Calculus for Economics, Commerce & Management
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Lecture - 35
Asymptotes, curve sketching

We had started looking at the notion of asymptotes for a function of one variable. So, we defined the notion now what are called a horizontal asymptote, the notion of vertical asymptote and the notion of oblique asymptotes. So, let us look at some examples of this, so let us look at the function f of x is equal to x divided by x minus 1 for x naught equal to 1. Because in the domain of the function is not; 1 is not included.

So, clearly as x approaches the value 1 from the left or from the right either way right then this difference x minus 1 becomes; if x is bigger than 1 and approaches 1, then this becomes positive and very very small. So, this will go to plus infinity and if x is approaching from the left side of minus 1; then this value will become negative and very very small. So, this quotient will be go to minus infinity; that means what? This will become very very small, but a very very large then negative number.

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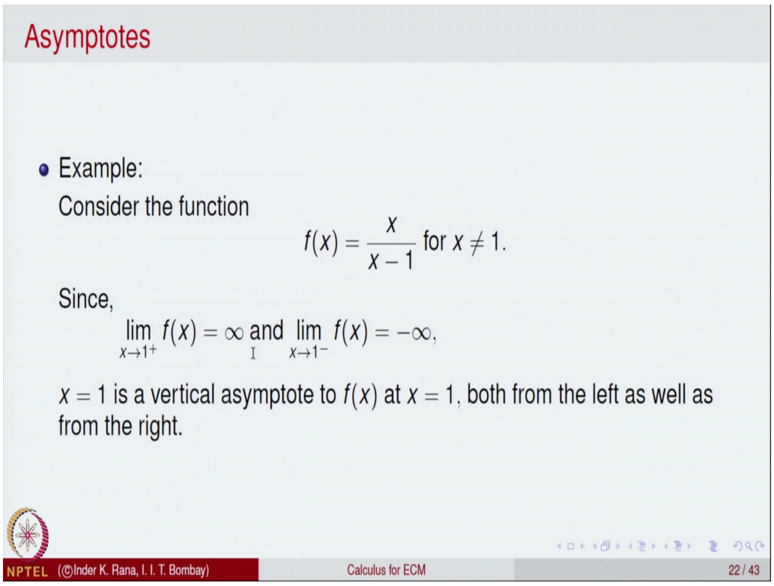

So, we write this as limit x going to 1 plus; that means, from the right side of 1 is equal to plus infinity and limit of x going to 1 from the left side is equal to minus infinity.

So; that means, what; so in our language x is equal to 1 is a vertical asymptote because what is happening? The value y is becoming plus infinity or minus infinity; is not becoming equal to, it is tending to plus infinity and minus infinity. When x is approaching the value 1; either from the left or from the right, so x is equal to 1 is the line x is equal to 1 is a vertical line. So, that is in an asymptote or the function from the left as well as from the right.

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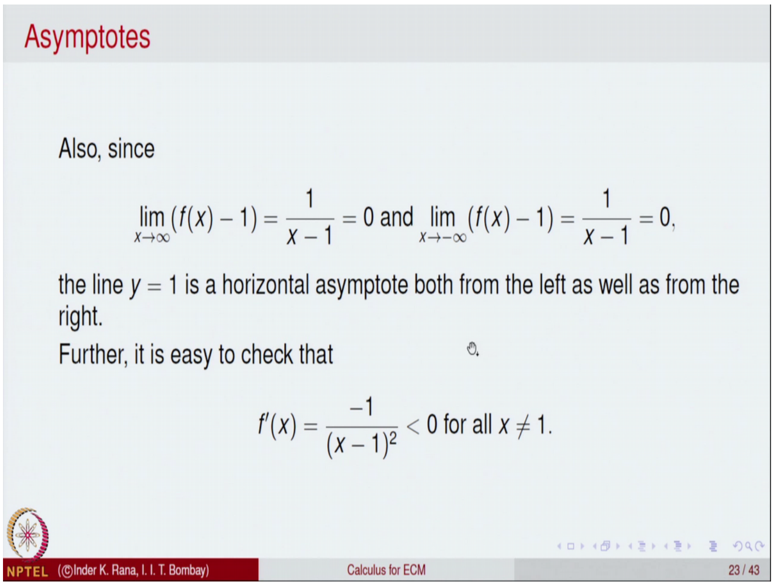

So, one can also if you look at f of x minus 1 then it is equal to 1 over x minus 1 that goes to 0; as x goes to plus infinity and similarly it goes to 0; if x goes to minus infinity.

So, what does that mean? That means, as x goes to plus infinity or x goes to minus infinity; y is becoming 0 so; that means, what? That means, the line y equal to 0; that is the x axis is a horizontal asymptote for; this is minus 1 sorry, not equal to 0; f x minus 1 that is going to 0; that means, f x is approaching the value 1 from the left as well as from the right. So, y equal to 1 or one could have written that this limit is equal to; so, this is a slight typo here, see f x minus 1 is 1 over x minus 1, so it is a limit of this.

So, limit of this is equal to limit of this; which is equal to 0 and similarly limit of f x minus 1 is same as limit of 1 over x minus 1 and that is equal to 0. So, y equal to 1 is a horizontal asymptote from the left as well as from the right. So we can look at this function slightly more explicitly; and if you look at the derivative of this function, so that is minus 1 by; 1 over x minus 1 whole square which is less than 0.

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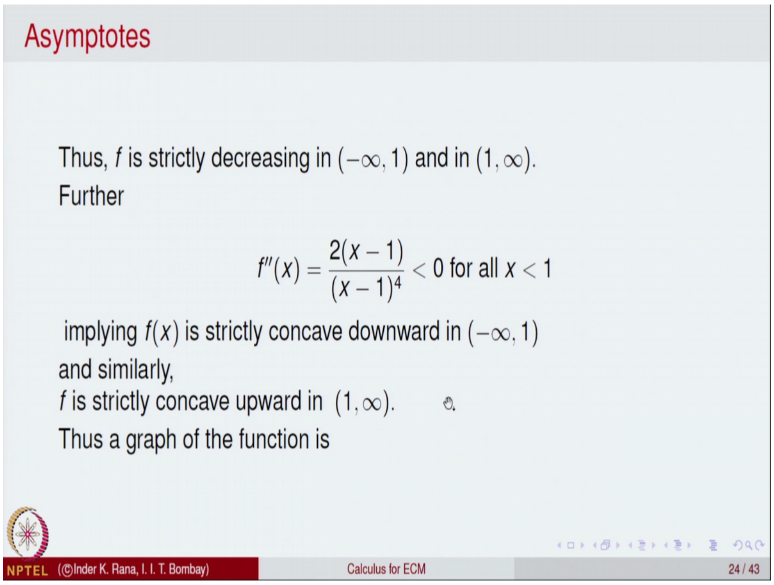

So; that means, what? That means, f is strictly decreasing in the interval minus infinity to 1 and 1 to infinity. And similarly, if you look at the second derivative of this then this comes out to be this quantity; you should check that second derivative comes out this. So, which is less than 0; for x less than 1 right; so implying it is strictly concave downward in the interval minus 1 to 1.

Because the nature of the second derivative gives you; concavity or convexity and for x bigger than 1, this will be positive, so it is concave upward in that inversion. So, this data can be all combined together to plot a graph of the function.

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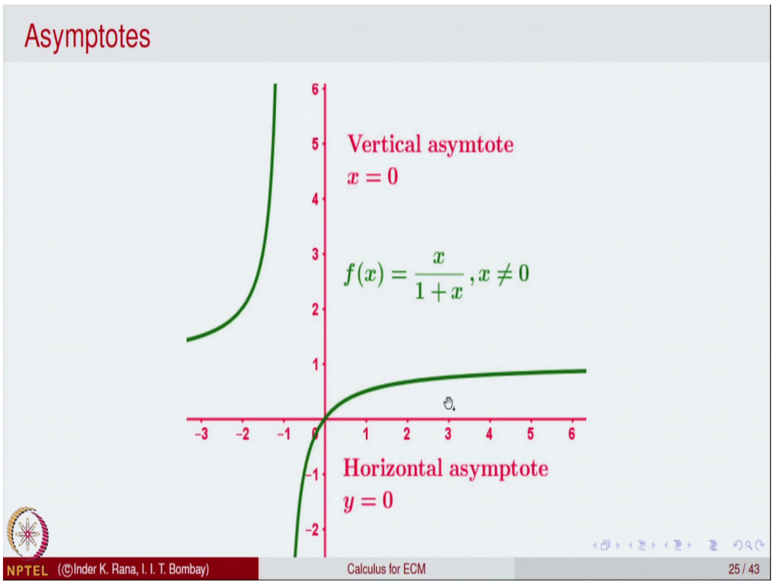

So, this is a graph of the function we saw right; that this line is not plotted here; it should have been a line here; that is x is equal to minus 1, is a line which is a vertical asymptote. And as you approach plus infinity or minus infinity; so that is looking at that, so for this is concave up; in the portion less than minus 1 and bigger than minus 1, it is concave down. So, this is the graph of the function.

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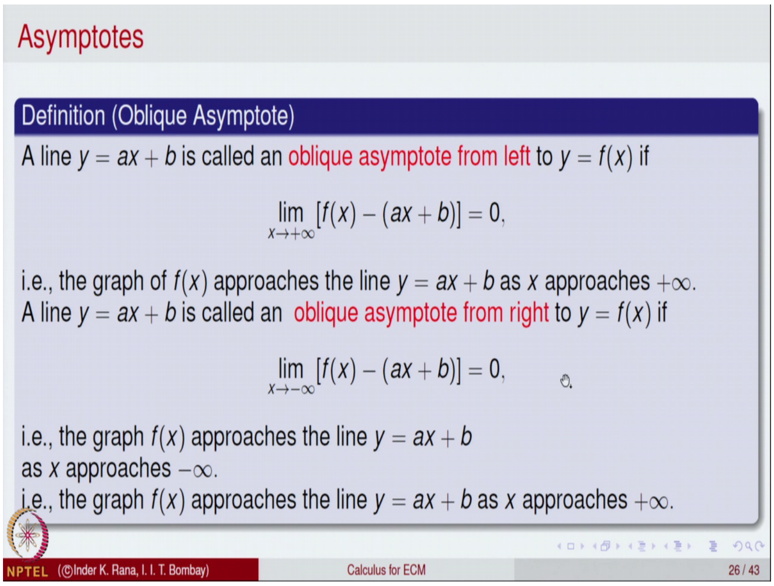

Let us look at; we define the horizontal asymptotes, we define the vertical asymptotes; there could be a line which is not either horizontal or vertical, but the function may be approaching that value as you approach x becomes plus infinity or minus infinity. So, we define what is called a oblique asymptotes, so a line y is equal to ax plus b; it is called a oblique asymptote to the graph of the function; that is y equal to f of x; if you subtract f x minus a x plus b and x to goes to infinity and that is equal to 0.

This is what we actually did in the previous example also; instead of this, we had the vertical line where it was equal to 1. So, let us look at what is called oblique asymptote; so y equal to ax plus b is called oblique asymptote. If the difference between f x and that line goes to 0; as x goes to plus infinity. So, that is same as saying that the graph f x approach is a line y equal to ax plus b, as x approaches plus infinity. So, this is from the left and similarly you can have from the right, if x going to minus infinity that goes to 0.

So, that is only way of saying which direction you are approaching; from the left or from the right, but the function is approaching the line ax plus b. So, that is what is called a oblique asymptote as x approaches minus infinity. So, basic idea is you are approaching some value or some line; as you approach plus infinity or minus infinity. So, the difference becomes smaller; distance between them. So, approaches plus infinity or minus infinity accordingly. So, these are called oblique asymptotes; I think best is to look at some examples to illustrate this.

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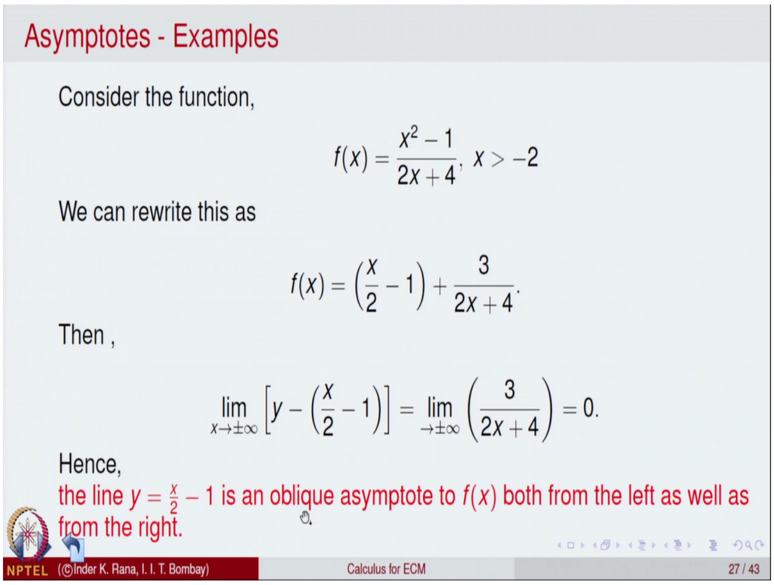

So, let us look at an example of a function f of x; this x square minus 1 divided by 2 x plus 4. Of course, 2 x plus 4 should not be equal to 0; so let us look at the domain when x is strictly bigger than minus 2. So, the no problem comes, so in this domain let us write this function as by using a bit of algebra; we can divide x square minus 1 by this actually and you can write it as x by 2 minus 1 plus 3; divided by 2x plus 4.

This is like something like a division of numbers, so this is the value; the quotient plus the remainder kind of thing. So, this is what you can write this expression as this, so now, if you subtract f x minus x by 2, minus 1; then the value is 3 divided by 2 x plus 4. So, as you approach plus infinity or minus infinity; this value will go to 0. So, that will mean that the line y equal to x by 2 minus 1 is a oblique asymptote or the graph of the function. So, that is what we write here; the limit of this or the limit of this minus right. So, this minus this; that is limit of this quantity is equal to 0, as x goes to plus infinity or minus infinity.

That means the line y equal to x by 2 is an oblique asymptote to the f x from; both from left and right because either way is ok; if you take go to infinity from the left or from the right.

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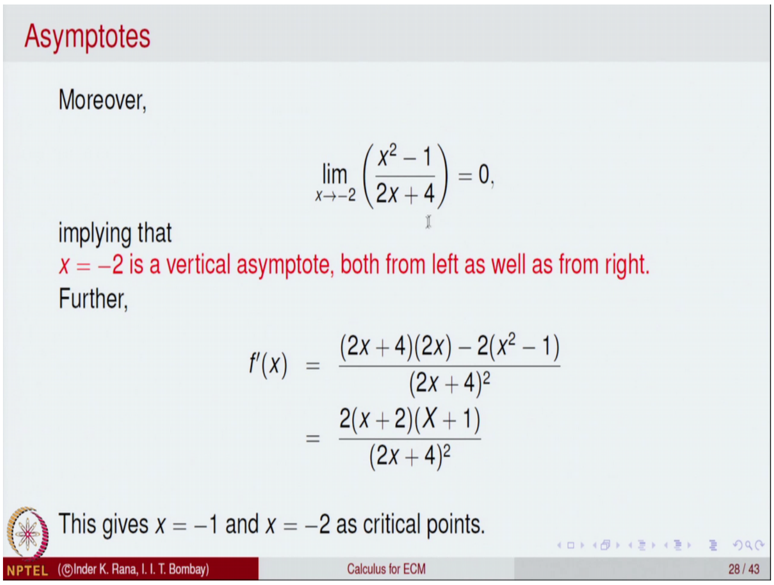

Moreover as x goes to minus 2; this becomes 0, because as you approach this becomes smaller and smaller; so this becomes 0, so the limit is 0. So, x is equal to minus 2 is a vertical asymptote; x equal to minus 2. So, as you approach the value x is equal to minus 2; your function becomes 0, so x is equal to minus 2 is a vertical asymptote both from left and from the right.

One can also look at the derivative of this function to analyze whether it is increasing or decreasing and what are the critical points. So, this is a function of the type p by q; numerator and denominator one function on the top, one function at the bottom. So, I can find out the first derivative, so first derivative f dash of x will be equal to by the quotient rule; 2x plus 4 to the power 2; 2x plus 4 into derivative of x square plus minus 1. So, that is 2x minus; x square minus 1 into derivative of the other function, so that is equal to 2; 2x plus 4 derivative is 2.

So, this one is simplify comes out to be 2 into x plus 2 into x plus 1. So, if you want to find the critical points; that gives you this is equal to 0; that means, x is equal to minus 2 and x is equal to minus 1; the critical points for this function. One has to now analyze whether this critical points are local maxima or local minima.

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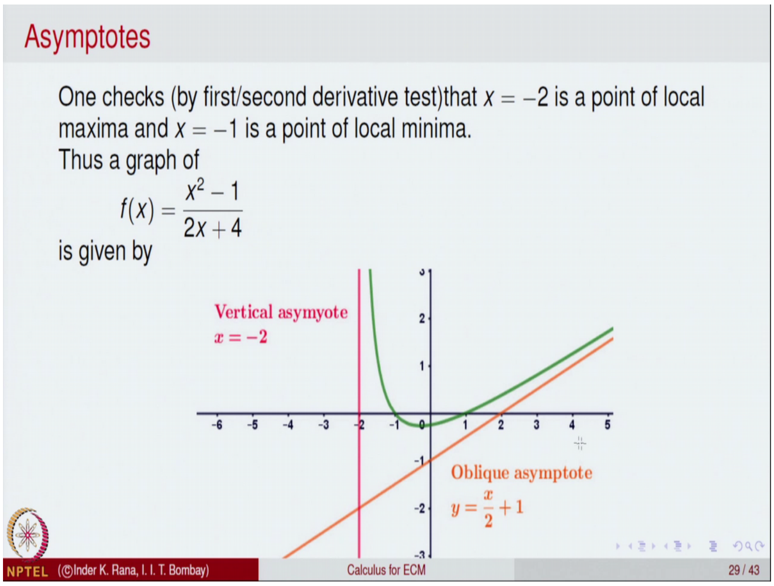

One can apply first derivative test, second derivative test at x is equal to minus 2 and minus 1 to check that the point x is equal to minus 2 is a point of local maxima and minus 1 is a pointer local minima.

So, in this picture; the bottom part of the graph is not visible, it is only the part in this. So, x is equal to 2; x is equal to minus 2 is a vertical asymptote, this is a oblique asymptote and this is one part of the graph; where x is bigger than minus 2; where x less than minus 2, there will be a bottom portion which will be something similar to this, but it will be coming on the other way round.

So, that is what the graph of the function and looks like; so, this is how asymptotes are used to find the graph. I would like to summarize this tools of calculus in sketching the graphs of a function. What are all the points one should keep in mind to sketch the graph of a function?

So, first of all given a function; one can look for some kind of symmetry is in the graph of the function.

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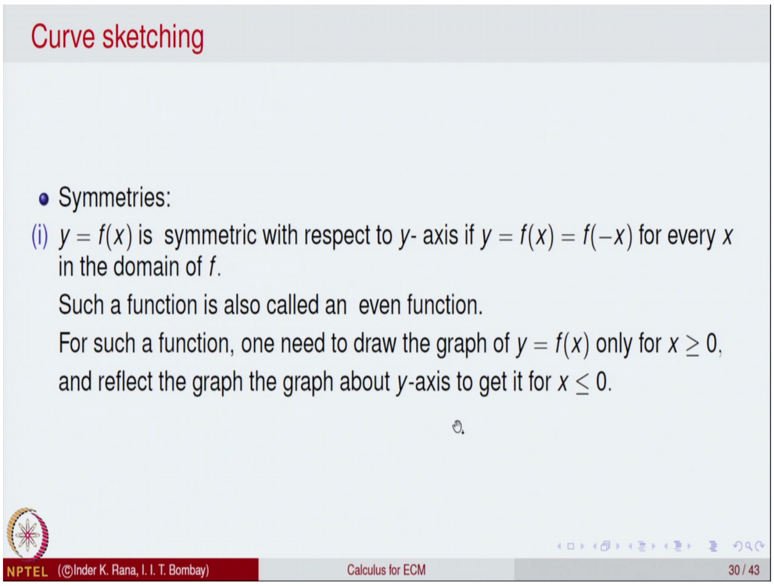

So, what does symmetry is mean? For example, function is symmetric with respect to y axis; that means, for a point on the left or right of the y axis; if you take x, I have take minus x; the values at both the points are same. So, this is symmetry with respect to y axis of the; so such a function is also given a name; such functions are normally called even functions. So, what is the advantage of this kind of knowing that the function is symmetric with respect to y axis?

The idea is that if you know on one side of y axis the graph; on the other side it is just the reflection of the graph along the y axis; as if y axis is the mirror and you can have the image of the graph from the left to the right or right to the left, whichever way you like. So, you have to draw the graph only on one side of the axis; positive axis or the negative axis, so that is what is called symmetry.

So, one needs to draw the graph only for x bigger than or equal to 0 say and reflect the graph about y axis to get the graph on the negative side, so that is what is called symmetry.

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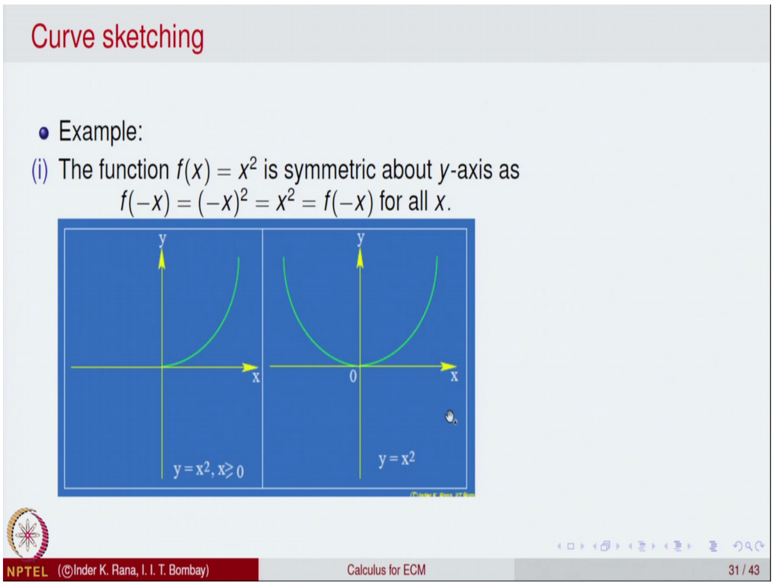

So, for example you look at the function f of x is equal to x square; it is symmetric around the y axis; f of x, if you change x to minus x; f of x remains the same equal to x square; the value x square remains the same. So, if you know the graph on the positive side; on the negative side, it is just reflection of this. So, that will give you the graph over the whole of the x axis. So, you need to draw only on one side; that is a positive side of the x axis or negative side. So, there is symmetry around y axis; which is important, it reduces the work in sketching the graph of the function.

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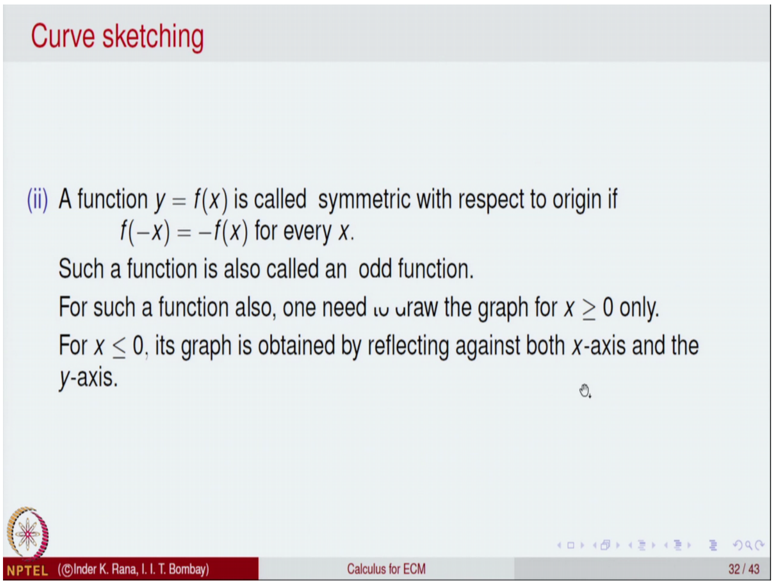

Another point in curve sketching is; knowing that f of x is called symmetric with respect to origin, if f of minus x is equal to minus of f of x. This is again a property of the function that at a point if you take the value and you change x to minus x and the value change is by minus of the value of f of x; this is called symmetric with respect to origin. Or one also called this as the odd function, so earlier we had a notion of even function and this is what is called the odd function.

So, for such a function again one need to draw the graph only on one side; on the other side, you first reflect along y axis and then reflect along x axis. So, you have to do it twice reflecting at both axis; will give you the graph on the other side.

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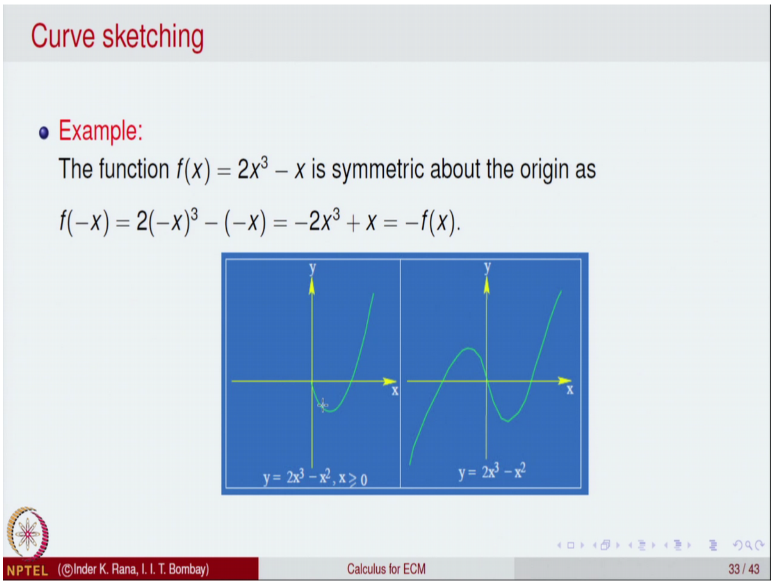

So, for example, if look at 2x cube minus x; this is symmetric around the origin because if you change x 2 minus x. So, 2x cube gives you minus of x cube if you change x 2 minus x and this gives you plus that is minus of minus; so, that is minus of f of x.

So, if you know on the graph on the positive side; to get the graph on the other side what you have to do is you first reflected. So, we will get a graph like this and then you reflect it down around x axis, you will get this point. So, this is a graph of the function; if you know on the positive side, you know it on the whole of x axis.

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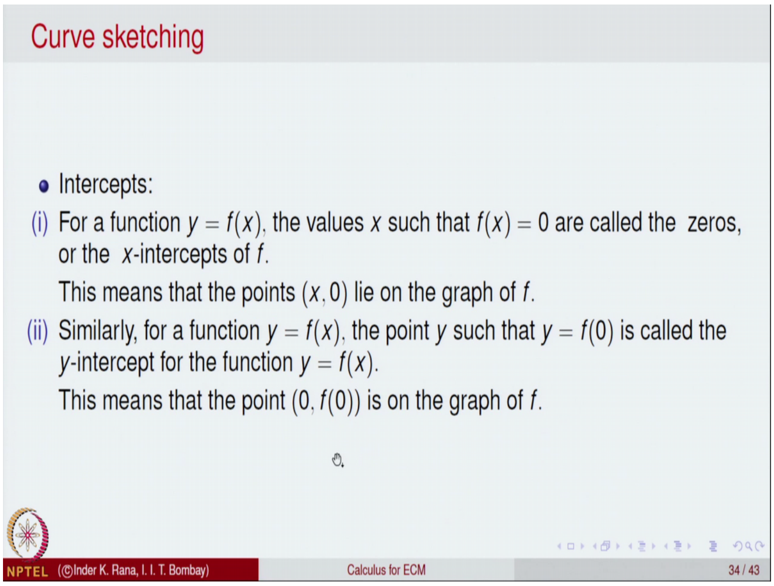

So, this is also a useful concept even and odd functions. Next is what is the all the intercepts; so, where does the graph cut y axis and x axis? So, if y equal to f axis is a function such that f of x is equal to 0; what is f of x? That is y, so y equal to 0; called the 0's of the function or called the x intercepts. So, those values which give you 0; right y becomes 0; that means, points on the x axis through which to the graph of the function has to cross, so that is what you called the x intercepts.

So, point x comma 0 will lie on the graph for all values of x for which this is true. And you can also have the intercept on the y axis; that means, on the y axis x is equal to 0. So, if you put x is equal to 0 and compute the value y; that gives you the y intercept. So, the graph will pass through the point 0 comma f of 0; that is a point on the graph of the function which cuts the y axis at the point f of 0. So, you get points through which the graph would pass.

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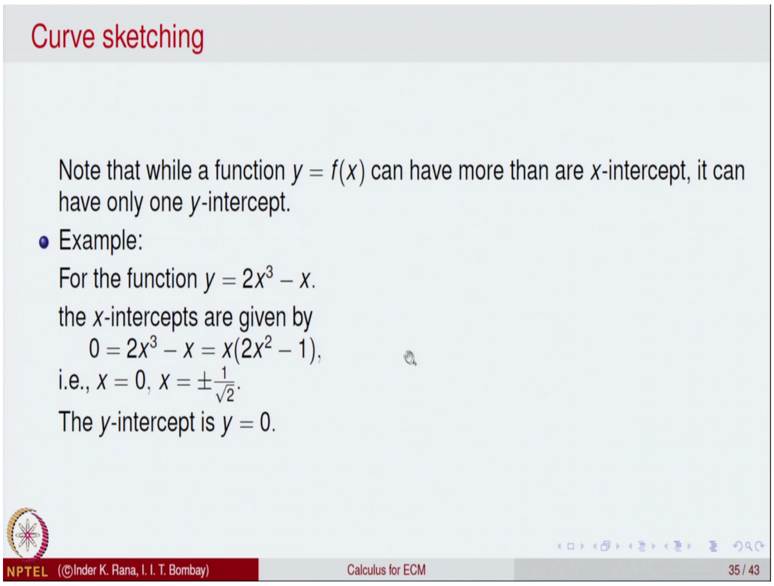

The function can have more than x intercept, but it will have only one y intercept because it is a function, so a function crosses y axis only at one point at the most, it cannot cross at more than 1 point because it is a function; but it can cross x axis at more than one point. So, for example, 2x cube minus x; you put it equal to 0, so you get x into; 2x square minus 1. So, x is equal to 0 and x is equal to plus minus 1 over square root 2 at the values. So, the graph passes through the points 0 comma 0 plus 1 over square root 2 comma 0 and minus 1 over 2 square root comma 0.

The y intercept is y equal to 0 because when you put x is equal to 0, you get y equal to 0. So, it passes through the point 0 0, so these are the points through which the function has to pass.

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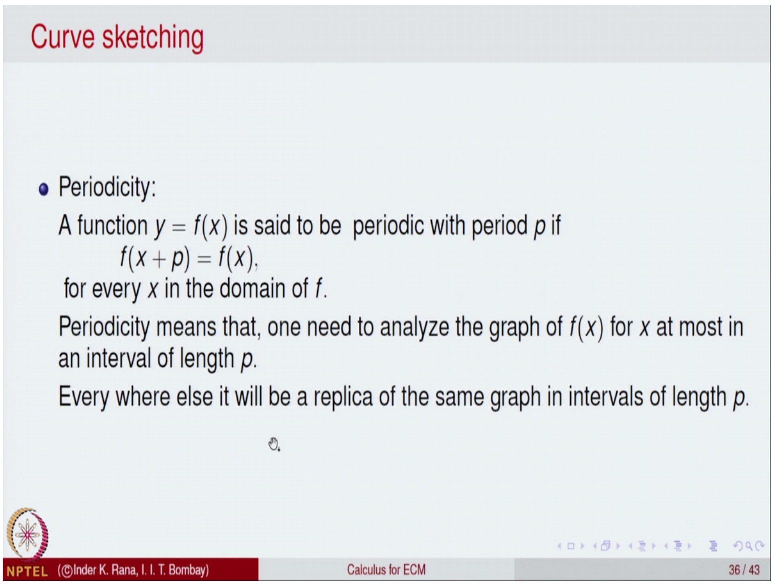

There is something called periodicity which is useful once you come across what are called trigonometric functions, but in any case let us define it. Periodicity means that there is a certain number p; say the value of x plus p is equal to x for all points x in the domain of the function, p is fixed.

So, in a sense that if you take the graph of the function; in some interval say 0 to p then you can go on repeating that interval graph again and again to get the graph everywhere. So this what is called the periodic function and then p is called the periodicity of the function. So, one has to know all the graph of the function in any interval of length p and everywhere else it will be just reproduction of that function again and translated next.

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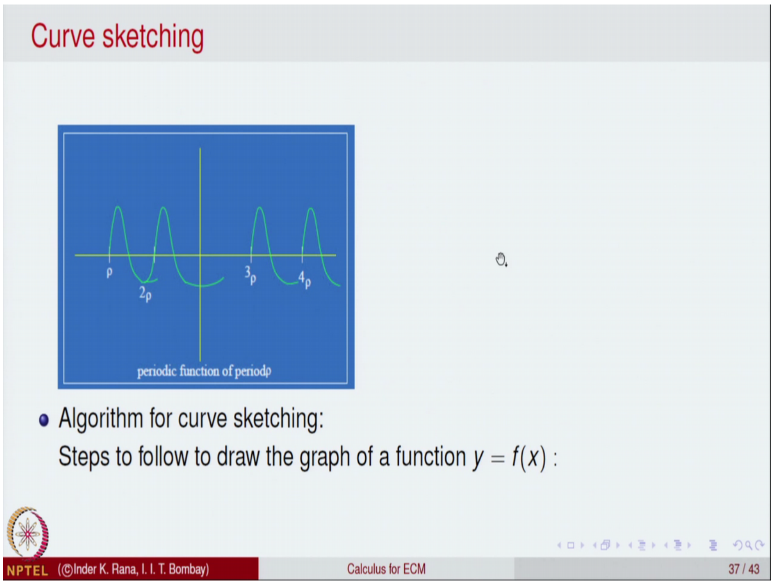

So, for example; if this is a graph then in the next portion; you use that portion is put here, that portion is put here, that portion is put here and so on.

So, graph is repeated in every interval of length p; that is periodicity. So, these are the important things which one should keep in mind along with other calculus things that we have done. So, let us summarize all these things to draw the graph of a function.

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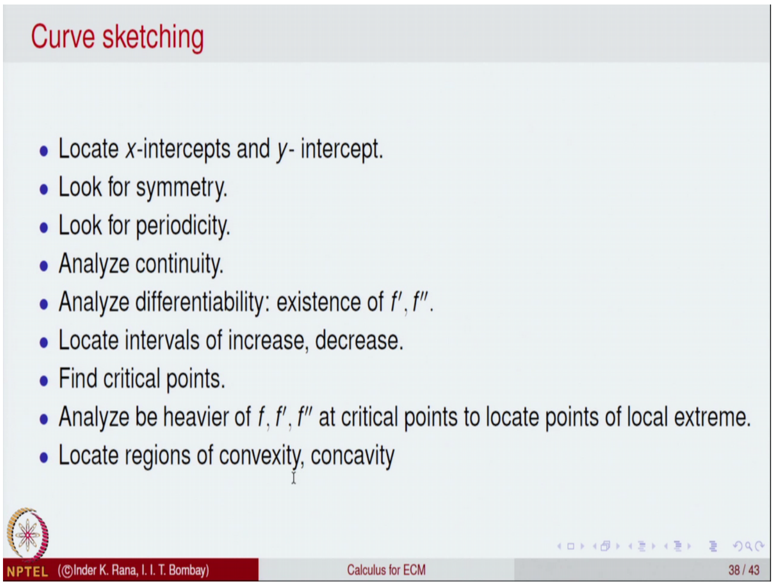

So, once you locate what is x intercept? What is y intercept? Look for symmetry; is there any symmetry; is odd function or a even function. Look for periodicity; is there any periodicity in the function, look for continuity whether the graph is continuous everywhere, are there any points where the function is discontinuous? What are discontinuities? What type of discontinuities are there and so on.

Then look at differentiability whether the function has derivative everywhere; what is the nature of the first derivative? What are the points where the first derivative is equal to 0; critical points, first derivative positive or negative gives you increasing, decreasing. Nature of the second derivative will give you concave up, concave down and points of inflection.

So, locate into those intervals where the first derivative is positive or negative increase, decrease; critical points where derivative is equal to 0, second derivative will give you locate; first derivative can help you to locate whether the points of extrema are local maxima or local minima and so on. And nature of the second derivative will give you whether the function is convex or concave, what are the intervals of convexity and concavity? And what are points of inflection?

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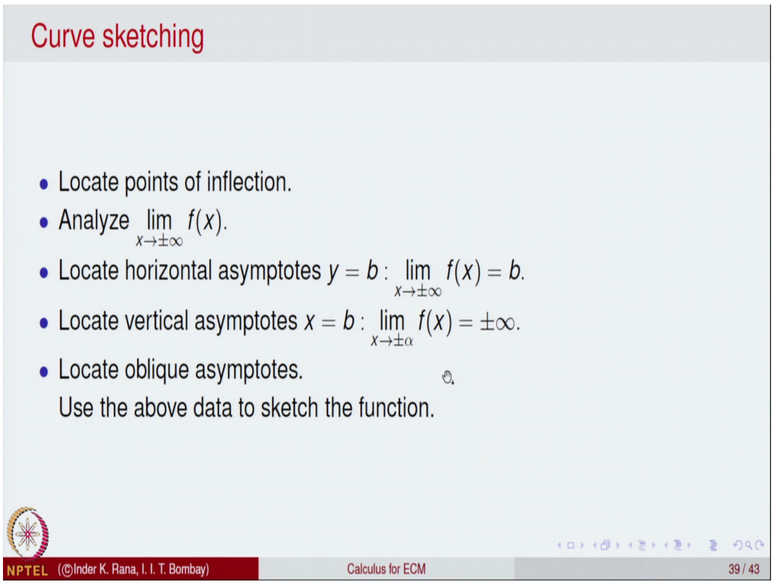

So, all these data and along with you can look at what happens to the value of a function as x goes to plus infinity or minus infinity? You can also try to analyze horizontal asymptotes and this will give you a vertical asymptote, this will give you equal to plus infinity minus infinity. This is the nature of the function as x goes to plus infinity or minus infinity; if it happens to be going to plus infinity minus infinity is x goes to plus; this goes to 0.

So, the vertical asymptotes will be there and oblique asymptotes; you should find and give all the above data to sketch the graph of a function.

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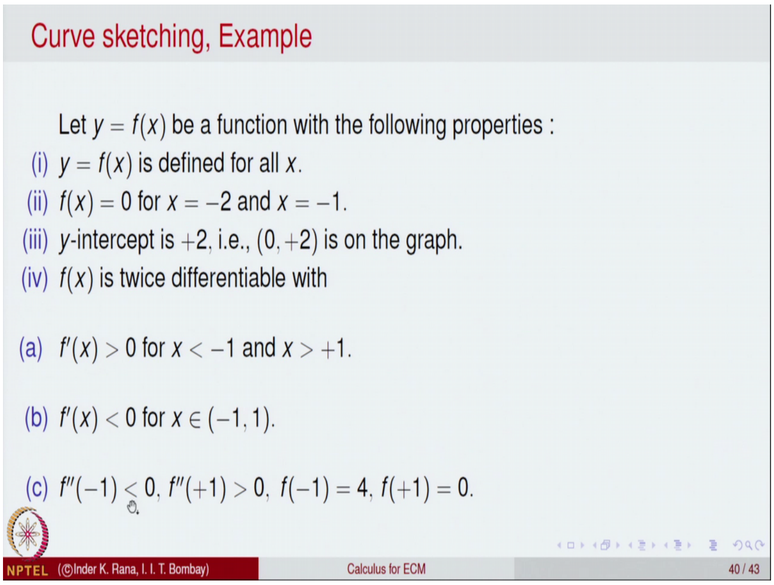

So, let us do one exercise; let us look at a function y equal to f of x, which has the following properties that f of x is equal to 0 for x is equal to minus 2 and x equal to minus 1.

That means, minus 2 comma 0 and minus 1 comma 0 are the points where which the graph will pass; y intercept is plus 2; that means, the graph will pass through the point 0 comma plus 2; f is twice differentiable with first derivative; with this property f dash of x is bigger than 0, for x less than 1 and for x bigger than 1; x less than minus 1 and f a bigger than plus 1, it is positive; that means, in this portion the function is going to be increasing and in minus 1 to 1; the function derivative is less than 0, so, it will be decreasing.

The second derivative is given as less than 0; at minus 1 is less than 0, bigger than 0 and f of 1 and so on; so this data is all given to us. So, from this data what do you get? We also are given the fact that second derivative is less than 0; for x less than 0 and second derivative is bigger than; the previous this data.

So, f double dash at minus 1 is less than 0 and f double dash 1 was not given to be equal to 0, but that indicates that at the point minus 1; say it is decreasing, it is increasing, it is differentiable so; that means, derivative at this point must be equal to 0 at minus 1 must be 0. So, that will tell you that minus 1 is a point of local maximum; plus 1 is a point of local minimum and these are the values at those points.

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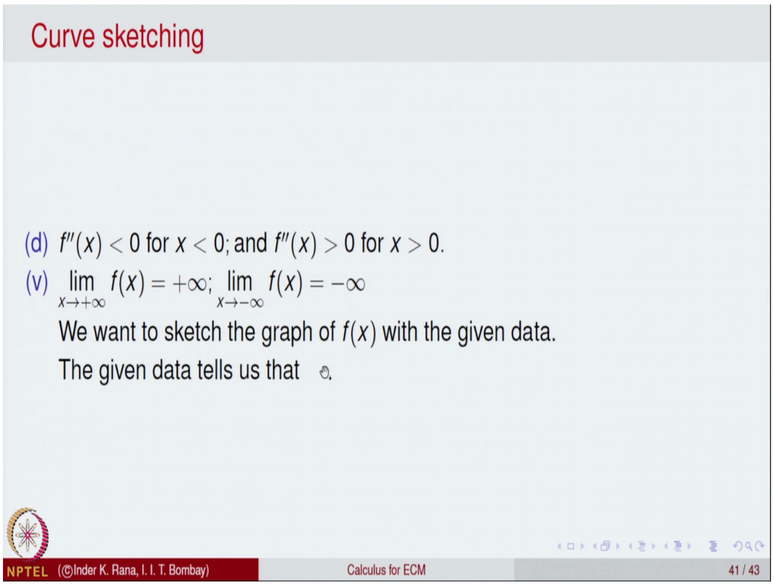

And the nature of the second derivative gives us whether the function is concave up or concave down. So, it says for x less than 0; it is given that the function is second derivative is less than 0; that means, the function is concave down and concave up for x bigger than 0.

And it says the x goes to plus infinity, the function goes to plus infinity and x goes to minus infinity; the function goes to minus infinity. So all these data going to sketch the graph of the function; so let us sketch the graph of the function from this data.

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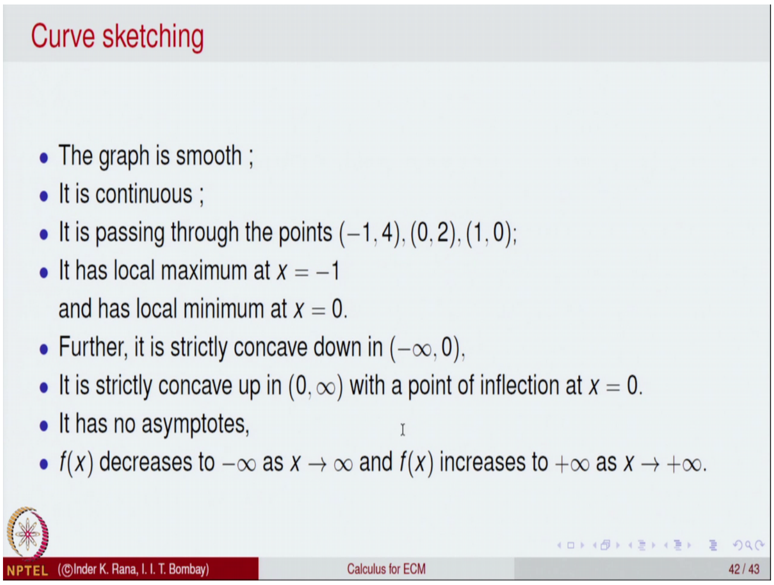

So, graph is smooth because twice continuously differentiable; it is continuous, it passes through these points; that is what we observed. And then it has a local maximum at the point minus 1; it has a local minimum at the point x is equal to 0.

Further, it is strictly concave down in minus infinity to 0 and it is strictly concave up in 0 to infinity; with a point of inflection at the point x is equal to 0. It has no asymptotes; it keeps on increasing or keeps on decreasing. So, f is decreases to minus infinity as x goes to plus infinity and it increases to plus infinity as x goes to plus infinity; this should be minus infinity here. So, all this data we can now plot a picture of the function; so the picture of the function is given as follows. So, this should be the graph of the function.

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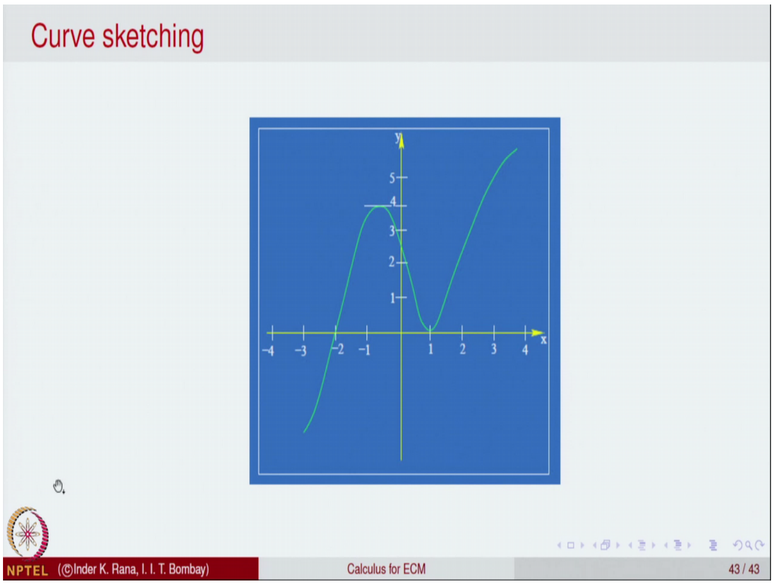

So, here is the point 1; which is a local minimum, here is a point which is a local maximum for the function; it passes through the point minus to 0, it passes through the point 1, 0; keeps on increasing as you go to the right side plus infinity, keeps on decreasing as you go to minus infinity. And in the portion minus 2 to 0; it is concave down, in this portion it is concave up. So, this is how you sketch the graph of a function.

So, till now what we have been doing is; we have been looking at function of one variable. Looking at the various tools of calculus that help us to analyze the function and apply to scenario of our economics commerce and management. We will start looking at in the coming few lectures, we will look at functions of several variables, functions of 2 variables and see similar analysis for functions of 2 variables.

Because in many problems in economics; the outcome depends on, not only on one variable; more than one variable. So, we will look at these functions of several variables and its applications in the next lectures.

Thank you.