Calculus for Economics, Commerce & Management
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Lecture - 36
Functions of 2 variables, visualizing graph, level curves, contour lines

Let us start recalling that, what we have done till now is looking at various kind of a functions in economics, that depended on one variable. And typical example was that of a production function which normally may depend upon a factor like a single variable like labour. It can depend on the capital and so on. So, but in general that is not the case, the production of a product depends on many things many inputs. So, for example, let us look at some scenario a pharmacy sells 2 brand of aspirin.

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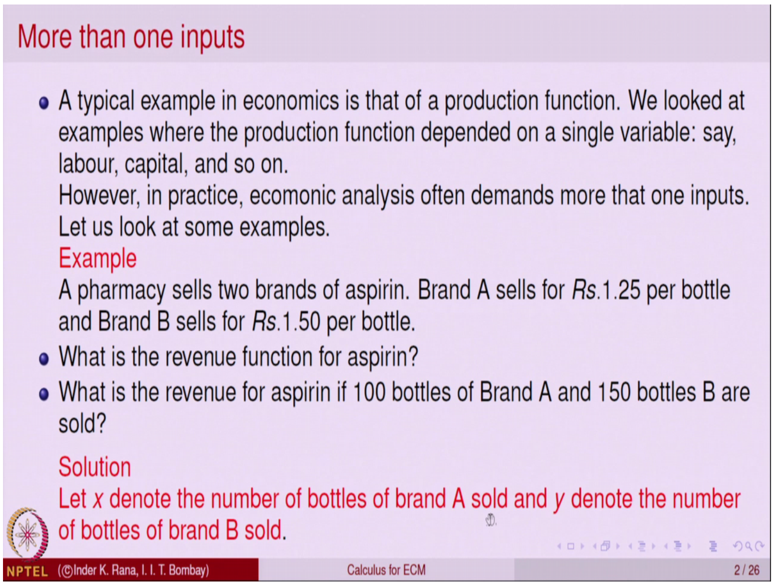

Brand A sells for rupees 1.25 per bottle, and the brand B sells for rupees 1.5 per bottle. So, some pharmacy selling 2 products of aspirin one at 1.25 per bottle, and another one at 1.5 per bottle. So, what is a revenue function for aspirin selling of aspirin for that particular pharmacy. So, to write down the revenue function, we will have to look at how many units of brand A are sold, and how many units of brand B are sold.

So, let us make in mathematics, that is when we do we do not know something we put x or y as the variables. So, and another question will like to answer later on is what is the revenue for aspirin, if one thousand bottles of A and 150 bottles of B are sold. So, let us assume that to solve the problem, that x denotes the number of bottles of brand A, which are sold and y denotes the number of bottles of brand B that are sold. So now, we will write down the revenue, if x bottles of brand A then the revenue from selling brand A would be x into 1.25. And the revenue from by selling brand B would be y into 1.5. So, total revenue you can call as the revenue, it depends on 2 variables x and y x has a number of bottles of brand A being sold and y is the number of bottles of brand B being sold.

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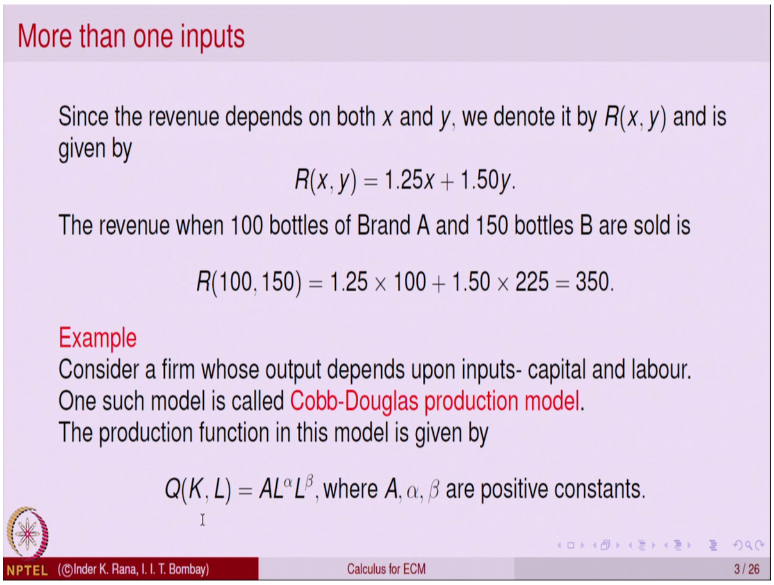

So, total is 1.25 into x plus 1.5 into y. So, this gives a revenue function which depends on 2 variables x and y, alright.

So, the relationship which is actually a function. So, let us now solve the what are as for that is for 100. Bottles of brand A are sold and 150 bottles of B are sold then the total revenue will be R, where revenue for x is equal to 100 and y equal to 150. So, we will put the values like in one variable now there are 2 variables 2 quantities. So, x is equal to 100 and y is equal to 150. So, put those and compute. So, that gives you total amount to be equal to 350. So, this is the revenue for those many bottles of A and those many bottles of B. So, the important thing that I am trying to motivate here is that the problems in general in economics commerce and management functions need not be of one variable.

The output may depend on lot of inputs of different kinds. So, here is one example that we looked at. Here is another example which is very, very important and common in economics scenario. It is called the cobb Douglas production model. So, a that model there is inputs are the labour, and the capital. And the output is the output function Q depends on K and L, K is the capital, and L is the labour. And the formula suggested by cobb and Douglas is that it should be equal to a times L raise to power alpha, and this should be actually K here, K raise to power beta. Otherwise it looks like a function of one variable.

So, the second one a raise over alpha and K raise to power beta. So, there is a typo, here this should be K raise to power beta. So, this becomes a so, cobb Douglas function production function or production model depends on 2 variables K and L. K is the capital L is the labour. It is a times L raise to power alpha K raise to power beta. Where this a alpha and beta are all positive constants. When we later on when we study this slightly in detail, we will probably analyze this model a bit more, alright.

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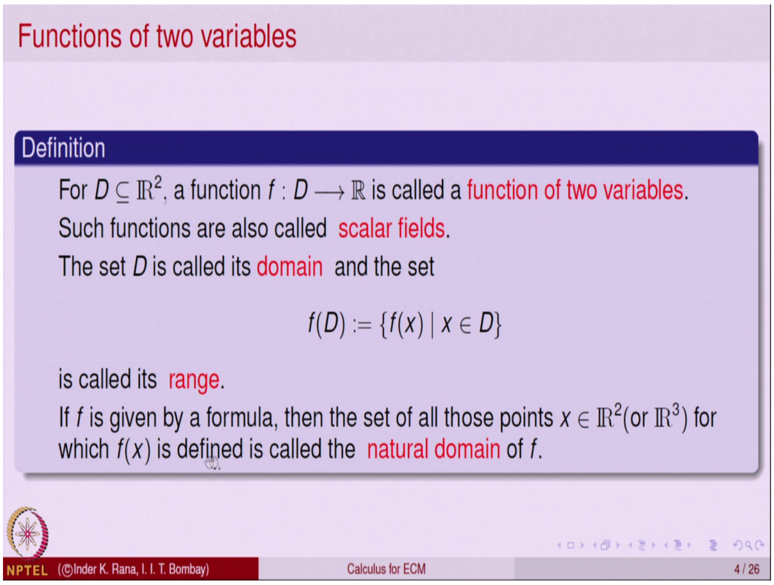

So, let us define formally that a function of 2 variables is a function with a domain D in the subset of R 2. So, it is a function with a domain D from which is a subset of R 2 taking values in R. So, this is called a function of 2 variables. So, here the domain has changed, for one variable the domain was a subset of the real line, for functions of 2 variables the domain is that of subset of the plane.

So, this is the function of 2 variables. Let us write such function are also normally called scalar fields. So, I will not go into details why they are called scalar fields. Basically, the idea is that the values of the point in D f of that value image of that is scalar. So, that is why it is called as scalar field. These are important importance in physics and mathematics actually. So, this D is called the domain of the domain of the function, and the set f of D means for a very point x in D, look at where alpha x goes that is a number. So, that gives you a subset of the real line.

So, that is called the range. So, domain for a function of 2 variables is a subset of the plane, R 2 and the range is a subset of the real line R. So, this is what function of 2 variables is. One can actually consider functions of more variables also. So, instead of taking D to be a subset or R 2. You can it to be a subset of R 3 R 4 and so on, and depending on how many variables how many inputs are there going to be there in your consideration in your analysis.

We will discuss only functions of 2 variables, most of the mathematics that we will do carries over to functions of more than 2 variables also. But let us and try to understand what we are trying to say for functions of 2 variables. So, sometimes the function is not explicitly given that every point, what is a image, it is specified by a formula. So, when a function is given by a formula, then the set of all those vales x in R 2 or if it is a function of 3 variables R 3, for which f of x is defined for which that form formula makes sense is called the natural domain of the function.

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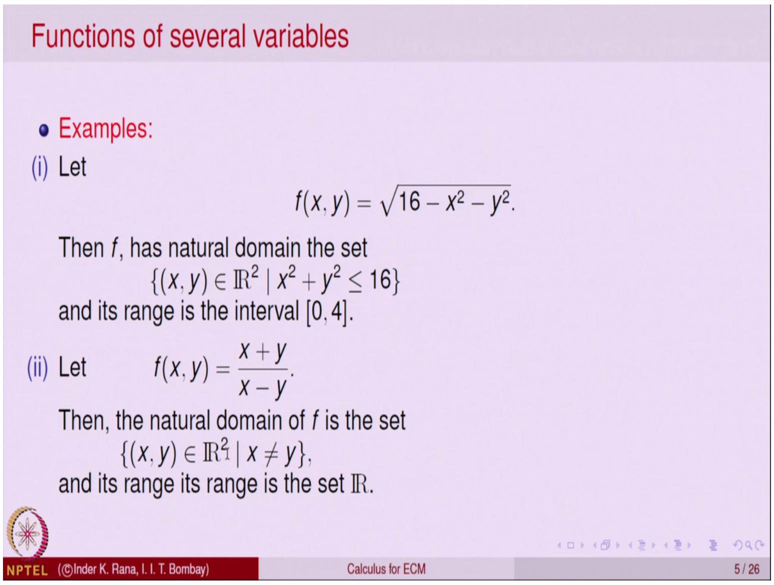

So, let us look at some examples of this. Let us look at a function say given by a formula f of x y is equal to square root of 16 minus x square minus y square. Where x and y are real numbers. Now because the right-hand side is the square root and we want f of x y to be a real number. So, square root of a quantity make sense only when that quantity is a number which is bigger than or equal to 0. So, the natural domain for this function is all x comma y belonging real line. So, is that 16 minus x square minus y square is bigger than or equal to 0. Or same thing I am saying x square plus y square is less than or equal to 16.

So, this is the natural domain of the function, and what will be the range of this in function? The range will be so, here now you should understand intuitively that if x square plus y square is equal to 16, then the value is 0. So, the smallest value that this will have is 0. And we can keep on the maximum possible value that you can give is and say for example, x is equal to 0, and y is equal to 0, right. In that case that will be the largest value for the function right f of xy. Because is minus x square and minus y square. So, whenever x is positive or negative, if it is non-0 then you will be subtracting something from 60.

So, it will be smaller than 4. So, the largest value is 4. So, that way you analyze and say that the range of the function is the interval 0 to 4. So, this is the function given by a formula with natural domain is x square plus y square all points in the plane, say that x square plus y square is less than or equal to 16, and the range of this function is the interval 0 to 4 is the close interval 0 to 4. Because both the values 0 is taken when x square plus y square is equal to 16, you can choose any values x is equal to 0 and y equal to 4. So, that will give you the value 0. And when x equal to 0 and y equal to 0 that gives the largest value that is 4. So, it is a close interval 0 4 that is the range of this function.

Let us look at another function f of x y is equal to x plus y divided by x minus y. So, for this formula to make sense, it is natural to have x minus y not equal to 0. So, the natural domain of this function is all points x comma y in R 2 size at x minus y is not equal to 0. So, that is the natural domain of the function. So, x is not equal to y. Geometrically this is the plane x y in R 2 that is a plane, and what is x not equal to Y; that means, you are removing the line x is equal to y from R 2 and the remaining part is the domain of the function. So, this is the all the points where x minus y is equal to 0 is a equation of the line y equal to x. A line passing through 0 0 with slope 45 degrees. So, the domain is R 2 minus the line x is equal to y. And it is range; obviously, is the whole of real line you can take everywhere. So, how do you find what is the range? So, let us put we want f of x to be equal to some number alpha. So, then you can solve for x and y, you can find many values of x and y says that this is equal to alpha.

So, the natural domain for this is natural domain is the R 2 minus line x is equal to y, and the range is the whole of real line. So, that is how mathematically you analyze what is a domain and what is the range of the function. So, let us start with a function of 2 variables one would like to know what is the graph of this function. See for a function of one variable f of x y right y equal to f of x, we said the graph is a subset of the plane R 2.

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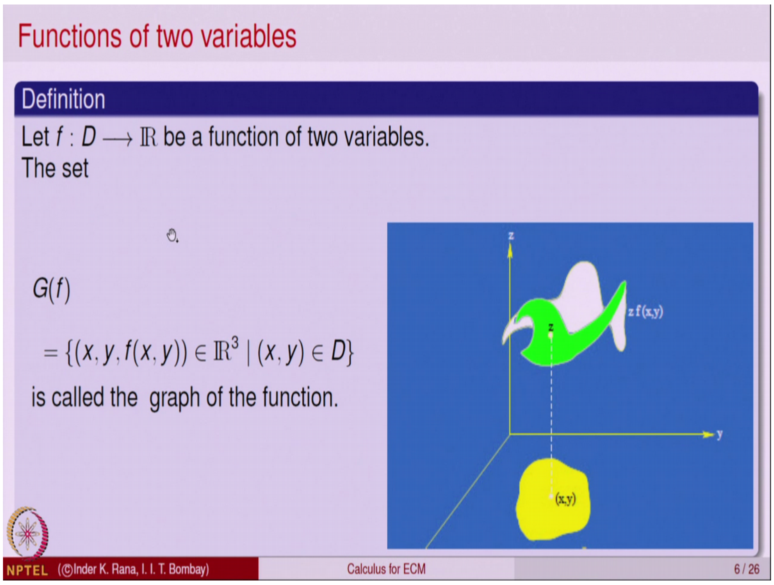

So, it was a region in the plane R 2. So, what was the graph it was x comma f of x, x belonging to the domain. So, that was the graph of the function. For a function of 2 variables graph is the set which describes everything about the function. So, if I take x and y in the domain, and I know where is f of x y, then I know what is the function for every x y in the domain if I know f of x y, right. I know what is the function. So, the graph is going to be a subset of R 3. So, for a function of 2 variables where domain D is a subset of R 2. The graph is a subset of R 3. I hope all of you remember or understand what is R 3.

So, let me draw a picture of R 3 for the sake of those who have not gone through who have not come across what is R 3.

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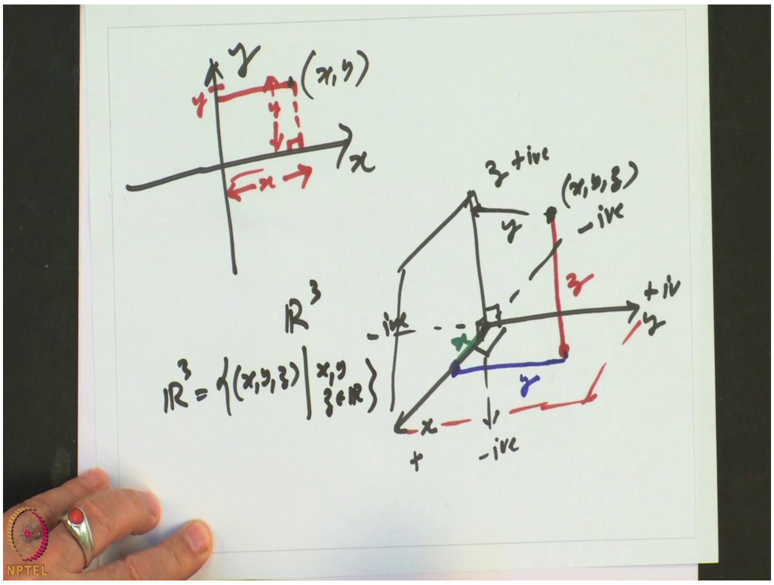

So, R 3 like plane is represented by 2 axis x and y axis. So, this is how you represent a plane. So, that is x axis and that is y axis. And every point has got 2 coordinates x and y. So, what is the coordinate? So, to find what are these values you drop a perpendicular. So, this value is x and this height is y. So, this is y and that is x. So, that is how we locate a point in the plane. The coordinates means x y means, x is the distance of that point from the y axis. So, that is the distance from the y axis that is x. And y is the distance of the point from x axis. So, that is so, in what is R 3, R 3 is normally represented by 3 coordinate axis. So, they are called x y and z. And so, they are 3 lines so, that each is perpendicular to each other. So, all the 3 lines, you can think it of as a corner in a room where the 3 one wall that is 2 lines are meeting in the corner, and the ground is meeting the 2 walls. So, that is the axis.

So, this is x axis. In fact, you can these are all lines which are on the (Refer Time: 15:08). So, this side is the positive side. This side is a positive and this is a negative side. And similarly, this is a positive side, and this is a negative side of the axis. And similarly this is a positive up is normally taken as positive down is taken as negative, right. Now how does this tell us? How do you locate a point in the space? So, these 3-coordinate system is enough to locate a point in space. So, let us take a point f with coordinates x y and z. So, what is x y and Z? So, first of all from this point if you are moving parallel to the z axis, you are moving parallel. So, when you move parallel, you will hit this plane the what is the x y plane. So, this plane x y plane, you will hit it at some point. So, that the that distance is the distance z. Now once you have reached that point, from this point you can move in the x y plane either parallel to y axis or parallel to x axis.

So, let us move parallel to y axis. So, this is the y axis, if you move parallel to y axis you will hit the x axis at some point. So, that distance is y. And if you move from that point to origin. So, that is x. So, that is how you locate the point x y and z. So, this is the perpendicular distance from the 3 planes basically you can call it. So, this is a perpendicular distance, that is the distance of this point from the x y plane, and this y is a perpendicular distance you can also think of this y is same as this y. So, that is the perpendicular distance of the point from the x z plane.

So, that is y and x is nothing but the perpendicular distance of the point from the y z plane. So, that is the this distance. So, that is how you locate points in space. So, that is what is R 3? So, in R 3 so, this is what is R 3 denoted as. So, R 3 is written as a triplet x y and z where each x y and z belong to real line. So, this is the coordinate system in the space which helps us to locate uniquely a point and given coordinates, the point is uniquely located and vice versa. So, that is how we use this.

So, the graph of a function. So, we were looking at the graph of a function of 2 variables.

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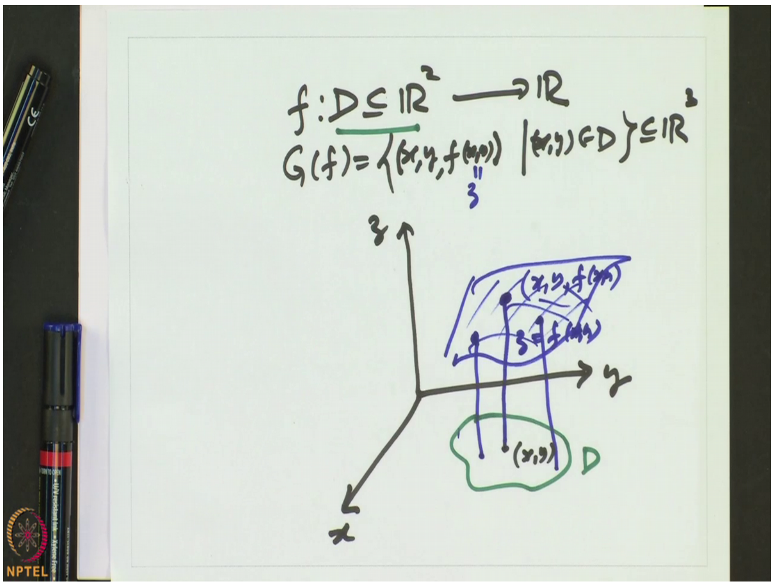

So, we are given a function f from a domain D contained in R 2 taking values in R. So, it is graph we are saying is nothing but look at a point x comma y, x comma y belonging to the domain. And look at it is value f of x y call it as the third coordinate z. So, look at this triple. So, this is a subset of R 3. So, geometrically you can visualize it as follows. This is x, this is y and this is z. So, what it a domain? Domain is a subset of the real line R 2. So, this is the domain.

So, let us say the domain of the function is this said D. So, that is a domain of the function. So, for every point in the domain of the function. So, this is the point x comma y, for every point in the domain of the function what we do we look at what is f of x y, that is my z. So, this is z equal to f of x y third coordinate. So, to locate the third coordinate from the x y plane, I will move a distance which is z equal to f xy. So, then we will get the point p. So, that is the point p x y, f x y.

So, for every point you go up you will get a point in R 3 for every point in the domain you will get a point, and all these points will give you some kind of a object in R 3 right. So, where are these dots will lie, and that is precisely what is the graph of the function in 3 variables. Now for graph of the function of 2 variables. So, that is what this picture is telling you, x y is the point in the domain; for every x y you look at the point z. So, collect all these points z. So, they will lie on some shape in R 3. In fact, that is what we call as a surface in R 3. So, that surface in R 3 is the graph of a function of 2 variables. In one variable the graph was a curve in plane for a function of 2 variables the graph is a surface. So, one can try to visualize this.

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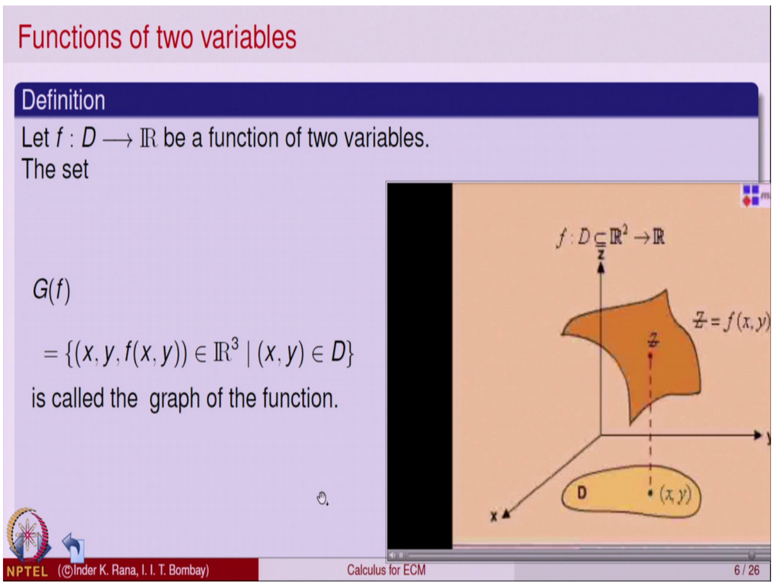

So, graph of a function of one variable. So, that is what. So, take a domain D and take a point in the domain, and go to the height that will give you a point z. And locate all these points put them together, that gives you a surface in R 3. That is what is the graph of the function of the 2 variables, right.

So, let us go over a bit more. So, how do you visualize this is the kind of a surface, right. How do you visualize what kind of a surface you will get for that as a graph of a function? For function of one variable it is not very difficult to visualize it is a curve only, but now it is a 3-dimensional object and to visualize that one looks at 2 things..

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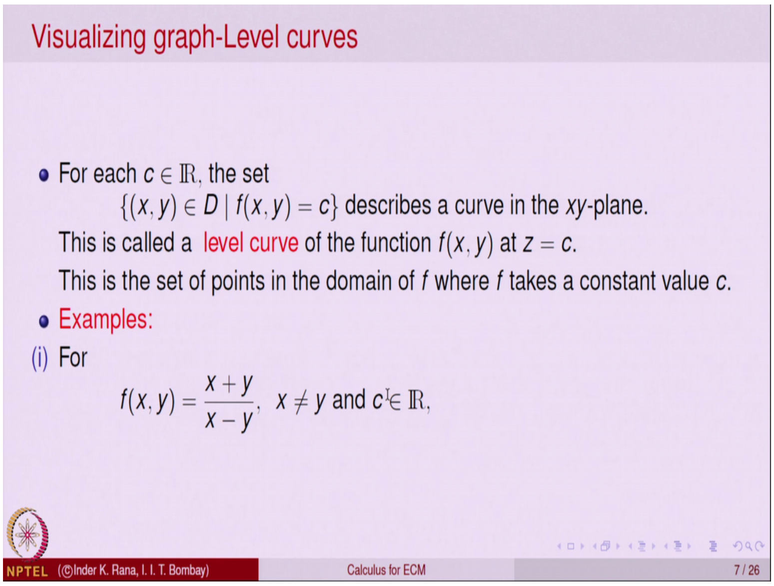

One is called the level curves for a point x y in the domain look at the points f of x y equal to c. So, look at all the points in the domain where which take a given values c. So, these are called these are the see when a f of x y is equal to c that gives you a relation between x and y. So, it will be a kind of a curve in the domain of the function. So, this is called a level curve for the function. So, level curve is a part of the domain for which the height right the f of x y is equal to c at a height c in the graph of the function. So, all those points c in the graph of the function, if we look at what are the points x y which are mapped into c, then that is called the level curve. So, this is a 2 dimensional. So, f of x y is a curve in the domain of a function. So, that is what is called the level curve.

Or the function at the point z is equal to c. This is one way of visualizing the level a function of 3 variables. This is the set of points of domain of f where f takes a constant value c. So, that is best way of describing. So, let us look at an example. So, look at that example we had x plus y divided by x minus y x not equal to y. And let us take a point c in R, and see what is the level curve for where the function takes the constant value c; that means, x plus y divided by x minus y is equal to c, and that is same as say x is not equal to y.

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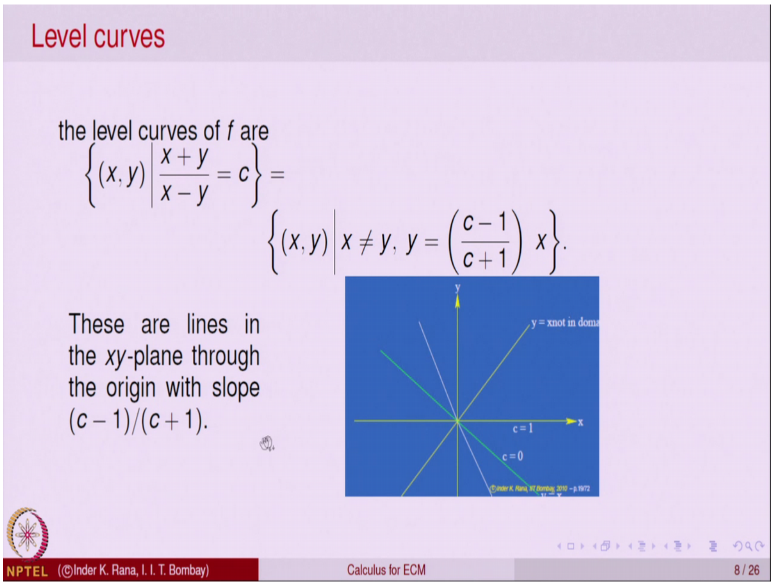

So, that is same as saying y is equal to c minus 1 divided by c plus 1 into x. So, this is again a equation of a line passing through as the origin with this as the slope where c is the given constant. So, these are lines passing through the origin as the level curves for the level curves for the function f of x y is equal to x plus y divided by x minus y. So, that gaves you give you some idea in the domain at this point, this is the value taken, right? At this point the value will be taken as c and given by this.

So, that is particular c for a so, these different lines are for different values of c. So, we will get curves as we change the values of c. So, for this function each value of c gives you a line through the origin. So, level curves for different values of c are nothing but lines through the origin.

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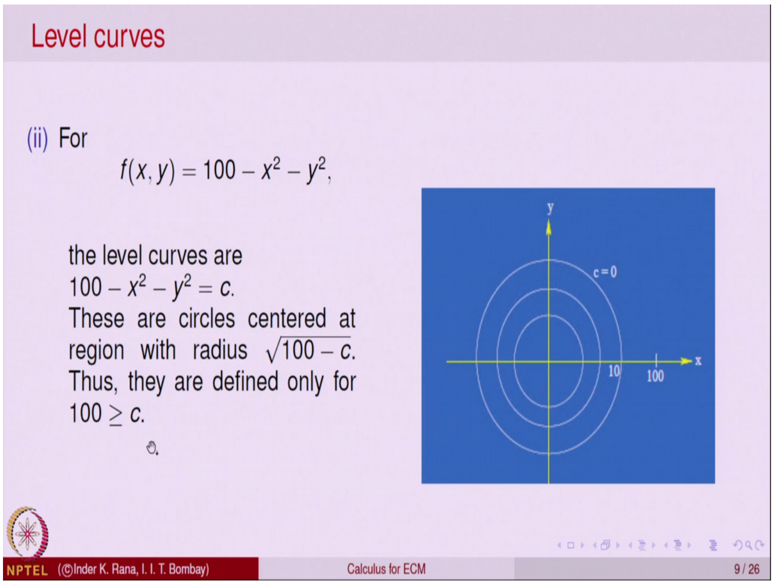

Let us look at the function f of x y is equal to x 100 minus x square minus y square. So, first of all what is the natural domain of the function, the domain of the function is f of x y is equal to 100 minus x square minus y square. So, domain is whole of R 2, right. Because all values are allowed. So, that is now we want to find out what are the level curves for this. So, level curve mean what we want those points in the domain, where this value is equal to c.

So, 100 minus x square minus 100 minus x square minus y square equal to c; that means, x square plus y square is equal to 100 minus c. So, that gives you a circle of radius 100 minus c. So, level curves are possible only when c is less than or equal to 100. Or equal to 100 will be the dot for all other points it will be the circles; that means, for different values of c these are the level curves in the for the function; that means, at this circle the value taken will be the that value of c. So, at 10 that will be the value taken at the circle.

So, all the points on the circle when c is equal to 10, that gives you square root of 10 and so on.

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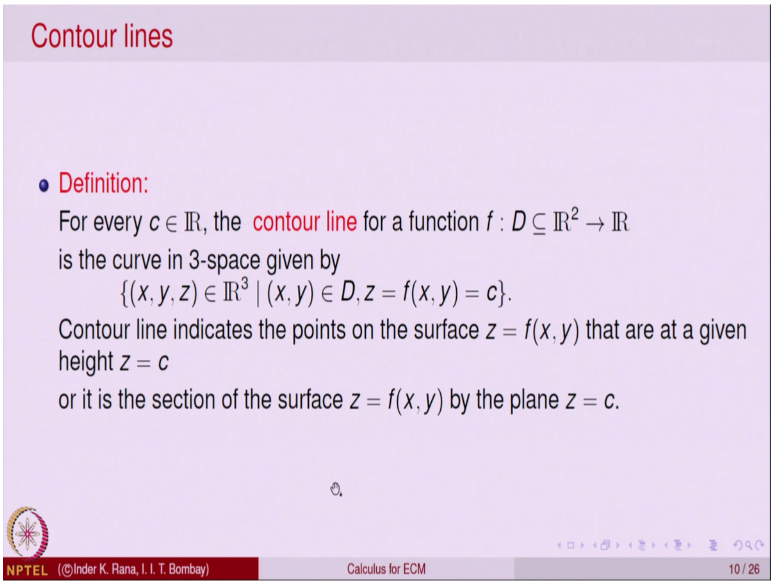

So, these are the one way of visualizing a function of 2 variables, the graph of a function of 2 variables. Another way is to look at actual part of the surface for that particular height. So, these are called contour lines for the graph of the function, a contour line for a function of 2 variables is nothing but is a points x y and z in R 3, say that x and y belong to the domain and z is equal to f x y. So, that means what? That means, for a point in the domain you are looking at all the points on the surface which are at a height c.

So, essentially what you are doing is contour lines the indicate the points on the surface on the graph that are at a given height z is equal to c. So, you can also visualize. So, these are the sections of that surface cut by the plane z is equal to c. So, you these are surface. So, if you take a plane z is equal to c that is a plane per parallel to the x y plane. So, at a height z is equal to c. So, see it what contour it will cut, at what curve it will cut.

So, that curve is a part of the surface part of the graph. So, contour lines are part of the graph those points which are at a given height c. So, if you have ever looked at a atlas. So, normally the topography of a place is described in terms of these level curves or contour lines. So, if you have not ever looked at a atlas. So, go back pick up an atlas and see what are contour line for example, if a mountain is to be described that is a mountain in a area then one describes the height, and the various points on the surface on the mountain at a particular height. So, that will give you a kind of contour lines on that object. So, this finds applications not only in economics commerce and management it finds applications in geography also, right.

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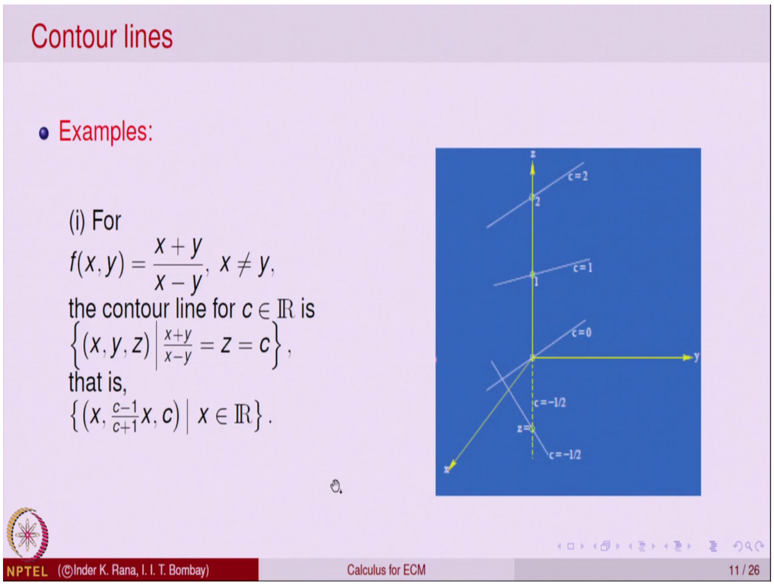

So, this is we have ways of visualizing. Similarly, for the function x plus y divided by x minus y, the contour lines right. Will be the lines z, z is equal to c.

So, these are the lines at a height z the equation of the line will be same. So, it is x comma y is computed in terms of c. So, that is c minus 1 over c plus 1 x, and height is z coordinate is c. So, these are the lines various lines for different values of c equal to 0 1 and 2 and so on. So, these are the parts of the surface at a particular height c. So, that is these are some of the ways one tries to visualize a 3 D object surface by contour lines and level curves.

This is one of this is a very important aspect of reconstruction problems for example, in all and you take a picture of a 3 D object you 2 D picture and constructing from the back the 3 D picture is something similar to that is been done here. So, let us just take the value take from this point, that a for a function of 2 variables the domain is R 2. The range is part of the real line. And the natural values natural domain for the function is wherever that formula given by the function make sense for all the points x and y in the plane that is called the natural domain.

All the values taken in the line images are called the range as before, and to visualize a function of 2 variables one has to look for a subset of R 3. So, the graph of a function of 2 variables is a subset of R 3. And there are 2 ways of visualizing them one is the level curves, another is the contour lines. So, with this introduction to functions of 2 variables we will continue our study in the next lecture.

Thank you.