Calculus for Economics, Commerce & Management
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Lecture - 34
Convex and concave functions, asymptotes

We have been looking at various examples of analyzing the notion of convexity concavity of functions of one variable and its applications to economics and commerce. We will start today with an example, we will discuss it completely and try to see what we can deduce about the function. So, the function is given as follows.

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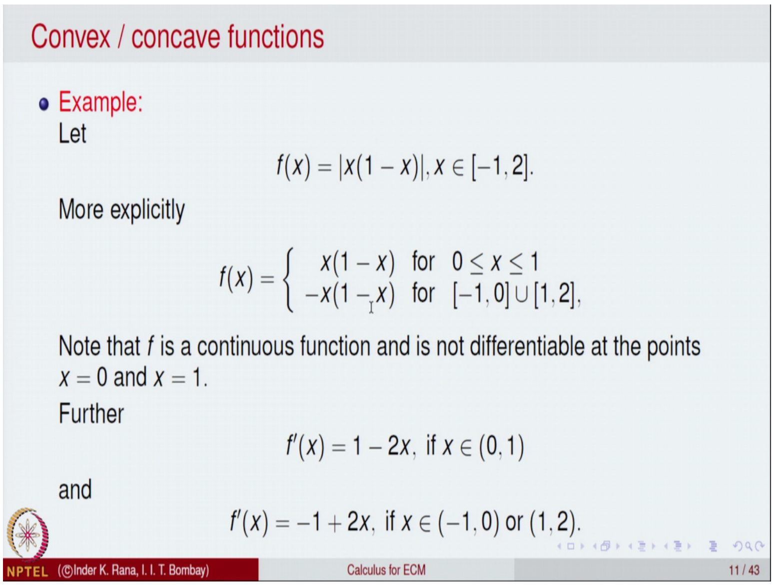

So, f of x is given as absolute value of x multiplied by 1 minus x for x belonging to the interval minus 1 to 2. So, the domain of the function is the closed bounded interval minus 1 to 2 and the function is defined as the absolute value of x multiplied by 1 minus x.

So, for such a function you should observe that if x belongs to minus 1 to 2, then x could be negative or x could be positive. And here is the formula for the function which involves absolute value and we know that the absolute value of a real number is the number itself, if the number is positive and it is minus of the number if it is number is negative. So, this value will change according as the; varies x, so let us assume x is between 0 and 1 because this 1 minus x that is going to cause a problem. So, if x is between 0 and 1; then x is positive and 1 minus x also is positive.

So, that will mean that the product x into 1 minus x is positive. So, the function will be defined as x into 1 minus x for x between 0 and 1. If x is outside, say x is between minus 1 and 0 then x will be negative and if x is bigger than 1 and less than 2, then again 1 minus x will be negative. So, both the terms x and 1 minus x are going to be negative when it belongs to between 1 and 2. So, the value of the function would be minus x into 1 minus x.

So, this is the function is defined differently between 0 and 1 and differently between minus 1 to 0 and 1 to 2. So, depending on whether the product is positive or negative it is differently defined. So, this is the composite; this is the formula given we expand it as for the interval 0 to 1; we write it as x into 1 minus x because both x and 1 minus x will be positive. And when x is between this or this; if x is between this interval then the product will be negative and this is minus of that thing right.

So, accordingly we will analyze the function; so first of all the function is continuous because in the interval 0 to 1, it is a continuous function; in the interval this portion also separately it is continuous function. Only we have to bother about the point at 0 and at the point 1. At the point 0, if you look at the values coming from left of 0 from here; then the value will be 0 times; 1 minus x, that is 0.

And similarly, if you coming on the right side then also the value is 0. So, it is continuous at the point 0 and similarly one can verify that it is continuous at the point 1 also. So, this is a function which is continuous everywhere between minus 1 and 2; claim is at this function is not differentiable at the points 0 and 1.

So, to analyze that one has to compute the left derivative at 0 and the right derivative at 0 and similarly left derivative at 1 and the right derivative at 1. So, let us compute this and check that the claim is true; this is a function is not differentiable at the points. So, let us look at the function.

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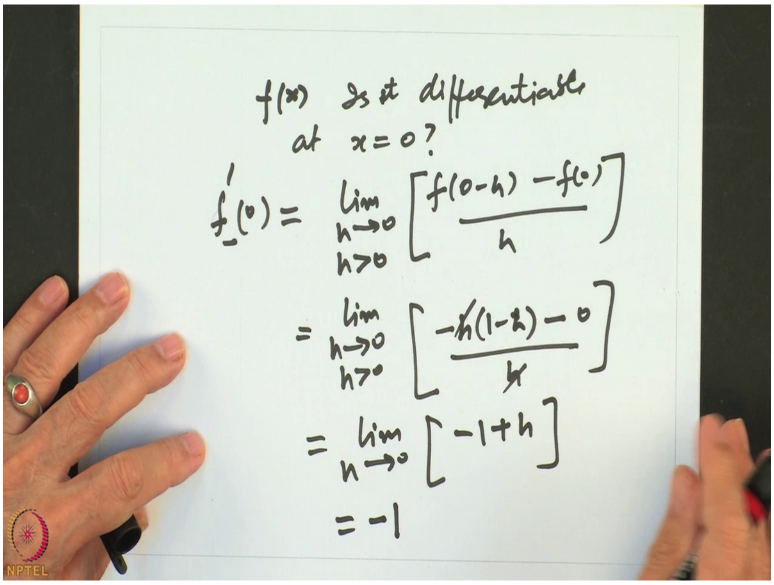

So, f of x we want to know is it is it differentiable at x is equal to 0. So, to find that let us try to compute what is going to be the left hand derivative at the point 0.

So, by definition this is limit h going to 0, so f of 0 minus h, let us take h positive; minus f at 0 divided by h. So, when you take 0 minus h even on the left side of 0, so you are computing this limit on the left side of 0. So, let us compute this; so it is limit, h going to 0; h bigger than 0. Now what is f of 0 minus h? 0 minus h will be a point in minus 1 to 0. So, it will be given by minus h; 1 minus h, that is the value of the function; minus f at 0 is 0, divided by h.

So, this comes out to be h cancels; so, this is equal to limit h going to 0 of minus 1 plus h. So, that limit we know is equal to minus 1, so the left hand derivative at 0 is equal to minus 1; let us compute the right hand derivative at the same point.

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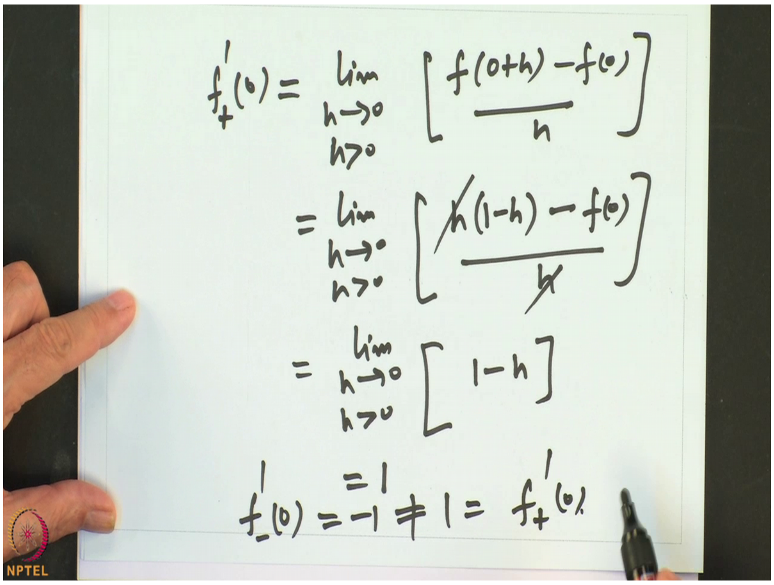

So, we want to compute f dash plus at 0. So, that will be equal to limit h going to 0; we want to be on the right side of 0, so f of 0 plus h; h positive minus f at 0 divided by h.

So, for that let us compute what are the values. So, h going to 0 h bigger than 0; at 0 plus h; that means, the function is on the right side of 0; so it will be in the interval 0 to 1, so the value is h; 1 minus h, minus f at 0 is 0 divided by h. So, that is equal to limit; h going to 0 h bigger than 0 of this h cancels with this, so this is 1 minus h; so, that is equal to 1.

So, we saw that the left hand derivative was equal to minus 1; the right hand derivative is 1. So, the left hand derivative f at minus 0 that is equal to minus 1 is not equal to 1, which is equal to f dash plus at 0. So; that means, the function this proves that the function is not differentiable at the point 0. A similar calculation you should try and prove that it is not differentiable at x equal to 1 also. So, this function which is given by mod x into 1 minus x; mod of x into 1 minus x; in the interval minus 1 to 2 is continues everywhere, it is not differentiable at the points 0 and 1.

Let us compute the derivative of this function wherever it is differentiable, it is differentiable in the open interval 0, 1 and the derivative will be x minus x square. So, derivative will be 1 minus 2; so, that is the derivative in the interval 0 to 1. So, once we have the value of the derivative in this interval, we can also compute the derivative in the other part of the domain. So, in this the formula is minus x into 1 minus x; so, if x lies between the open interval minus 1 to 0 or the open interval 1 to 2, then the derivative will be equal to minus 1 plus 2x.

So, from this it gives minus 1 plus 2x in these two parts of the interval; in these two intervals. So, now to find out what are the points of local maxima minima, one can put derivative equal to 0. So, if you put this derivative equal to 0; in the interval 0 to 1, we get x is equal to 1 by 2. And in this if you put; this again gives you x equal to 1 by 2, but that it is not part of this domain anyway.

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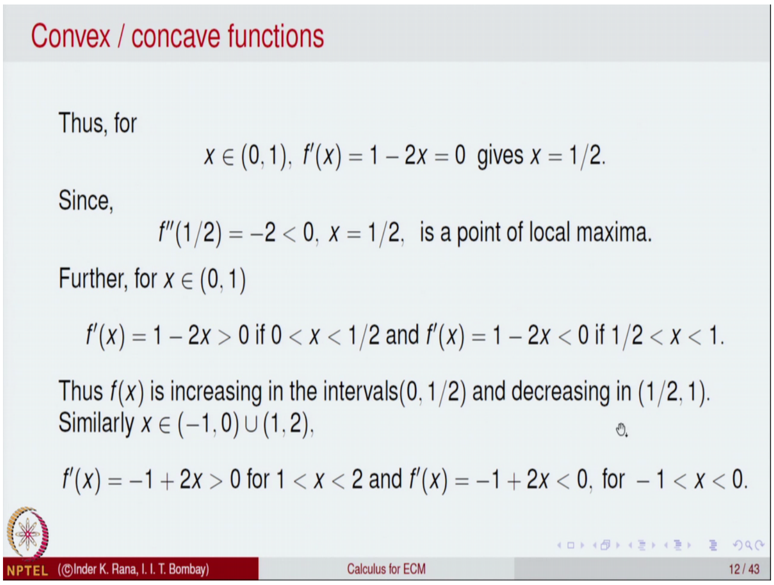

So, the only critical point is x is equal to 1 by 2 and one can easily check that at the point 1 by 2; the second derivative at the point 1 by 2; look at the second derivative. So, at the point 1 by 2; the function in this interval derivative is 1 minus 2X; so, second derivative will be equal to minus 2.

So, the point x is equal to 1 by 2 is a critical point and the second derivative at that point is minus which less than 0. So, that gives you that x is equal to 1 by 2 is a point of local maximum. We can also analyze the nature of the derivative on the left and on the right of this point that we know already, but let us analyze the nature of the derivative in all the parts of the domain. So, derivative if is between 0 and 1; the derivative is equal to 1 minus 2x. So, that will be positive; that means, 1 is bigger than 2x; so, this will be bigger than 0, if x is between 0 and half and it will be negative if it is between 1 by 2 and 1.

So; that means, the function will be increasing on the left side of half from 0 to 1 by 2 and decreasing on the right side of 1 by 2 to 1. We can do a similar analysis, so f is increasing in the interval 0 to half and decreasing in the interval half to 1. We can do a similar analysis for the other parts of the domain. So, namely in minus 1 to 0 and 1 to 2; we know the derivative is minus 1 plus 2x.

So, that will be bigger than 0; that means what? That means, 2x is bigger than 1. So, in this portion this is going to be positive and it is going to be less than 0 in minus 1 to 0. So, from the nature of the derivative; we deduce that the function should be increasing in 1 to 2 and should be decreasing in minus 1 to 0.

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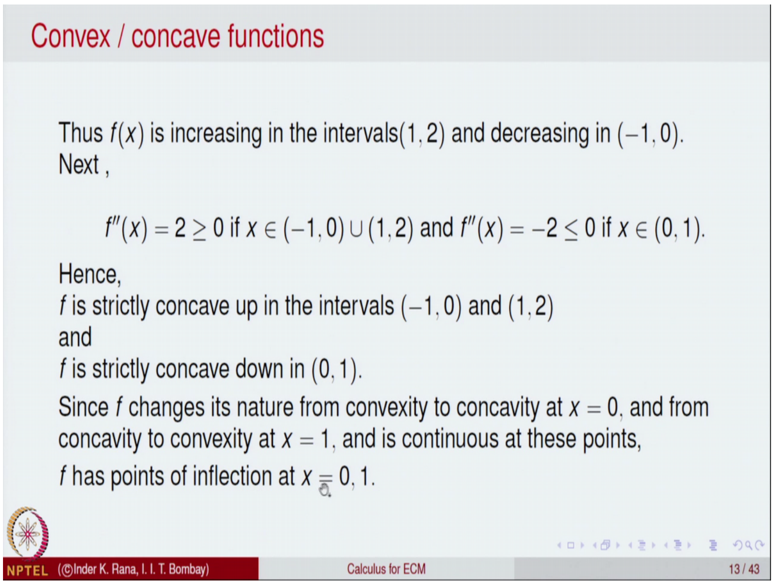

So, these are the conclusions withdraw from the looking at the derivative, the sign of the derivative. So, once that is done we know increasing, decreasing; we can look at the second derivative of the function. So, the second derivative of the function is equal to 2, if x is between minus 1 to 0 or the portion 1 to 2 and it is negative, if x is between 0 and 1.

So, it is strictly concave up in the interval minus 1 to 0 and 1 to 2 because of the derivative is strictly bigger than 0 and strictly concave down in the portion 0 to 1. So, this is an properties that we deduce for the derivative function by looking at; about the concavity and convexity of the function by looking at the second derivative of the function.

So, one side is done; we can also look at, it strictly concave up in this portion and on the left of 0 and on the right of 1. So, it is changing its nature from strict convexity to concavity at 0 and from concavity to convexity at x is equal to 1 and it is also continuous at these points.

So, these are the points of inflection; so f has points of inflection at 0 and 1. So, all these data can be combined together to plot a graph of the function.

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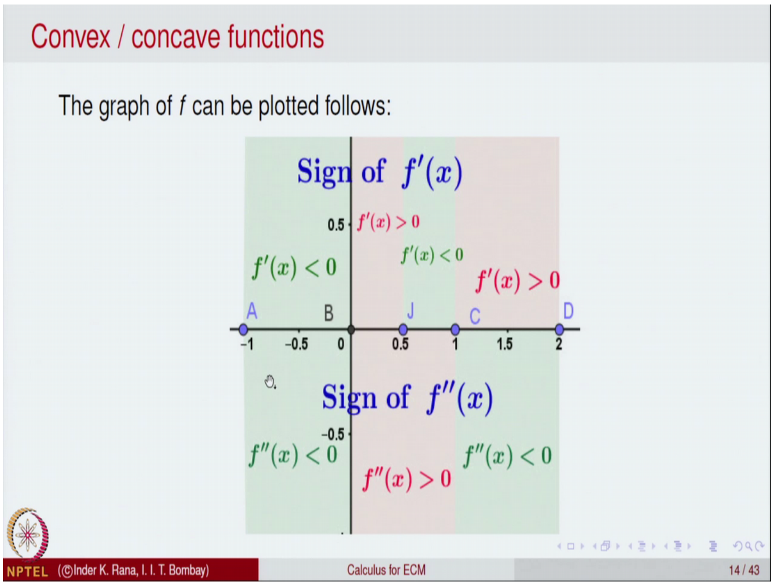

So, let us see what the graph looks like. So, this is first of all to draw the graph; it is advisable to look at the interval domain that is the interval minus 1 to 2. In that portion, look at the intervals where the derivative; if it exists is bigger than 0 and less than 0. So, we are looking at the sign of the derivative function. So, we get minus 1 to 0; the derivative is less than 0 and in the portion 0 to half, the derivative is strictly bigger than 0.

In half to 1; the derivative is strictly less than 0 and in the portion 1 to 2, the derivative is strictly bigger than 0. So, once this data is available in this form we can see that in this green portion the function should be decreasing. In this red portion, the function should be in increasing. Again the derivative is less than 0; so in this green portion the function should be decreasing.

And in the red portion derivative is bigger than 0, so the function should be increasing here. So, it is decreasing; it is increasing, it is decreasing it is increasing. So, that indicated that the 0.5; that is half, at this point there should be a local maximum for the function. For the second derivative, when you plot the signs; what is the sign of the second derivative? It is less than or equal to; the second derivative is strictly less than 0 in the interval 0 to minus 1 to 0, so; that means, the function is concave up right and concave down and so on.

So, this data can be plotted for the function and you can plot a function this way.

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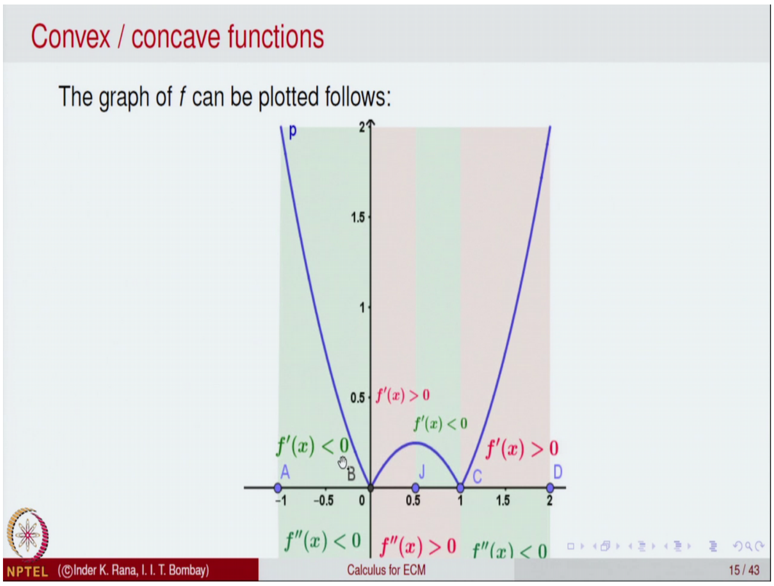

So, in this portion derivative is less; so this concave down; it is concave up and so on. So, this is the graph of the function by looking at the various properties of the functions, so that is how the graph is drawn. So, it should be clear; what is the process for sketching a curve or sketching a function?

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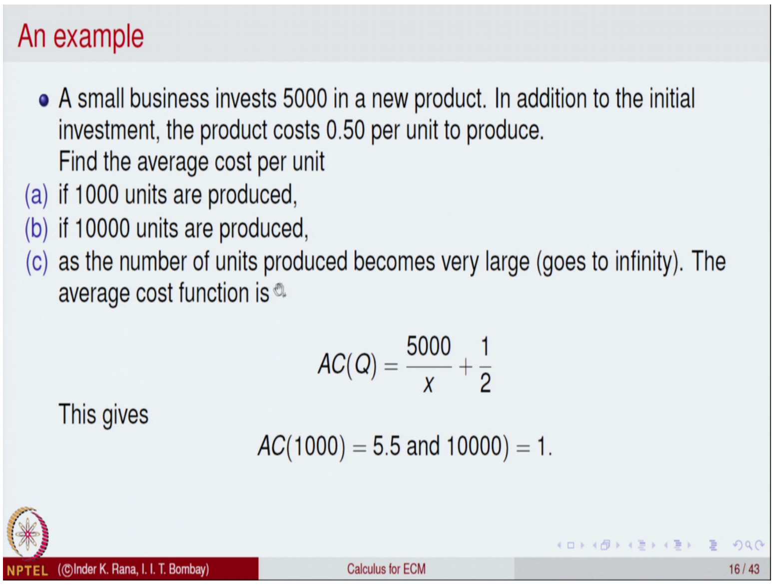

Let us look at slightly more ideas about calculus; how they are applied to plotting the function or analyzing the functions?

Let us analyze a small business which invests rupees 5000 in a new product. In addition to the initial investment, the product costs is 0.50 per unit to produce; it costs. So, that is the cost per unit that is being; so, to produce each unit one has to invest 0.50. So, let us see what we are trying to do? In this if you want to find out what is the average cost function, so we want to find out what is the average cost per unit? Question is find the average cost per unit, when 1000 units are produced or when 10000 units are produced; what is the average cost? And what is the average cost when the number of units produced becomes very very large?

So, we keep on increasing the number of units produced; what is the behavior of the average cost of the production? So, to find that let us first write down what is the average cost function? So, the average cost function we know is defined by invested money is 500, so number of units being produced very very large, so we want to find out.

The average cost of producing, it should be x actually here or one should write Q here because a variable is denoted by Q. So, this 5000 invested and x is the number of units or Q or X, where Q is equal to x is a number of unit being produced and 0.5 per unit to produce. So, 0.5 per unit that gives you additional average cost of half per unit.

So, this is a average cost function for the production of that whatever product is there. So, for this you know that to for this product function, if x want to calculate what is the average cost for 1000? You put the value x is equal to 1000 or Q equal to 1000; so, average cost at 1000 will be, so this is 1000. So, that is 5000 plus 0.5 and at 1000; it will be equal to 1.

So, that is a average cost; so that is only just plugging in the values Q equal to 1000 or x is equal to 1000 and 10000. Now we want to compute what happens? See if you look at the average cost function, then as the number of units being produced increases. So, if this x is increasing; then it is 5000 divided by X, so this will be a decreasing. So, average cost, you can already see that for 1000; it was 5.5 and for 10000; it will came down to 1.

So, if you keep on producing more number of units; the average cost keeps on coming down. So, that is the conclusion that we get from here; so average cost is becoming smaller and smaller and this quantity will become smaller and smaller. A if x is very very large, this quality becomes very very small; so, the average cost comes closer to the value 2. It will never become equal to 2 because that is not possible mathematically.

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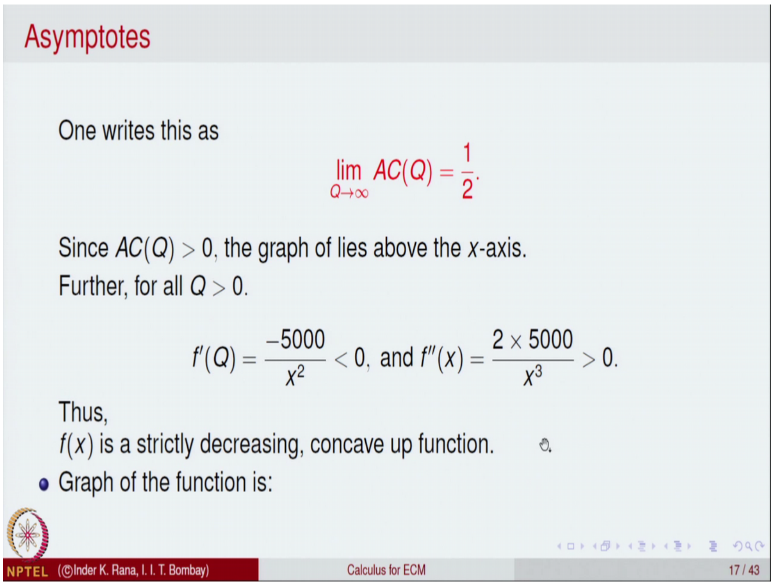

So, we can bring it as close to half as you want it, so this is written mathematically as the average cost of producing Q units as Q becomes infinity is equal to 2. This is only a symbolic way of writing that as Q becomes very very large, this value of the function average cost comes closer and closer to the value 2. So, there is only Q is not equal to infinity; do not make that mistake of saying that Q is equal to infinity and something divided by infinity should be 0; that is wrong, we have saying things you cannot divided by infinity.

So because infinity is not a number; anyway so, a mathematical way of saying that as you increase Q; AC the average cost is decreasing and is approaching the value 1 by 2. So, this is what this symbolic representation of that is and let us observe something more that; when you look at the first derivative of the average cost function that was average cost was 5000 divided by x plus half. So, the derivative is minus 5000 by x square, which is less than 0; whatever be x, it is less than 0. Here keep in mind x is same as Q, this is the variable which is not typed properly on the right hand side; it should be Q or it should be written as x here.

And the second derivative of this will be equal to this minus 2 goes up. So, that is plus 2 into 2 into 5000 divided by x cube and for all x bigger than 0 or Q bigger than 0, this is also bigger than 0. So, what does that indicate? That means, that the average cost function that is f. So, here this AC is denoted by f, so the average cost function is decreasing and it is always positive; so, that is concave up function. So, one can look at it strictly decreasing and concave up function. So, let us look at the graph of this function looks like this.

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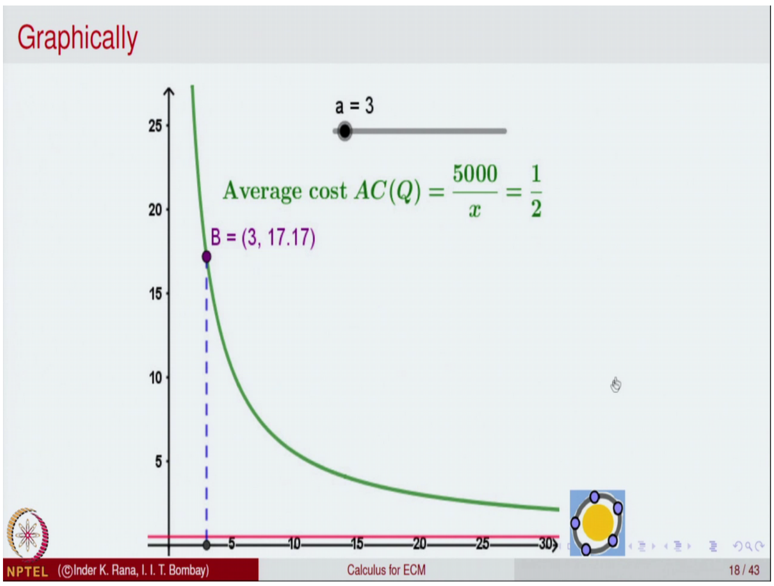

So, this is the graph of the function; the average cost function and this line; red line is a line which is y equal to 1 by 2, so this is a y line y equal to 1 by 2. So, this value of the function as u become smaller and smaller; this will become, so average cost function; there it is a minus 1 by 2. So, as x becomes large and large; it will become 1 by 2, so we can show it dynamic geometry software; the value if you like how this is happening.

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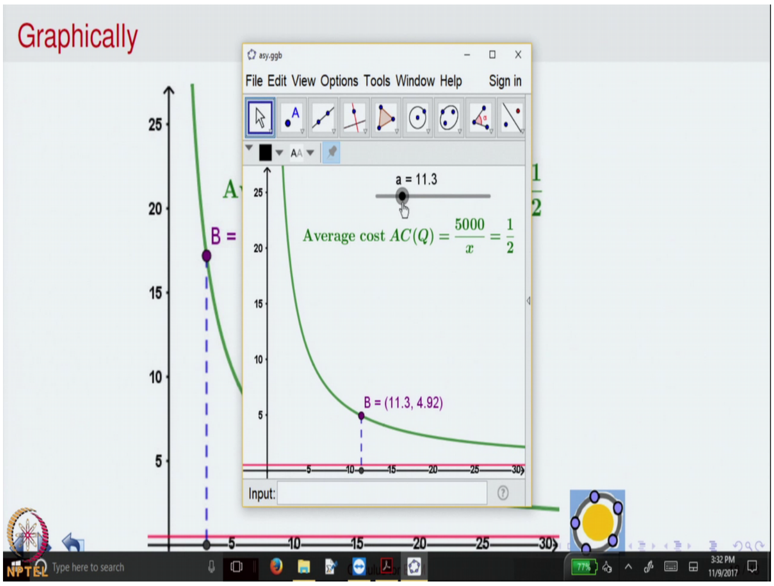

Basically the idea is that this function will become, the value of the function becomes smaller and smaller. So, this is a graph and at the value B is equal to 3; at the point B when the value x is equal to 3, here is a minus sign here, so which can be corrected. So, if I move the point; if Q becomes larger and larger, the value is becoming smaller and smaller. So, this is indicating that the value is becoming smaller and smaller; so, that is the value becoming. So, you can see the value is changing here, so if you have closer to 0 it is increasing; and if you are going this side then it is decreasing.

So, this is a dynamic way of looking at it, but it should be obvious from the graph that as you go on the right side, this value or the function will decrease and it will approach the value 1 by 2.

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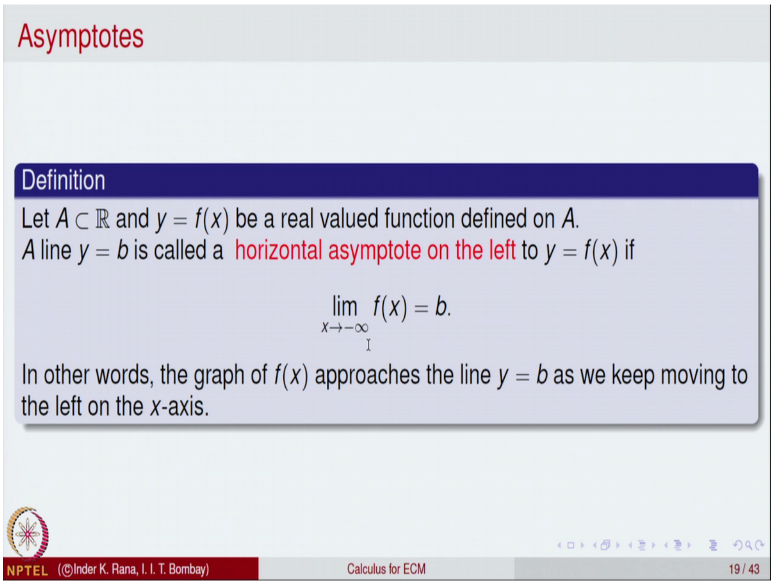

So, we want to make this notion precise; so, let us define what is called a asymptote for a function of one variable. So, let f be a function defined in a domain A and given by y equal to f of x. So, another way of writing f colon A to R be f be a function, one just write sometimes let y equal to f x be a real valued function defined on a.

That means A is the domain of the function and x belongs to A and the values are y. So, we say that a line y equal to b is called a horizontal asymptote. So, it is called a horizontal asymptote on the left of the function, if as you approach minus infinity x goes to minus infinity; f of x approach is b. So, f of x is the value that is y; so y is approaching b. So, the line y equal to b, so as you approach minus infinity, as x becomes minus infinity; f of x is approaching b.

So, y is becoming b; so you say y equal to B is a horizontal line. So, we say y equal to B is a horizontal asymptote; on the left side because x going to minus infinity. So, horizontal asymptote on the left side of the graph; if the limit f x; so, as x becomes infinity, x becomes closer and closer to infinity; that means, x becomes larger and larger; f x approaches the value b.

So, this is what we are saying; this symbolically says the graph approaches to the line y equal to B, as we move; keep closing to the, as we keep moving on the left side of x axis. So, this is called horizontal asymptote on the left.

Similarly, you can have a horizontal asymptote on the right; the only difference will come is you are approaching x is equal to plus infinity.

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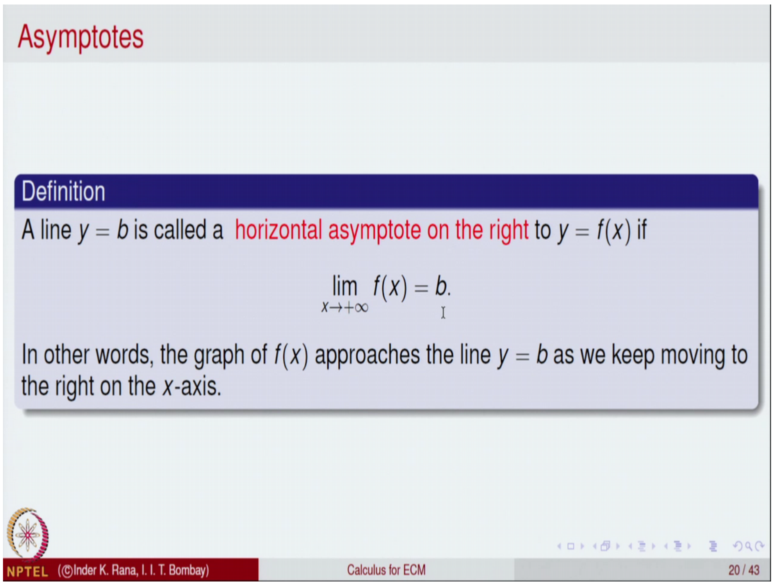

So, horizontal asymptote on the right side of f of x; if x going to plus infinity f of x is equal to b. So, as you approach plus infinity; as you go away and away on x axis, your value f of x becomes b. That is what happened in the previous example, as the average cost came closer and closer to the line y equal to half. So, in the previous example you can say y equal to 2 is a horizontal asymptote on the right side. We can also have what are called the vertical asymptotes.

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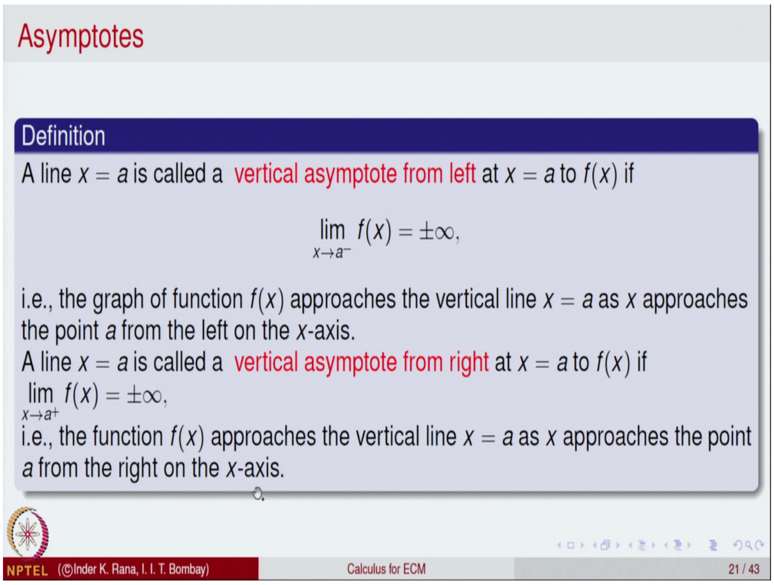

So vertical meaning x; x is approaching some value. So, x is equal to a is called a vertical asymptote from the left; at the point x is equal to a, to the function if x goes to a from the left side. If you approach the value x is equal to a from the left side, the value y should become very large; it should become plus infinity or minus infinity; that means, the value of y becomes very very large or very very small; as x approaches a from the left side.

And similarly so the graph of the function approaches the vertical line, as you approach x from the left of the axis. And similarly vertical asymptote from the right as you approach the point a from the right side; the value of the function becomes larger and larger or smaller and smaller.

So, that is at same as saying f approaches the vertical line; graphically approaches the vertical line x is equal to a, as x approaches the point a from the right side of the x axis. So vertical asymptote from left and right, horizontal asymptote from left and right; so, why these are necessary? We saw an example that is the average cost tends to become smaller and smaller; never become 0, but approach is the line x is equal to. So, asymptotes do play roll in analyzing problems in economics and we will study some more examples in the next lecture.

Thank you.