Calculus for Economics, Commerce & Management

Prof. Inder K. Rana
Department of Mathematics
Indian Institute of Technology, Bombay

Lecture - 31
Property of marginals, monopoly market, publisher v/s author problem

In today's lecture we will start looking at some other properties of marginals, which lead to one of concern one of deep concepts in calculus namely convexity and concavity. So, before introducing that concept, let us look at the scenario in mathematically.

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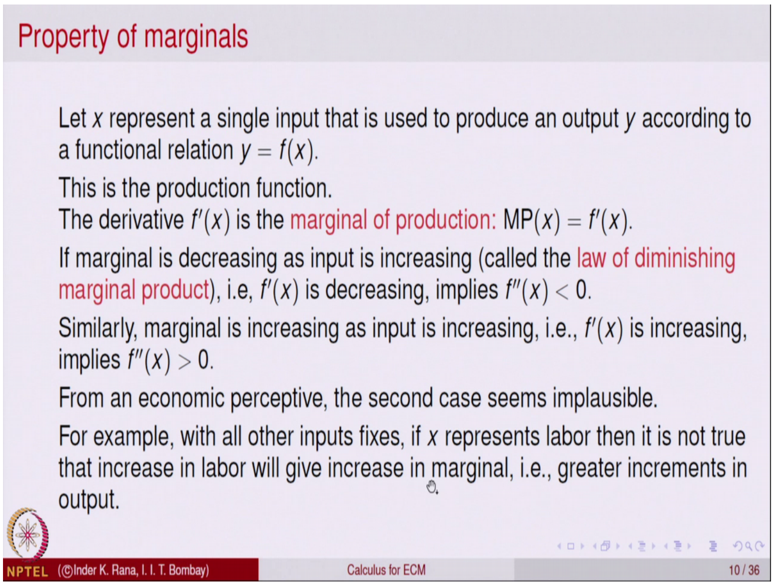

Let x represent a single input that is used to produce an output y; that means, y is a function of x, and the functional relationship is represented as y equal to f of x. So, x is x is input and y is the output of some quantity whatever it maybe.

So, we will later on see examples to illustrate this. So, in that case we the product function, the production function is given by this is what is called the production function, right fine. So now, look at the derivative of this. So, if this is called the product function or the production function. So, marginal of the product will be the derivative of this function. So, marginal m P of x is the marginal and that is equal to f dash of x that is a derivative of the function f of the product function of the production function.

Now, one would like to know if the marginal is increasing if the marginal is a function of x again it is the derivative of we have assume that the function f is differentiable everywhere. So, derivative is again a function. So, marginal is a function of the input x again. So, if the marginal is decreasing, right if this marginal is decreasing as input is increasing called the law of diminishing marginal product; that is, f is decreasing right implies the derivative of that derivative of f dash should be less than 0. So, that is what so, the point we have trying to make is that saying something about the marginal is revealed about the property of the second derivative of the marginal of the function product function. So, that is a second derivative.

So, this we want to study mathematically more deeply. So, what we do is the following similarly, marginal is increasing if the marginal is increasing right; that means, the input is increasing if the marginal is increasing that the means here there is a mistake input is decreasing. So, if the marginal is increasing as output is increasing, then f dash is increasing function and; that means, what f double dash must be bigger than 0. So, all this we want to make it mathematically very precise. So, let us look at mathematically what this means of course, the second scenario in economics does not seem to be plausible situation, where it says that the marginal is increasing as output is increasing.

So, that does not seem to be very good situation in economics. So, let us come back to mathematics and then see how it do we imply. So, for example, if in economics for example, if all other inputs are fixed and if x represents, for example labor then it is not true that increase in labor will give you increase in marginal; that is, greater increments in output that may not always happen that we know. So, that is why this scenario may not always happen right.

So, let us continue with our examples of monopoly market, that we have been looking at. So, we have looked at an example of monopoly firm with demand price function P Q equal to 10 and cost function as C Q equal to 5 Q.

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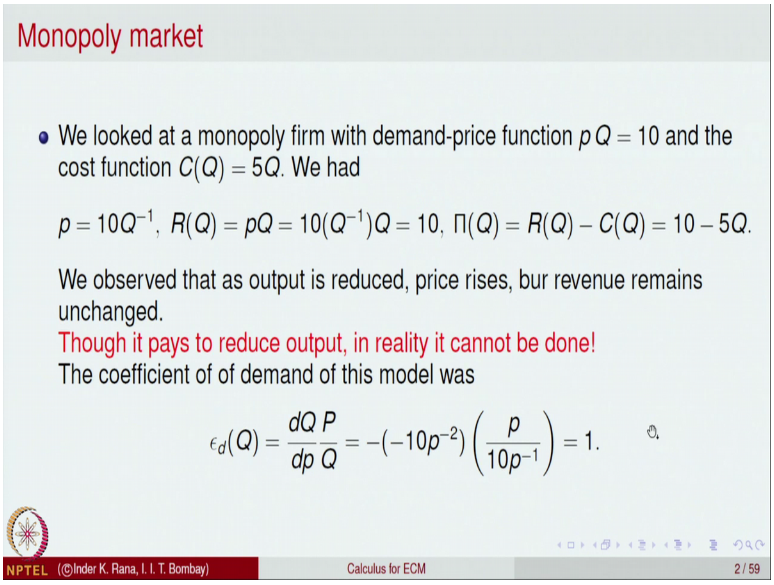

For this we wrote down the various functions P equal to 10 Q minus 1. So, here P Q was 10. So, Q on the other side if you take it. So, you get price demand relationship P is equal to 10 Q minus 1. The revenue function is P Q. So, that comes out to be multiplication is equal to 10 that is a constant function, and then you look at the profit function the revenue function minus a cost function that comes out to 10 minus 5 Q.

So, the we have done this I am just revising it for. So, we observe that the output is reduced the price rises, but the revenue remains unchanged. So, in this example this is what the observation we are made the output is reduced then the price is rise if the output is reduced because it is 1 over Q. So, if the output is reduced the price will go up, but the revenue remains constant. So, though it pays to reduce output, it is not really it cannot be done, because you cannot reduce output as much as you like you cannot make it a fractional kind of a thing.

So, in reality this scenario is not really useful that is what we had observed and for this model we had computed the coefficient of elasticity which gave it out to be 1. So, coefficient of elasty this is a unit coefficient of elasticity example. So, let us look at one more example.

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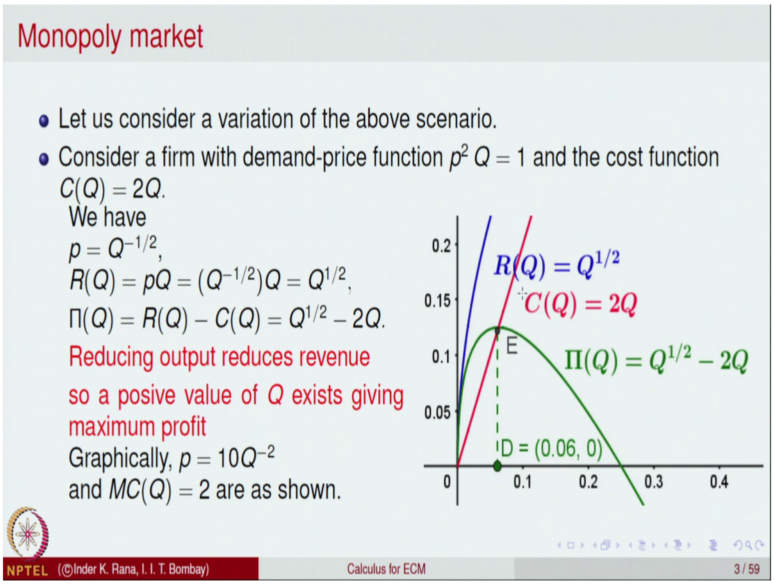

So, consider the variation of this example namely consider a firm with demand price as P square Q equal to 1, and the cost function as C Q equal to 2 Q. So, here the cost price function demand price function is given by this.

So, if you want to write P as a function of Q, then P square is equal to 1 over Q. So, P will be equal to square root of 1 over Q. So, that we can do; we can write P as a function of Q to be Q square root of Q. So, we are written it as Q raise to power minus 1 by 2. So, the revenue function is P into Q. So, this is P and into Q. So, that gives you Q raise to power Q raise power half into Q raise to power 1. So, total power becomes Q raise to power half. And the profit function is given by R Q minus C Q.

So, R Q is here Q raise to power half and C Q is equal to 2 Q. So, when you put that values you get the profit function to be equal to Q raise to power half minus 2 Q. So, all the basic functions are there. So now, we can observe the reducing the output, if you reduce the output see Q is equal to Q raise power half. So, this is a increasing function has such. So, if you increase the output R Q will go up. So, reducing the output if you other way around if you reduce the output, then the revenue is also reduced, but you can do it only when till Q remains a positive value.

So that means, if you increase Q then R Q is going to increase, and if you reduce Q then the revenue is going to decrease. So, profit is going to increase. So, at some positive value the maximum profit can occur. So, that is the this model says that positive at some positive maximum profit at some positive value will occur. So, graphically let us plot these 2 functions P is equal to 10 Q minus 2. So, P Q is not 10 it is just Q raise to power minus 2 and MC Q. MC Q is equal to 2. So, let us plot this. So, this is the this blue one is the revenue function which was equal to Q raise to power half. So, that is a revenue function, and the cost function is c 2 Q that is a cost function that is a linear function. So, that is it. And pi Q the profit function is Q raise to power half minus 2 Q and there is a graph of it.

So, if you look at the graph as the production increases it keeps on increasing, and then it starts decreasing after some point onwards. So, the profit occurs at the value when D is equal to 0.6 not a very realistic situation, but still from the graph that is what we get.

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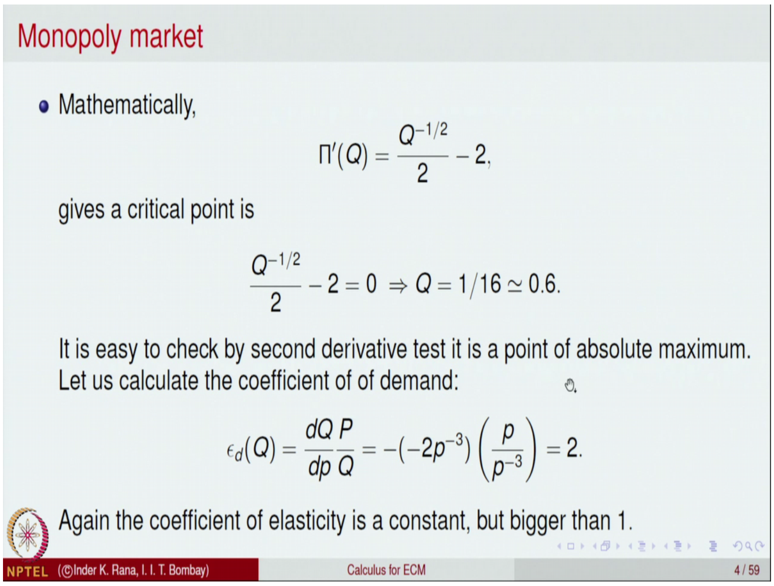

So, if you want to verify it mathematically will look at the derivative of the profit function, so that is equal to this, right recall the derivative pi was equal to this. So, derivative of this will be half Q raise to power half minus 1 into 2 minus 2, derivative of 2 give is 2.

So, we can put this value in the derivative function. So, we get derivative equal to this. So, which font want maximize this we have to put derivative equal to 0 to look what the possible values. So, when you put it equal to 0 Q comes out to be 1 over 16 from this equation. So, 2 into 2 4. So, Q raise to the power half is equal to 4. So, Q will be equal to 1 over 16. So, that is approximately 0.6.

So, that is a point where possibly the maximum value can occur.

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One can verify it by the second derivative test if you like, or you can check it from the first derivative. So, we will leave it for you to check that this point is a point of absolute maximum for the function. So, there is how you will verify it mathematically from this what is the visible from the graph can be verified it mathematically. So, let us calculate the coefficient of elasticity for coefficient of elasticity of demand for this function. So, that comes out to be by formula it is D Q by D P into P by Q.

So, put the values and simplify that comes out to be equal to 2. So, again this is an example of a constant demand constant coefficient of elasticity of demand that is 2 which is bigger than 1, and in this model the profit is maximized at some point.

Though not very realistic. So, that is the mathematical way of saying things, and what the imply in terms of economics. So, again the coefficient of elasticity is constant, but it is bigger than 1. So, let us look at just recalling then in the example of unit elasticity that we are discuss in the previous lecture, and we have revised today also in the beginning profit could be increased as much needed theoretically.

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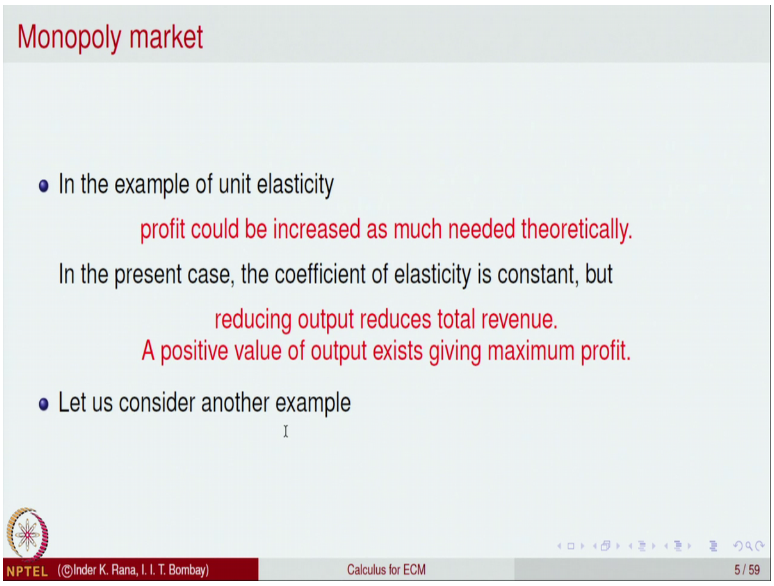

So, there we said that if you make output smaller and smaller, the profit keeps on increasing, but that is only theory in practice you cannot make the production, how much you produce to be very, very small.

So, that was unit elasticity scenario, and the present case we have again the coefficient of elasticity is constant, but the value is equal to 2, it is more than 1. And in this case reducing output reduce is total revenue. So, that means, if you increase the output you will increase the revenue and a positive value output exists giving the maximum profit and that is what we observe happened at the point Q equal to 0.6. So, we are relating coefficient of elasticity with existence of profit maximum profit and so on.

So, this is to comparison of the 2 scenarios. Let us consider one more example still.

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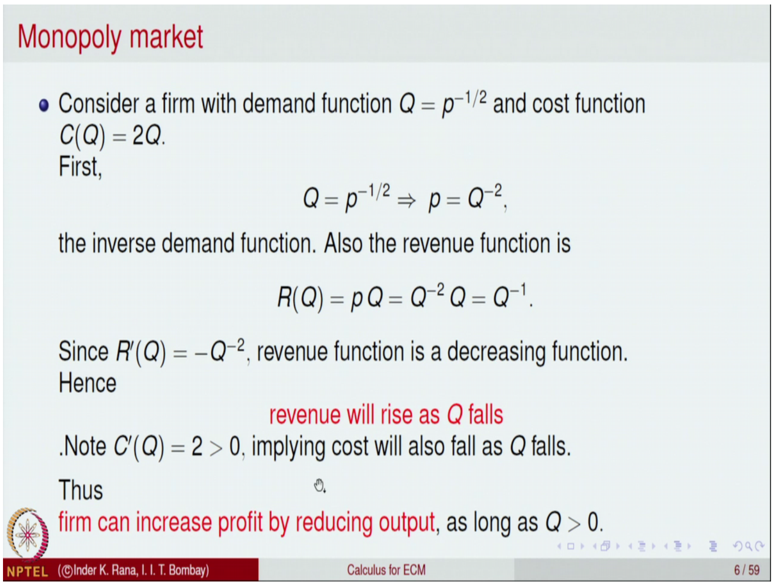

Namely, consider a firm with demand function to be equal to P raise to power minus half, and the cost function to be equal to 2 Q. So, with this scenario, let us write down the functions. So, if you Q is equal to P raise to the power minus half. So, if you take it on the other side P raise to power half will be equal to 1 over Q. So, when you square both sides you will get the relation P is equal to 2 Q P is equal to Q raise to the power minus 2.

So, that is the relation between the price and the quantity being produced, once again note that this Q P to the power minus 2 means it is equal to 1 over 2 to the power Q 1 over Q square. So, as Q decrease is P is going to increase. So, once again the decrease in decrease in the quantity produce increase is the price. So, this is what you we call as a inverse demand function. So, the revenue function for this will be equal to P into Q as usual price in to the quantity is being produced, P is equal to Q raise to power minus 2 into Q. So, that becomes revenue is equal to Q raise to power minus 1.

So, if you look at the derivative of this revenue function; that is, Q raise to power minus 1 it is derivative will be minus 1 into Q raise to power minus 1 minus 1. So, that is equal to minus Q minus 2 minus of Q raise to power minus 2. So, whatever we Q it is 1 over Q square. So, that is going to be negative, so that means what? So, the derivative of a function being negative the function is going to be a decreasing function. So, revenues will rise as Q falls. So, this is going to be a decreasing function. So, because is 1 over Q right or Q. So, this one we are deducing it from here look at it.

So, revenue is equal to 1 over Q. So, if Q falls; that means what? Q becomes smaller 1 over Q is be going to become bigger. So, revenues will rise as Q falls. So, that is a conclusion from this equation this only says that it is it decreasing function. So, all the marginal of cost is equal to 2; that means, implying that cost will also fall as Q falls. So, that is positive, right. So, firm can increase profit by reducing output as long as Q remains positive. So, profit can be increased by reducing Q. So, that is what is the outcome of this says.

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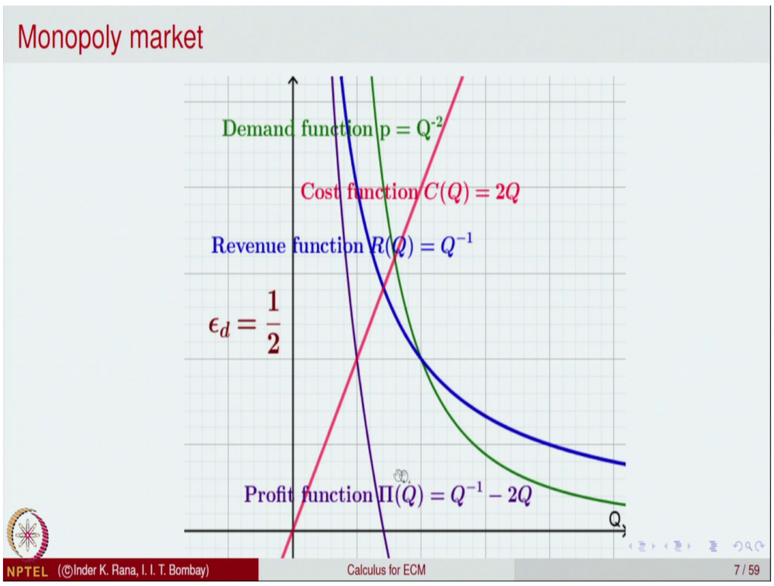

So, let us look at the various graphs. So, we can plot them. So, for this this is C Q that is a linear function. So, that is a straight line. The demand function is P equal to Q raise to power minus 2 that is a green one. So, this is the green function. So, this is a function which is Q raise to power minus 2, and the blue one is the revenue function that is Q raise to power minus 1. So, the profit function is pi Q. So, that is Q raise to power minus 1 minus 2 Q. So, that is this graph that is this profit function.

So, these are the various functions related with the economic scenario, which help us to say few things. For this model the coefficient of elasticity is 1 by 2, which is less than 1. So, that is that one can compute, and we will do that and c that comes out to be equal to 1 by 2. So, here there is a nice observation here that if you look at this green graph the green is the demand function. And the blue one is a revenue they intersect here.

So, what does that intersection mean; that means, at this point the quantity produce that price is equal to the revenue. So, that will be the conclusion of this there are other various other interpretations one can make that depending upon what you are looking at.

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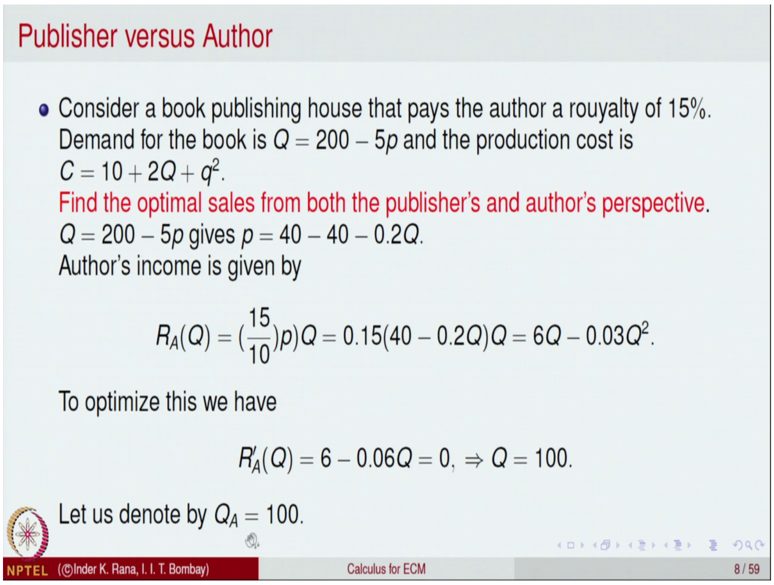

Let us look at a one more example for our scenario, and that is as follows. Consider a book publishing house that pays the author a royalty of 15 percent. So, when you publish a book the publisher prints the books, sells the books, and then the author is given a part of the selling price of the book as the profit as the royalty.

So, this publisher is offering author a royalty of 15 percent. The demand for the book the demand function for the book is given by 200 minus 5 P, right. And the production cost for the publisher to produce the book is given by 10 plus 2 Q plus the Q square this is a typo here this is a small q, but that is same as notation is Q; that is the quantity is being produced right. So, it is 2 Q plus Q square. So, this scenario being given, what is it to one is interested in. The author would like that there should be maximum possible profit for him from the royalties; that means, he would like to know the number of books that should be sold by the publisher. So, that it profit is maximized.

On the other hand, the publisher would like to know what is the optimal scenario, where his revenues will be his profit will be maximized. So, this is profit of the publisher versus the royalty of the author. So, we would like to know this. So, find the optimal sales from both the publishers and the authors perspective. So, we would like to know from authors point of view, and publishers point of view how much the author thinks publisher should be selling. So, that his share forever royalties is maximum, and on the other hand the author would like to know how much he should sell how many books he should sell. So, that his profit is maximum because he has to produce the books.

He has to spend on the cost he has to give a royalty and so on. So, let us try to solve this 2 problems. The first the Q is 200 minus 5 P right. So, that gives you P in terms of Q as 5 P is equal to 200 minus Q, there is something 5 P is equal to 200 where Q. So, that is so, let us see what is a typo here. So, here 5 P goes that side is equal to 200 minus Q, right. So, that is 200 minus Q. So, that means, divided by 5 that is 4 minus 1 by 4 Q. So, that that should be 1 by 4 Q and so, there is a typo here. So, let us see if that is corrected.

So, that is 40 minus. So, this is this part is extra here. 0.2 Q that is there that is that part is so, this typo that minus 40 is extra term here from this equation P is equal to 40 minus 0.2 Q, right. When divide by 5. So, authors income will be given by say 15 percent of the royalty, right. So, again it should be 15 by 100 here not 10. So, this should be 15 by 100 here 15 by 100 into P that is selling price. So, that is his profit and Q units are sold so, that will be his royalty of the author.

So, once you do that. So, that is 0.15 and 0.15 of P into Q. So, P is 40 minus 0.2 Q into Q and Q the value of Q is equal to 200 minus into Q itself, right. So, when you simplify this equation you will get 6 Q minus 0.03 Q square. So, simple calculation will tell you this is say modulo this typo here it should have been 100 here. And once you simplify it should be come out to be this.

So, what we want to do is optimize this function which is 6 Q minus 0.03 Q square. So, to maximize that we look at the derivative; so that will be 6 minus 2 comes down. So, 0.06 Q and to find out the possible point, where it can have a maximum, we have to put it equal to 0. So, this equation gives you Q equal to 100. So, at Q equal to 100 possibly the maximum of the revenue of the author will be there when there is sale of 100 books. How do you know this is the maximum point of maxima? Well, one can differentiate this again if you like. So, double dash second derivative of this function will be minus of 0.06 and that is a negative quantity. So, if the derivative is negative at a point is a negative constant everywhere. So, the point of critical point should be a maximum.

So, Q is equal to 100 is the optimal which author can hope that if the publisher sells 100 books will make a his profit or his revenue or his royalty will be maximum. So, let us call this as Q a to be equal to 100.

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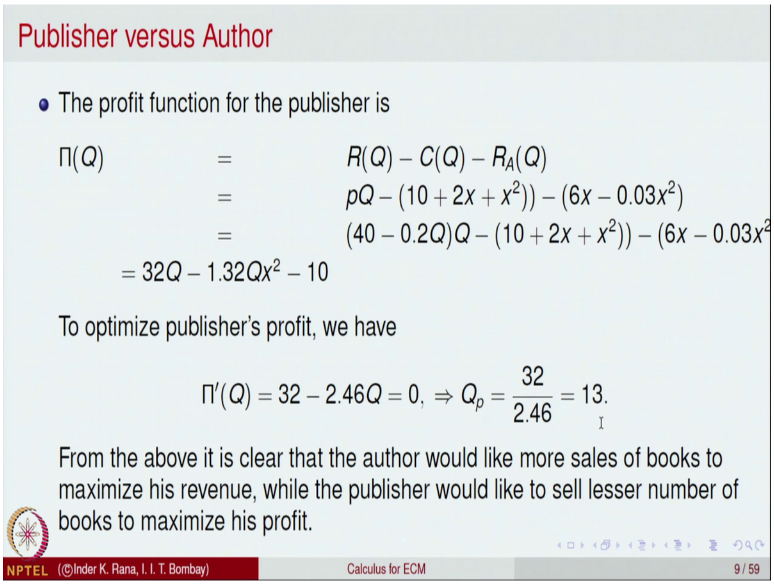

Let us look at from the point of view of the publisher; the profit of the publisher will be R Q minus C Q, right minus the royalties he has to pay to the author. So now normally the profit is R Q minus C Q, but here he is also incurring a running a expenditure depending on how many books he sells he is to give a profit he has to give 15 percent of royalty to the so, that means, R Q is minus C Q minus r a Q.

So, let us put the value R Q is P Q, C Q is here there is a typo again here it should have been Q instead of x instead of x is Q squares of 10 plus 2 Q plus Q square minus r a Q we got it 6 Q minus 0.03 Q square. So, in this calculation please treat x as capital Q. So, once you do that and you simplify this equation. So, this comes out to be 32 minus Q minus 1.32 Q it is not Q x square it is Q square. So, x is extra here minus 10.

So, I am sorry for the typos here. So, that comes out to be equal to this. So, when you do to optimize the profit function. So, there is no x here it is Q square. So, once you want to maximize this you have to differentiate 32 Q derivative will be 32. 1.32 Q square will give you 2 times 1.32 that is 2.46. And Q and that equal to 0 gives you the value maximum profit being Q. Anyway, modulo this minor arithmetically typos, you will find out by the derivative test first derivative equal to 0, find the derivative of the profit here the only it thing to be noted is profit is R Q minus C Q the revenue minus the cost revenue is P Q cost function is given minus the revenue of the author, that has to be subtracted from to find out the profit.

So, get gives you Q equal to partial derivative to be sorry, the derivative first derivative equal to 13. So, that means, the possibly the maximum can occur, at the point of 13 once again if you take the second derivative that comes out to be negative. So, this is a point of maximum for the function pi Q also. So, as far as author is concerned, the publisher should be selling 100 books while the publisher to maximize his revenue things you should be selling only 13 books. So, there is a conflict of interest between the 2. So, the publisher would like to sell lesser number of books to maximize his profit, whether right in the lesser number of books than the author things.

So, this is a comparative statement we can get from so, all this analysis uses the properties of derivative and it is applications.

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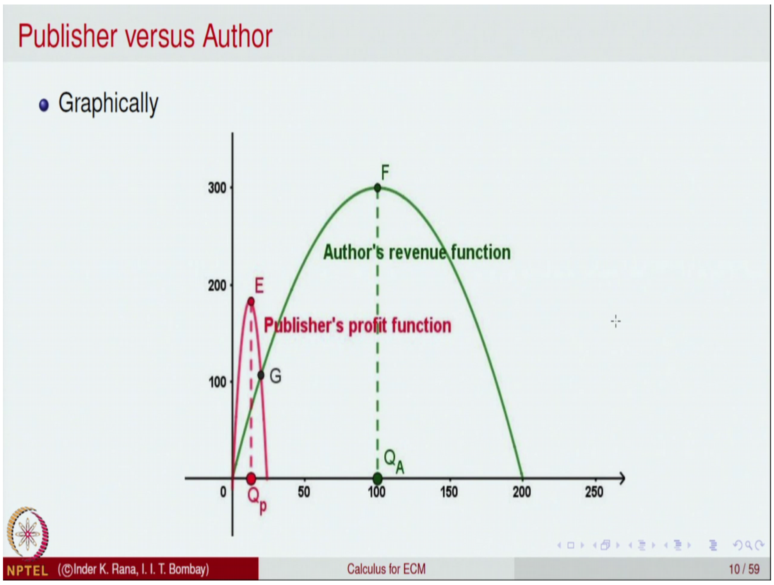

One can look at the graphical representation of this and that is nothing but saying that you look at this this is a publisher's profit function this red one. And the green one is author's revenue function. So, that maximizes at the point of 100. So, when 100 books are sold authors profit is author's royalty is maximum, and when it is only 13, then the publishers profit is maximum. Here is the interesting scenario that graphically these 2 intersected the point G. So, G is the value which will give you equal amount of profit for the publisher as well as author. So, both will have same amount of take home or whatever we want to have publishers profit will be equal to the royalty.

But that rarely happens always publishers would like to make more money on the sale of books than the author can get. So, this is how we will use the calculus, and it is derivatives properties of the calculus to maximize or minimize the problems. So, let me just have a recap of what do we have been doing till now, namely we looked at the scenario of how calculus is used in optimization problems we looked at increasing decreasing a functions we looked at the critical points in terms of the derivative.

We had the condition in the beginning that if a function has a local maxima or a local minima at a point and if the function is differentiable at that point the derivative must be equal to 0. So, this is crucial theorem that helps you to locate the possible points where the function can away extremal value. So, given a function and look at the domain of the function, see if there are any endpoints in the domain of the function, look at the points where the function is not differentiable and the points are interior or look at the interior points where the derivative is equal to 0.

So, that gives you the possible points where the function can have a extremal values the maxima or the minima. To analyze which of them is a maxima which of them is a minima you have to apply various test, you can apply the continuity test, you can apply the first derivative test or you can apply the second derivative test. So, in all this test please be sure that conditions are satisfied. Because for the first derivative test for the first continuity test the function should be continuous, and if it is continuous at that point, and it is increasing on the left decreasing on the right, then it will be a local maximum and other way round.

For the first derivative test we do not need the function to be differentiable at that point we only need that on the left side the derivative should be say positive on the right side of the funct point the derivative is negative, then there will be a point of local maximum and similarly, for local minima. And the second derivative test is that once you are located a point where possible it can have a local maxima or minima. If the function has second derivative at that point and it is bigger than 0, then it is a point of local minimum, and if it is less than 0 it is local maximum.

And this critical points also give a help you to analyze absolute maxima and absolute minima of a function the only condition is that you should ensure that absolute maximum and the absolute minimum exists. So, out of all the critical points, look at the points where the largest value of the function takes place and the smallest value takes place. So, we will look at the points of absolute maxima and minima. So, this is how calculus tools are used to locate optimal solutions or optimization will continue with our study of applications of calculus tools in some finer aspects of analyzing functions, in the next lecture.

Thank you.