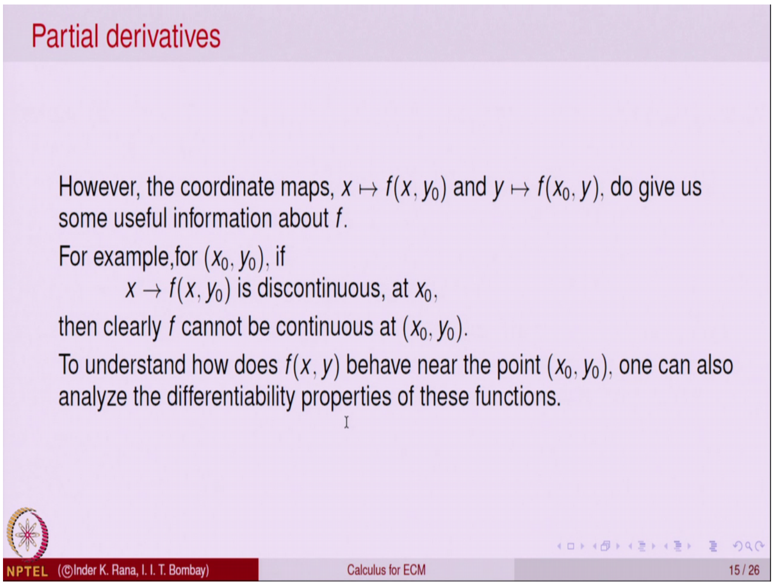

And other is y going to f of x naught y, when x is fixed at x naught these are the coordinates of maps. They give us some useful information, we will see what are that information. For example, look at this the following way, if a function if one of the coordinates maps is discontinuous at a point x naught, then the function itself cannot be discontinuous cannot be continuous at that point x naught y naught. Because there is a break in some sense in the graph.

So, these are the kind of informations which are useful that if a function is not continuous, one can prove these things rigorously. We are not going to do that, but I am just trying to give you a illustration that fixing one variable at a time does give you some information if not about continuity it gives you information about discontinuity. Namely if a function of 2 variables when one variable is fix at a point say y naught and it is discontinuous, then it cannot be continuous at a point x naught y naught and so on. So, to understand; how does fxy behave near a point x naught y naught, one can analyze differentiable properties of this function and so on.

So, let us what we are going to do next is going to look at the we looked at, we fix one variable at a time, and look at the coordinate functions and look at their differentiability properties. And let us see what information can we get out of that.

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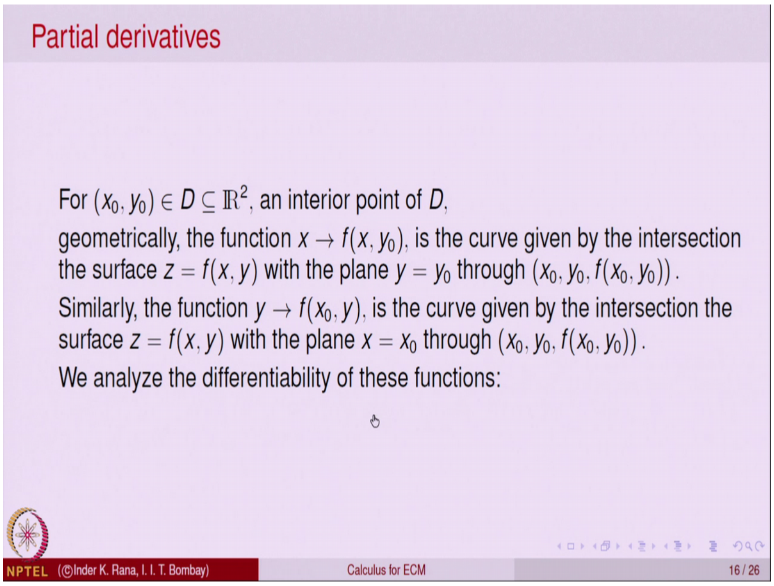

So, here is a function of 2 variables, x naught y naught belonging to d is an interior point in the see whenever we want to look at continuity differentiability, we normally look at the interior points in the domain not the endpoints and not the boundary points kind of a thing.

So, that you are able to move around in a neighborhood of that point. Geometric there functions so, this when you fix a variable y naught what are you getting y naught is fix; that means, at the point x goes to f of x naught y naught, that is a function of one variable. So, that is a curve in R 3 right. So, this is no longer a curve in the plane, it is a curve in R 3 and what is that curve? This curve is nothing but the intersection of the surface that is equal to x fxy with the plane y equal to y naught see.

So, visualize there is a surface, and the graph is a surface; when you are fixing a point y naught, right and you are looking at x we are varying x. So, you are actually moving along a line an x is varying. So, you are moving along a line x is varying y naught is fix. So, moving along a line in the domain, and you are looking at the values of the function at that height by the plane when it cuts that surface. So, this is a curve in the so, z is equal to. So, x goes to fxy naught this xy naught is a point which lies on the curve on a curve which is in the is a point on the is a image.

So, it is a image point for the function. So, it is a point on the surface. So, as x varies this gives you a curve on the surface. And geometrically this curve is nothing but the intersection of that surface that is a graph with the plane y equal to y naught. So, similarly when you fix the other one other variable as x naught, you get a curve on the surface which is the intersection of the plane x is equal to x naught. So, we have to now visualize what we are doing.

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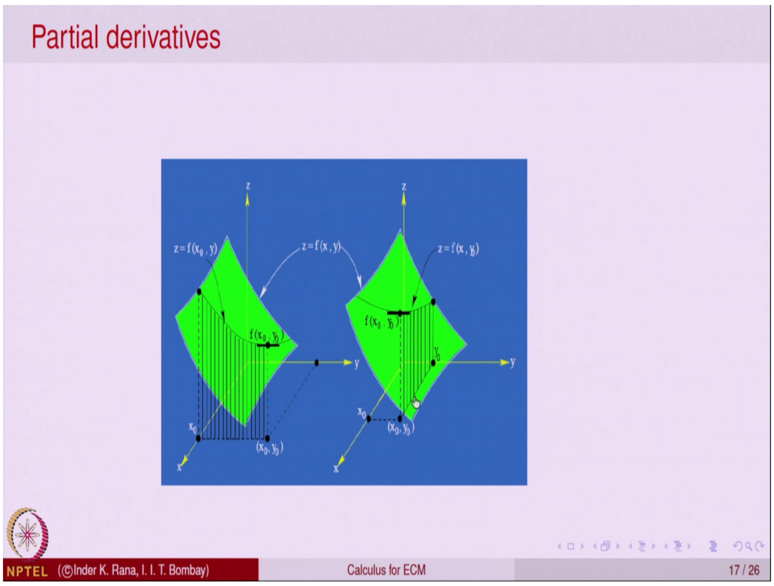

So, let us look at a picture of this. So, this is a picture. So, I imagine this is the R 3 right here is a domain. So, this is a surface. So, this is surface the green one is a graph of that function in the domain. So, at a point x naught, y naught if you fix x naught; that means, this point x naught is fix you are able to move along this right. You are able to move along this point x naught is fix. So, what is if x naught is fix y is allowed to vary. So, y is allowed to vary means you are moving along this line, you are moving along this line x naught is fix.

So, when you move along this line for every point, you will get a image for every point you will get an image for every point, you will get an image. So, you will get a curve on the surface. Then what is that curve? That is as if you take this plane passing through this point and going up. So, look at the plane x is equal to x naught raised up and intersecting with the surface. So, that is a curve right. So, that is the you should how that is how they visualize the range of a function of 2 variables when one variable is fix. Similarly, when you in this when you fix y naught, then what you are allowed to move is along this line in the domain.

When you move along this line, the point each point will give you a point on the graph of the surface on the graph of the function that is a surface. So, you will get a curve here right. So, this is a and what we want to know is; so, you get 2 curves right. So, this is the domain and that is the curve. So, it is the function of one variable you can ask whether this is differentiable. At the point x naught y naught similarly whether this is differentiable at the point x naught y naught; that means what? So, when you are fixing x naught you are looking it as a function of y. You get this curve and at this point whether it is differentiable meaning at this point as a function of y. So, y equal to y naught whether there is a tangent here or not.

Similarly, for the other case, when you are fixing y naught you have fixed y naught you are moving along x. So, you get a curve. So, whether you are getting a tangent at the point x naught y naught. Let me try to show it visualization of this the partial derivatives.

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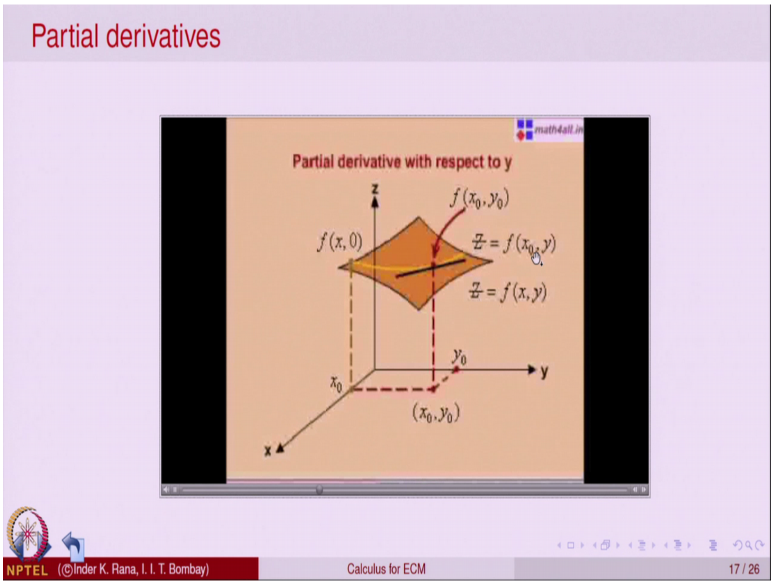

So, look at so, this is a surface. So, when you fix x naught. You are moving to move along this curve. And that curve is the one which is x naught y. So, at this point you want to know the tangent.

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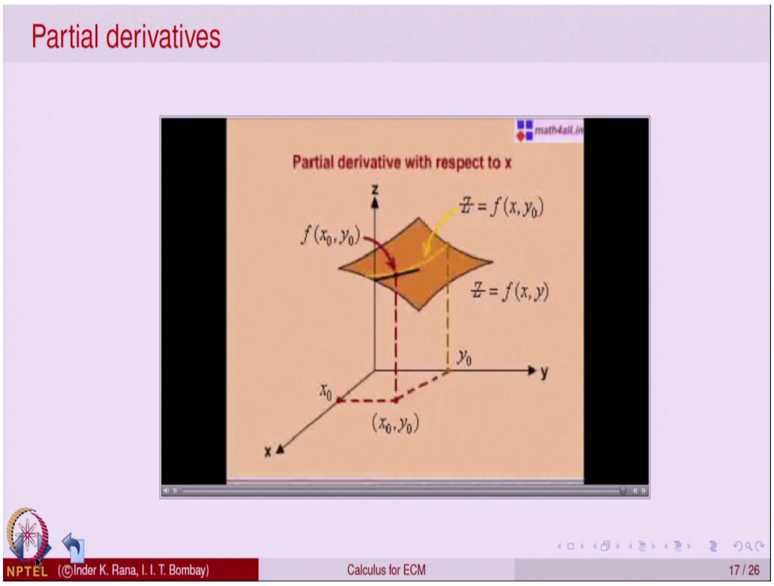

Similarly, for this you are moving, moving from a point y naught. You will get a curve right as x varies, and you want to know whether that point there is a tangent possible or not yeah. So, let me show you once again. This is a surface that is a graph of the function, at a point x naught y naught right if you fix x naught if this point x naught is fixed you are allowed to move along this line only. So, once you move along this line, you will get a curve which is z is equal to f of x naught y, y is allowed to vary. This is a point f of x naught y naught. And saying whether this is differentiable at this point or not, means whether at this point you are able to draw a tangent or not.

So, that is what meant by that. So, whether a tangent is possible at that point or not. So, that is a notion of differentiability for a function of 2 variables, when one variable is fixed and the other variable is allowed to vary right. So, similarly you will have that one-point y naught as fix and x is varying. So, you will be allowed to vary along this line. So, once you vary along this line you get a curve, and at this point on this curve which is f of x naught y naught you want to know whether a tangent exists or not.

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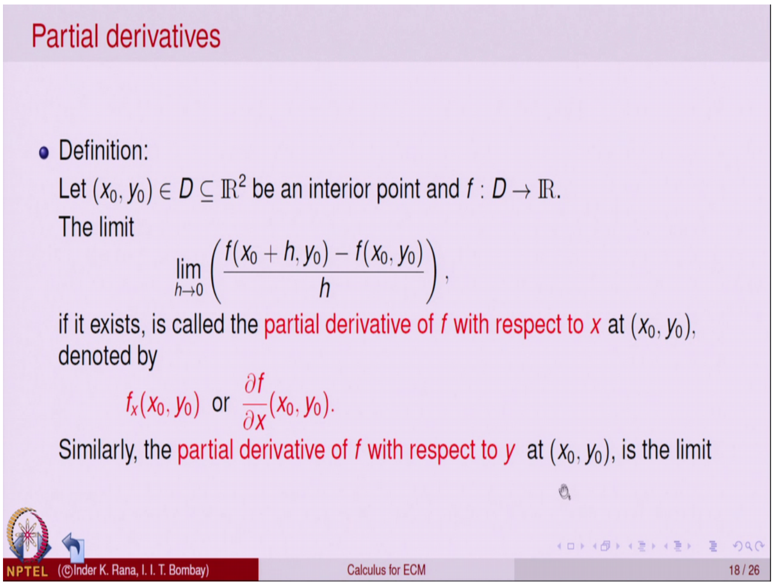

So, this is so, let us make it mathematically precise what we are saying is for a point x naught y naught in the domain an interior point of course, look at f of x naught plus h y naught. So, that means what? You are keeping y naught fix at both you are looking at the value at the point f of x naught plus h comma y naught, and the difference f of x naught comma y naught. So, we are looking at the increment in the function in the direction of x axis only y naught is fix. How much is the increment? H. So, this is the rate of change of the function in the direction of x axis at the point x naught y naught. So, limit h going to 0.

So, this we will call as the partial derivative of the function f at the point x naught y naught in the direction of the variable x, because x is varying y naught is fix. So, once again we are trying to understand what is the rate of change of the function of 2 variables in the direction of along x axis. So, when you say along x axis you will change the values only along the x axis, but keeping the y you will go parallel to the x axis keeping y naught y equal to y naught fix. So, at a nearby point the values f of x naught plus h comma y naught.

So, this is the increment in the function when you move from x naught y naught to a nearby point x naught plus h comma y naught and divide it by h. So, that gives you the ratio of the change and h going to 0. If it exists that is called the partial derivative of the function f at the point x naught y naught. And we denote this partial derivative as f of x, x naught y naught. So, there is one notation use they remains for the function x when the lower x; that means, we are looking at the derivative partial derivative derivative with respect to the variable x at the point x naught y naught. And that means, this limit should exist.

So, that is a value equal to this. Or this is also written as something similar to dy by dx. So, instead of writing dy by dx, or df by dx we write this Greek letter called partial. So, del f. So, this is also called del. So, del f by del x. So, this is called the partial derivative of f with respect to x. So, this symbol is called the partial. So, partial f divided by partial x the partial derivative of f with respect to x at x naught y naught. That is this limit if it exists that is called this. So, this is the partial derivative of the function f in the along the variable x at a fix y equal to y naught. So, whenever it exists this is denoted by this value. You can have something similar for the other variable.

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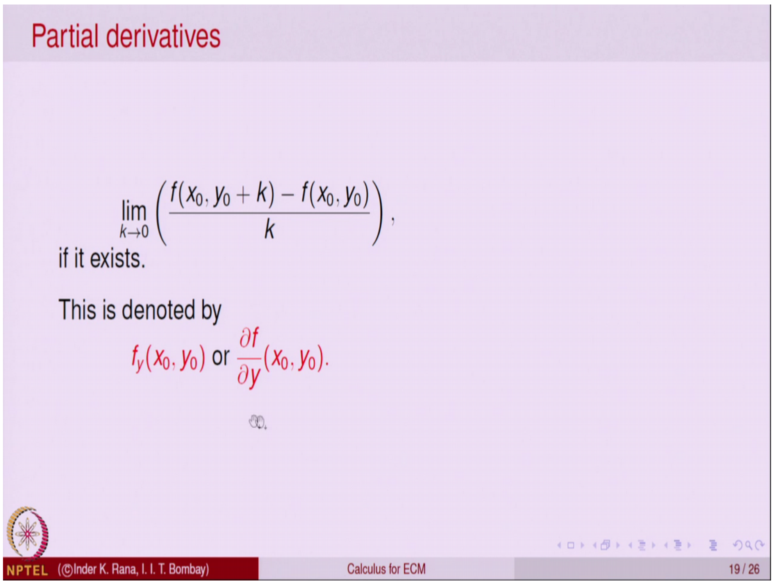

So, let us similarly we can define the partial derivative of f, with respect to y at the point x naught y naught to be the limit that is x naught is fix. So, y is allowed to vary from y naught y naught to y naught plus k. So, that is a increment in the value of the function as you go from x naught y naught y naught y naught plus k increment is k. So, take the ratio and take the limit like in one variable. So, we are treating this is a function of one variable. If this exist this is called the partial derivative of f with respect to y at the point x naught y naught. Or it is also as the variable partial del f by del y. So, this is either read as del f by del y or just say partial of f with respect to y at x naught y naught.

So, these are numbers by the way. So, for like for a function of one variable the derivative at a point is a number is a scalar. Similarly, the partial derivative of a function of 2 variables with respect to either variable at a point is a number, alright.

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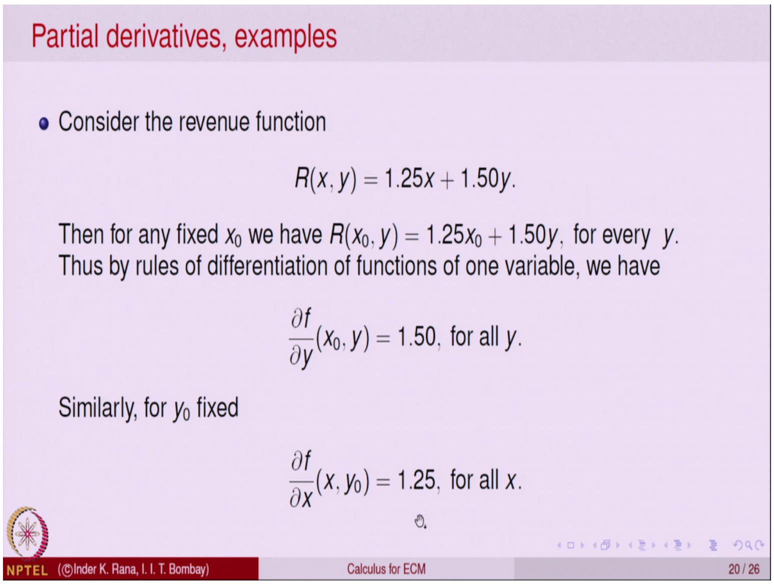

So, let us consider our revenue function that pharmacy example, where the revenue by selling 2 different kind of products was R x y equal to 1.25 x plus 1.50 y. Then for any fix value of x to be equal to x 0, right. If x is equal to x 0 then the function is 1.25 x 0 x is equal to x 0 1.5 y. So, with x is equal to x 0 fix this is the value of the function of 2 variable when y is allowed to vary. Now x 0 is fixed so; that means, this quantity is a constant as far as the variable y is concerned. So, this is only a function of one variable. So, we can differentiate apply the rules of differentiation of one variable, and calculate it is derivative.

So, that tells us because this as a function of y is differentiable. So, we get the derivative partial derivative of this R x y with respect to y at x 0 y, whatever be x 0 the value comes out to be this is this gives you the derivative to be 0, and this part gives you the derivative 1.5 for all y. And similarly, the partial derivative with respect to y equal to y 0 fix, if this is fixed at y equal to y 0 you will get the value 1.25 as a partial derivative at the point. For every all values of x the partial derivative at a point x 0 y 0 or any point is equal to same.

So, it is. So, this is how you calculate the partial derivative the idea is. You fix one of the variables see what the function looks like, and see whether rules of differentiation of one variable are applicable and accordingly calculate the derivative. Sometimes you may have to do it by the first principle as the limit, sometimes just looking at the function and you can apply the theorems of differentiation. So, this is we applied.

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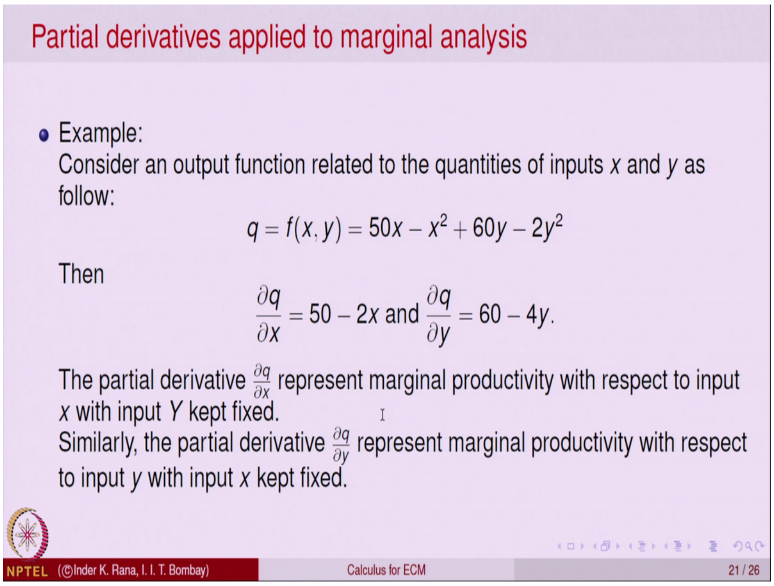

So, let us look at consider an output function related to the quantities of inputs x and y as below. Where q is a function of 2 variables x and y say that is 50 x minus x square plus 60 y minus 2 y square. So, this q that is the output is a function of 2 inputs x and y and is related by this formula. So, this is defined for all x and y and belonging to and R. And it is a nice function it is only (Refer Time: 24:22) powers of x and y. So, this is what is called nominally a polynomial in 2 variables. So, let us fix one variable and calculate the so, let us the partial derivative of q with respect to x. So, with respect to x means, we are fixing the variable y right. So, y is going to be fixed. So, when y is fixed 60 y is fixed as a constant minus 2 y square is fixed as a constant.

So, as far as x is concerned, these are the only terms where x is appearing this is the constant as far as x is concerned when x is fix. So, we are looking at a partial derivative of q with respect to x. So, we will treat 60 y and 2 y square as a constant. So, what will be the derivative of this with respect to x. So, dq by dx will be 50 minus 2 x. And similarly, dq by dx, dq by dx will be equal to treat this as constant and compute what is the value of the derivative with respect to y. So, this being a constant differentiating with respect to y gives you 60 minus 4 y.

So, these are the partial derivatives of q, this output function with respect to the input x and this is with respect to the input y. So, if you want you can relate it with the marginals now. In one variable we looked at the derivative to be as the marginal of that quantity whatever the function we were looking at. So, here is the production function q as a function of 2 variables. So, we can call this partial derivative of q with respect to x you can say it is the marginal or productivity with respect to input x and with the input y as fix.

So, keeping the input y fix, what is the marginal of the product function with respect to the variable x. So, that is partial derivative of q with respect to x. Similarly, if we keep x fix the input x fix we do not change the input x, keep it fix at some particular value. And look at the partial derivative of q with respect to y, that gives you the marginal of cost marginal of the product function with respect to the input y for the other variable x as fix. So, these are the interpretation of the partial derivatives as the marginals. So, we will continue our study of partial derivatives and it is applications in analyzing functions of several variables in scenario of economics so, we will.

Thank you.