Calculus for Economics, Commerce & Management
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Lecture - 32
Convex and concave functions

Welcome to today's lecture. In the previous lecture we had seen how various properties of the derivative help us to analyze properties of the function, and analyze it is nature. Today will be starting to look at some of the finer aspects of how calculus helps in looking at over deeper properties of a function. Which will have applications in our economic scenario, how the marginals increase and decrease. So, this properties are captured by properties of the function called being convex and concave. So, to motivate this concept let us consider the following examples of 2 functions.

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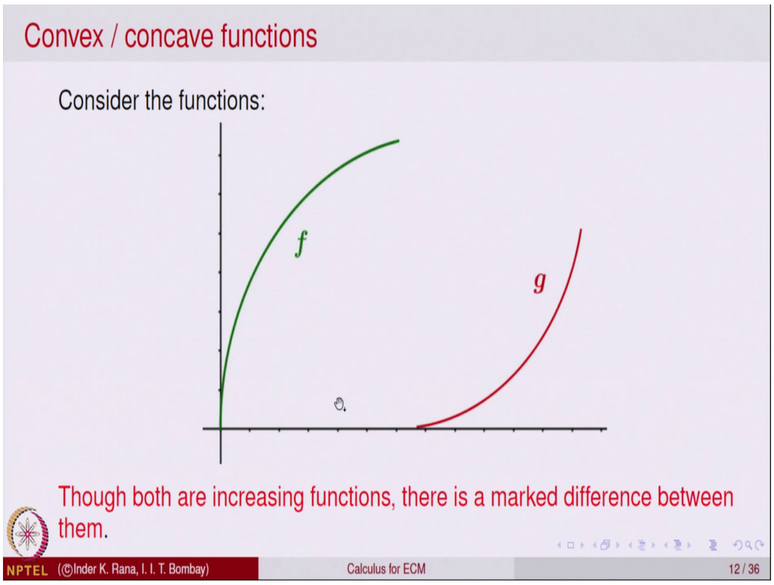

So, mathematically let us look at this graphs of this 2 functions, one function is f that is as green one, and g is another function which is the red one. So now, the question we would like to ask is what are the similarity between these 2 functions. And what are the differences between these 2 functions. So, from the picture it looks like f is a nice, continuous, smooth function right. So, there are no corners, and there are no breaks; that means, one says mathematically that the f is a continuous and offers stronger condition namely f is a differentiable function. And similarly, g also looks like a nice, smooth graph without any breaks.

So, one says that g also is a differentiable function. So, both the functions are differentiable functions. This function f is increasing. So, as you move from left to right, the graph of the function is rising. And same is happening for the function g as you move from left to right the graph of this also is increasing is also moving upwards. So, both this graphs are graphs of differentiable functions and both are increasing, but still there is a categorical qualitative difference between the graph of f and the graph of g. And what is the difference? So, if you look at the picture itself, the graph of f seems to be bending away from x axis, and the graph of g seems to be bending towards x axis.

So, this property of functions one being bending away, and one bending towards you would like to make it more precise. So, what we are saying is there is a marked difference, between the quality of the 2 graphs, though both are differentiable increasing. So, geometrically we can say that the graph of f is bending away from x axis, where as the graph of g is bending towards x axis. So, how do we analyze this mathematically.

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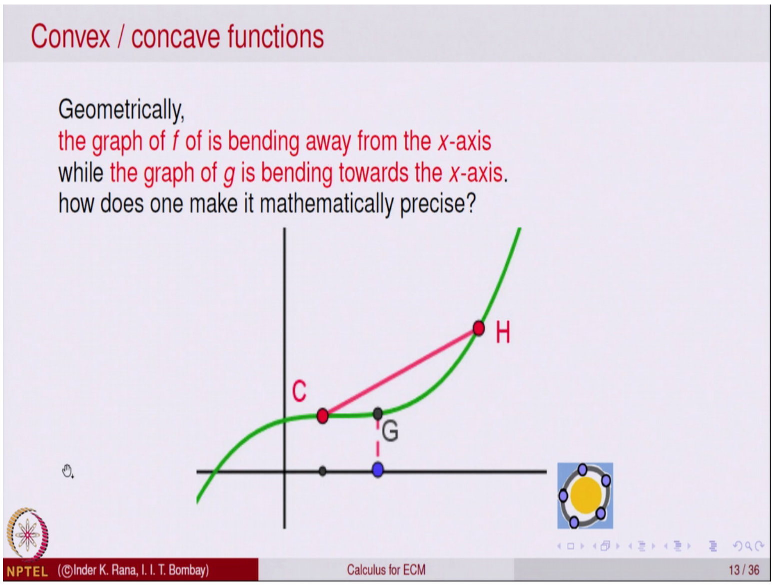

So, this is the kind of scenario we are looking at. For example, look at this graph green graph. Here in this portion the graph is bending away from x axis, while in this portions graph seems to be bending towards x axis. So, we would like to make this more precise.

So, let us look at geometrically what is happening, and then under try to understand this and make a mathematical formulation of this property.

So, I am going to open this graph GeoGebra. And trying to see dynamically what is happening in that.

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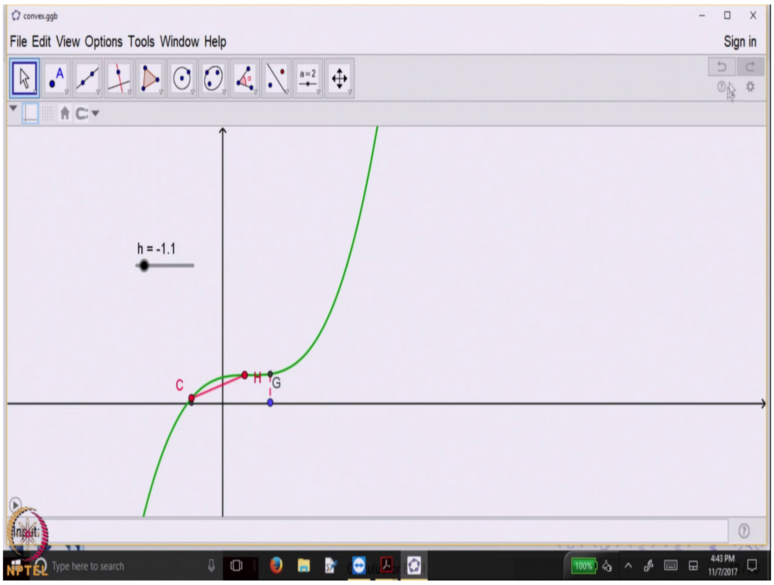

So, here is the graph of the function. So, this is the graph of a function. So now, let us see what I have done is; because this I want to understand this bending towards x axis and bending away from x axis. So, what I have done is; I have taken 2 points say C and H on the graph of the function. And I am going to move this points. So, let us move this point say for example, let us move this point C, and see what happened. So, if I move this point C on the x axis. So, the C up. So, what is happening to the red line as a move the graph of the function as a move with a point C?

The red line stays above the graph of the function same will happen if I keep the point C fix, and move this point H. So, remove the point H, you see that the state the line joining the points C and H on the graph of the function always stays above the graph he never comes below the graph, right. It always stays above the graph of the function. While if I go on the other side. So, let us take the point C on the other side here, and bring this point H closer. So now, you see, up to certain point up to this point for example, it is above part of it is above and part of that is below. If I move this point a bit more. So now, you see it is started staying below the graph, after this point for example, after this point G it starts going up.

So, what we are saying is that there is in this graph, there is a point right which separates out bending towards and bending away. So, what is bending away; bending away means that the line joining any 2 points on the graph should stay below the graph of the function. And saying that it is bending towards is saying that if I take 2 points on the graph of the function, and take the line joining them that the line joining will stay above the graph of the function.

So, that is what categorically characterizes geometrically characterizes the property that the graph is bending towards and bending towards away. So, bending towards x axis will mean, that if I take any 2 points on the graph of the function, then the segment joining the chord joining those 2 points should stay above the graph of the function, and other way round on this side when it is bending away then the chord joining these 2 points should stay below the graph of the function. So, that is what should happen.

So, this property let us try to put this property mathematically. So, this property you want to saying that when saying bending towards x axis should mean that if I take any 2 points, then the line joining them should stay above the x axis. So, this property we want to characterize.

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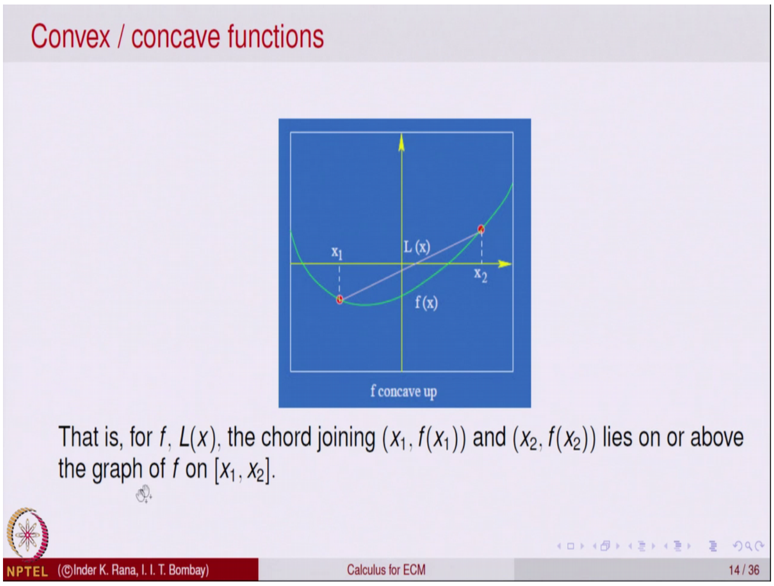

So, this is another picture of the same thing, was in belief the graph is bending this is bending towards. So, the line joining should be away, should be above right. So, the L x the line joining the chord joining should be lies above on or above. The similarly it should I am sorry. So, yeah, it should stay above if it is to be bending towards the x axis, and the chord should stay above if it is to be chord should stay below the graph of the function if it is to be bending away from the bending away from the x axis.

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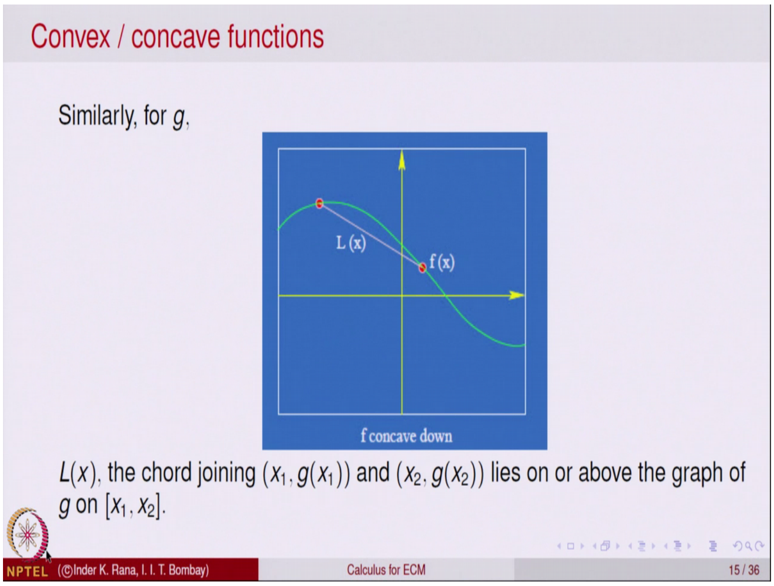

So, this we want to write mathematically. So, let us put a definition.

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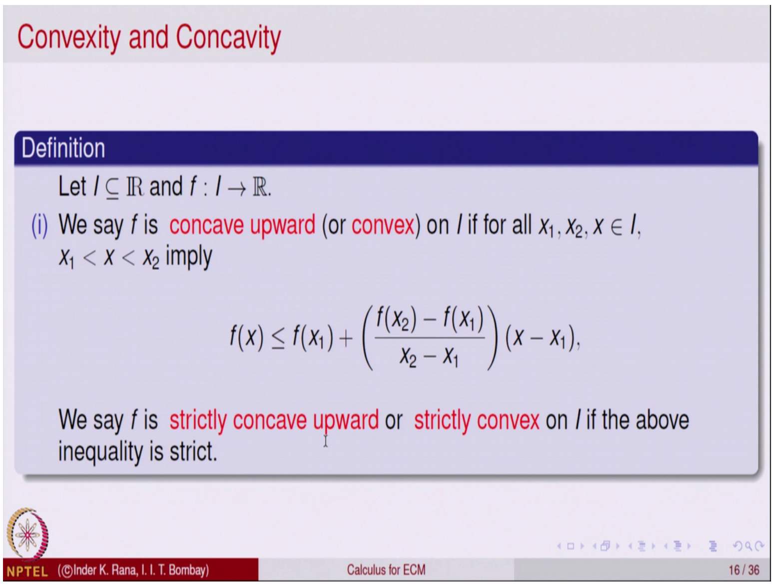

That let f be a function defined on a interval I taking values in r the real numbers. So, we say f is concave upwards or it is also called convex, if for any 2 points x 1 and x 2 lying on the interval. And a point x in between what should happen? The value of the function at that point we want to say that the graph is below. So, below means the value of the function at that point should be less than or equal to value of the chord joining. So, this is if you recall, we had the 2-point formula for the line joining 2 points. So, this will be the value of the function at that point. So, if you want to look at the picture once again. So, this is L x if you want to say it is below. So, if we take a point x here and take. So, this is a value of value on the line joining and this will be the value on the function joining on the graph of the function.

So, the value on the line joining should be below that is concave down. So, this property we are writing. So, we say can came up, if f of x is less, right. The graph is below the chord joining, and then we say it is concave up or it is called so called convex and. Which I strictly concave up or strictly convex if this inequality is inside of less than or equal to it is strictly less than.

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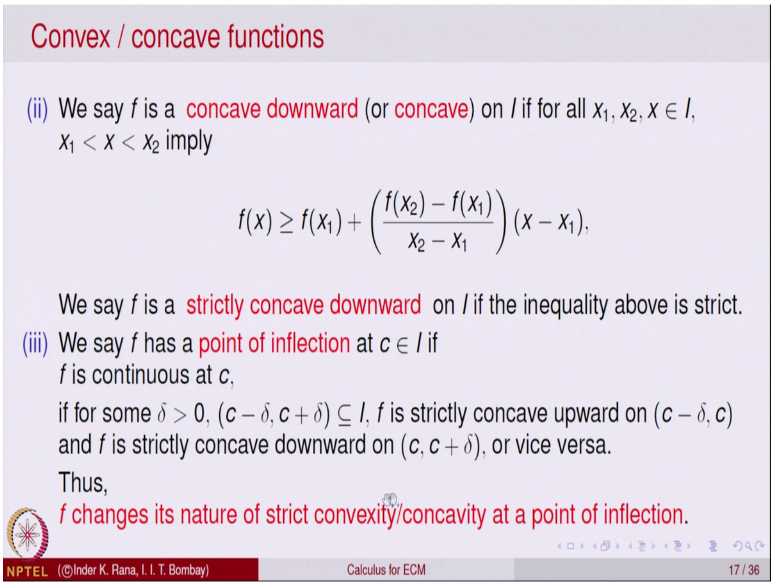

Similarly, will say is concave downwards, if the inequality is reverse f of x is bigger the chord joining is below the chord joining this is the value of at the point in the chord joining and this is a value of the function. So, graph is always above the chord joining, then it is called concave upward and we can say is strictly concave when f of x strictly bigger than then one at the word strictly concave up or concave down right.

So, as we saw in the previous example there was a point on the graph, where the nature of the function changes from concave to convex and other way round. So, we say a point C is called a point of inflection if f is continuous at C first of all first of all, they shouldn't be any break in the graph of the function, and on the there is a interval C minus delta to C plus delta, such that f is strictly concave upward on C minus delta to C; that means, what on the left of C it is strictly concave up. And on the right of it on C to C plus delta it is concave downwards. So, this is the point C is a point where strict convexity changes to concavity or other way round. So, the function changes it is nature from strictly convex to strictly concave or vice versa.

So, that is called a point of inflection. So, this is a point changes it is nature. So, point of inflection for a function is a point on the graph or point in the domain, where the function changes it is nature from strict, the word strict is important otherwise constant function will have are very point as point of inflection. So, this is important that strictly concave upward, and strictly concave downward on the left or vice versa of that. So, it should change it is nature from strictly concave up to concave down, or strictly concave down to strictly concave up as you cross the point C.

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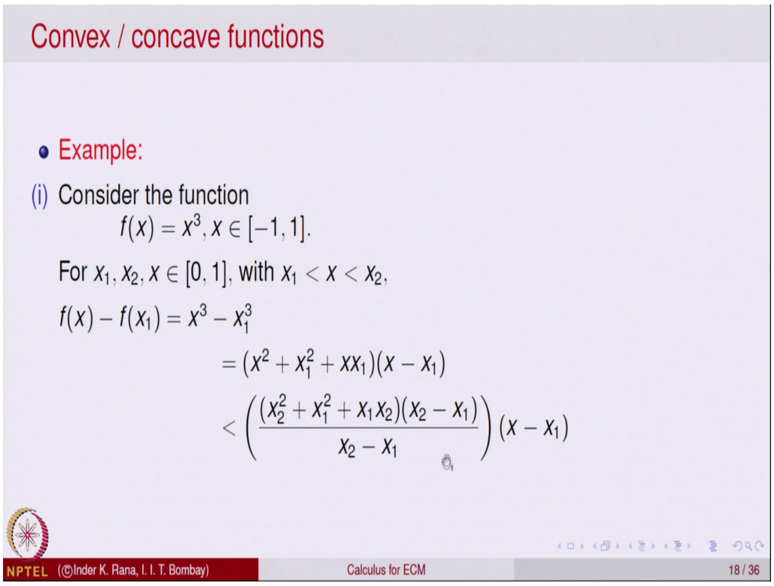

So, let us look at some example. So, let us look at an example of the function f of x is equal to x cube. X is between the interval minus 1 to 1. So, this function we know that f of x cube is a differentiable function differentiable everywhere. So, no problem comes, right? We are not going to use the property of differentiability, see in the notion of definition of convex and concavity; we just want to compute given 2 points whether the value of the function at a point is less than or bigger than the value at the line joining them. So, this can be done very nicely without any calculus tool. Let us take 2 points x 1 and x 2, and a point x in between them in the interval 0 to 1. Then let us look at f of x minus f of x 1. So, what is f of x? F of x is x cube f of x 1 is x 1 cube.

Now, using a bit of algebra will factorize this thing. So, this can be factorize as x square plus x 1 square plus x x 1 into x minus x 1. So, this is called the factorization of a cube minus b cube identity in algebra. Hopefully, you have studied in tenth standard, if not look at the factorization of various algebraic identities. Namely, what is the factor of a square minus b square or a cube minus b cube and so on. So, this is the factorization of a cube minus b cube, it is equal to a square plus b square plus a b into a minus b. So, here it is x 1 minus x minus x 1 that is factor here; so x square plus x 1 square plus x into x 1. So, that is a factorization of this.

So, using that we have gotten this f of x minus f of x 1 is equal to this. And this x minus x 1 we can keep it out. Here is x 2 minus x 1 that cancels out. So, we are dividing by x 2 minus x 1 and multiplying by x 2 minus x 1. So, instead of x we written x 2s now. Because and the sign has change to less than because x is less than x 2. So, in this if I change x 2 x 2 will get less than right. So, x 2 x 2 x is less than x 2. So, will get a less than, and multiply and divide by the same quantity x 2 minus x 1, you get this. Now this expression looks familiar. So, let us write down this expression is it.

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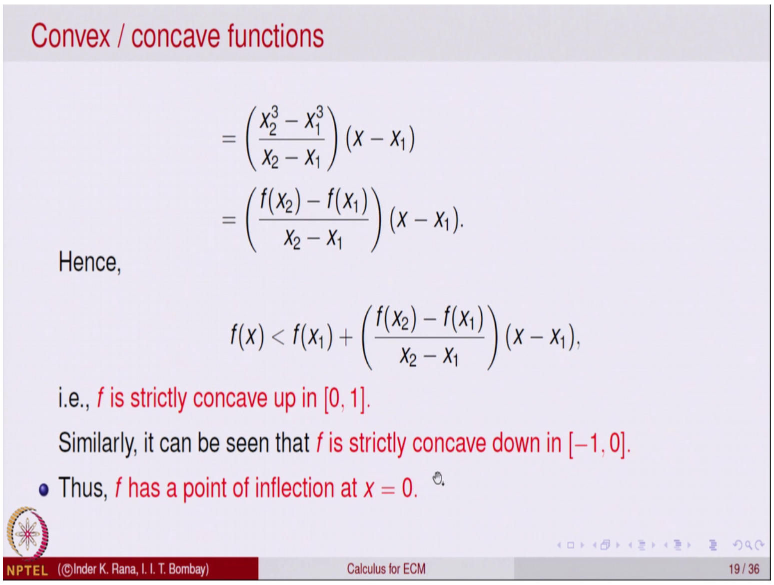

Equal to x 2 cube minus x 1 cube divided by x 2 minus x 1. And that we put back the value in terms of the function f; so f of x 2 minus f of x 1 divided by x 2 minus x 1.

So, what we get is that f of x which was on this side is less than or equal to f of x plus f of x 2 minus f of x 1 divided by x 2 minus x 1 into x minus x 1. So, that gives us that the function f of x 2 equal to x cube, right. Here we have assume of course, let us just go back and see one should be a bit careful here. So, we have let to be taken in x between 0 and 1. That is why every inequality was maintained. So, this gives that in the interval 0 to 1, the function is strictly concave up in the interval 0 to 1. A similar analysis algebraic analysis can be made to show that it is strictly concave down in minus 1 to 0. So, the function is strictly concave up in this strictly concave down in minus 1 to 0. So, 0 is a point of inflection that is a point is continuous at that point, and it changes it is nature. So, that means that, that is a point of inflection.

So, here is the graph of the function. F of x is equal to x cube. So, note that so, if I look at the points on the right side, right 0 to 1 onwards.

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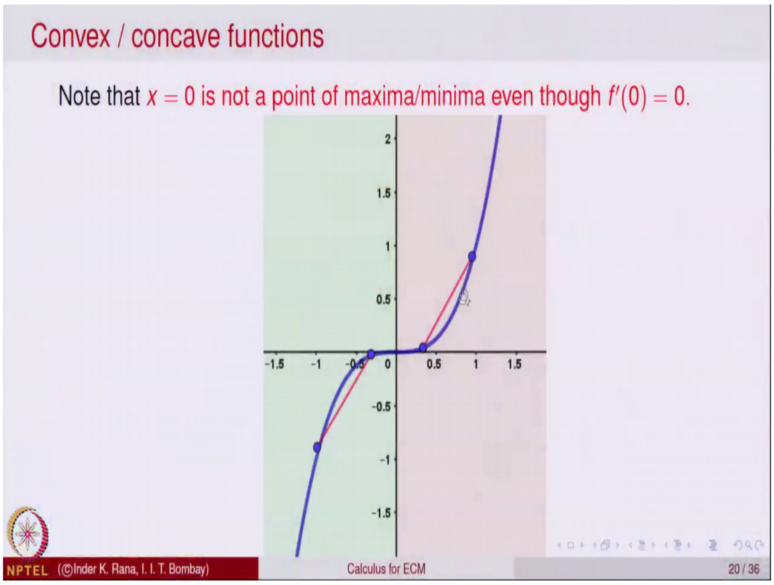

Then the chord joining them stays above, if I look on the left sides the chord joining the points stays below the graph, right. And the point 0 is the point where the function changes it is nature from strictly concave down to strictly concave up. And note that at this point 0, even though f dash of 0 is 0, this is not a point of local maxima or local minima. The tangent here becomes horizontal. So, the function is everywhere increasing function. So, it as you move from left to right, the function is rising. So, it is everywhere increasing function, strictly concave down, strictly concave down in this area on the left side in the green and concave up in the region which is the red one on the right side of it.

So, this is how you decide whether a function is concave up or concave down, by looking at purely algebraic nature of this we could do because our function f of x is equal to x cube is a very simple function. But for more complicated functions such kind of algebraic identities may not be possible. So, one has to device certain tools which help us to analyze convexity and concavity of a function in terms of the derivative over the function.

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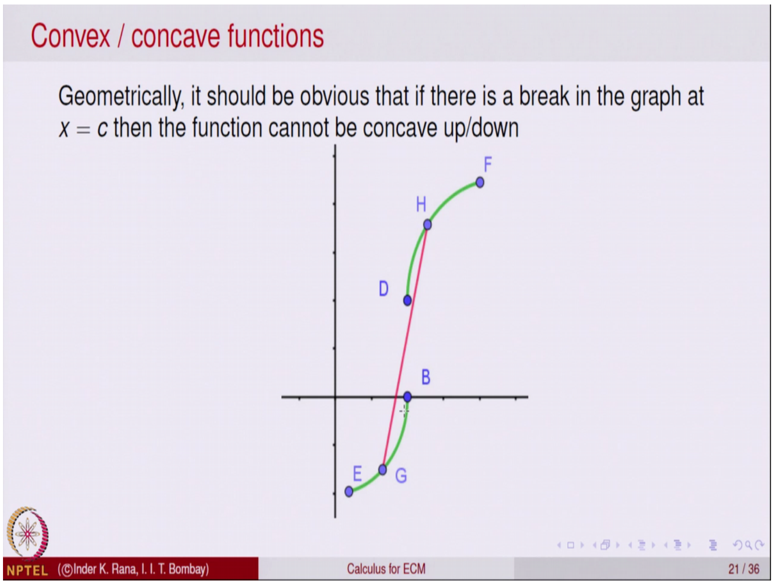

So, we will like to find out those criterias which will help us to decide whether a point what are the points where what are the intervals of convexity and concavity, and what are the points where the function can have a point of inflection. Here is one question you should have the geometrically it should be obvious; that if there is a break in the graph of the function that then, then the function cannot be in that interval a convex up or concave down. So, what we are saying is supposing this is a graph of the function the green one, and at this point there is a break in the graph and graph starts here and goes this way.

So, if I look at this part of the graph, then that part of the graph looks like bending away from x axis, right. And this part of the graph looks like bending towards x axis. So, in this this part of the graph is concave up, that part of the graph is concave down, but in the whole of the interval I cannot say the function is concave up everywhere or concave down everywhere. For example, if I take a point G in this part of the graph, and take a point H and take the line joining them, then what do you observe you observe that in this part of the graph the red chord the segment is below the graph, and in this part, it is above the graph. So, you cannot say that in the whole interval domain of the function the function is concave up or concave down. So, that indicates that if a function is concave up or concave down, then it should be continuous at every point. So, this is only indication of a result which is actually true.

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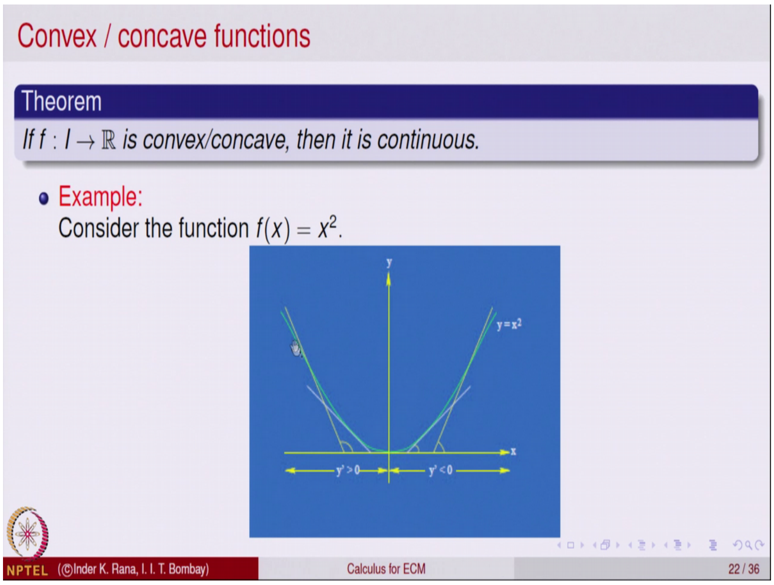

But we will not be proving that. So, it says that if a function is convex or concave in a interval I, right. Then it must be continuous in that interval.

So, whenever you are analyzing convexity or concavity, you have to ensure that the domain that you are looking at is a interval and in that interval the function is continuous. Otherwise you cannot analyze that. So, let us look at one more example to understand what we are saying. Now let us look at the example of f of x is equal to x square. So, we know that graph of this goes like is a parabola. So, it is like so, it is like a cup is cup up, a cup imagine a cup which is kept in a on a table, and it is at the top is not the inverted cup it is a cup kept on the table as such open cup you can think of.

So, this is parabola can be thought of that way as right. So, this we know it is if I take any 2 points on the graph of the function clearly that chord will stay above. So, this is a function which is concave up. So, concave up also is rhymes very well with cup up. So, physically cup up is like concave up. So, that is like u or v or whatever want to say. So, way of remembering is associating that concave up is like cup up

So, what we are say trying to say is this function f x is equal to x square is concave up. So, let us look at the slope of this at a different points. If we take a point on the left side the derivative is 2 x, and if x is negative the slope will be negative right. So, as you come closer and closer to 0, the slope is negative stays negative, but at the point 0 the slope becomes tangent becomes horizontal. So, slope is 0. And as you go on the right side, the graph of the function. So, if you look at a point on this side, then 2 x the derivative will be positive. So, tangent is having a positive angle. Here the angle is more than 90. So, it is a negative slope, on this side the slope becomes positive. So, as you go from left to right for this concave function, the slope goes from negative to positive. So, that means, it is slope is a increasing function for this kind of a graph which is concave up.

So, this is a general phenomena which happens for functions which are concave up, and gives us conditions in terms of the derivative. So, we are saying that if the function has is concave up in an interval, then the derivative must be increasing function. So, that theorem will state it in the next lecture. So, this example f of x is equal to x square is indicative of the result that if is function is concave up, then the derivative must be increasing. So, we will look at it in the next lecture.

Thank you.