Calculus for Economics, Commerce & Management
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Lecture – 29
Absolute maximum and minimum

Welcome back to our discussion about the absolute maximum and the absolute minimum of a function of one variable. We had defined the concept of absolute maximum and the absolute minimum. So, the point of absolute maximum is a point where the functions take the largest value in the domain of the function and absolute minimum is the value of the function at a point where the value of the function is the smallest as compared to the value of the function at all other points in the domain of the function. That is why, it is called the absolute maximum and absolute minimum and we are said that to locate the possible points where the function can have absolute maximum or absolute minimum is same as that of local maximum and minimum because absolute maximum is also a point of local minimum.

So, we collect together all the points possibly where the function is not differentiable in the domain of the function and look at the points in the interior points in the domain of the function where the derivative is equal to 0. This is the second setup points and the third set up points is the points which are the end points. For example, the domain could be a interval and closed interval or and the endpoints of that interval are possibly also the points where the maximum value of the function can occur.

So, among these three setup points which are same as that of for analyzing local maxima or minima one locates this points compares the values of the function at this points and see which one is the largest and which is the smallest and of course, the critical point or the crucial point in this analysis is that the function should have absolute maximum and absolute minimum and the only theorem that we have discussed and stated is what is called the Maxman theorem for continuous functions on closed bounded intervals.

So, that said that a given a function on a closed bounded interval if it is continuous then it will always have a point of absolute maximum and a point of absolute minimum. So, whenever we do problems in our discussions one should ensure that the functions involved are all continuous functions. So, let us look at some examples. So, let us look at the example.

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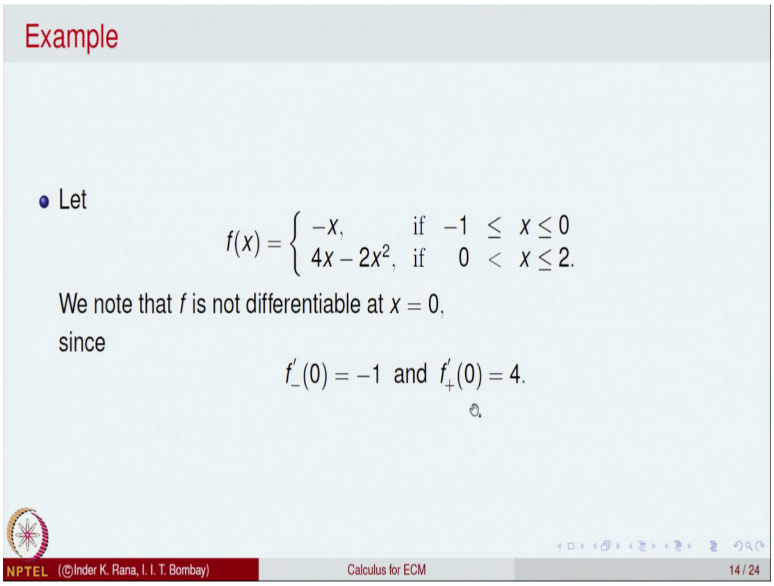

Where the function is given by f of x is equal to minus x if x is between minus 1 and 0 and it is equal to 4 x minus 2 x square, if x is strictly bigger than 0 and less than or equal to 2. So, the function is defined differently for the values minus 1 to 0 and it is defined differently for values bigger than 0 and less than or equal to 2.

So, this function is defined the domain of this function is a closed bounded interval from minus 1 to 2. So, the function is defined in a closed bounded interval minus 1 to 2 is it continuous, let us analyze look at the function in the interval minus 1 to 0 at all points, it is minus x which we know is a continuous function also in this interval the open interval 0 and close at the point 2, it is defined by this quadratic function.

So, which we know from our illustrations discussions in continuous functions that this is a continuous function. So, the only point left is the point x is equal to 0 is it continuous at that point to find that one analyses the left limit at the point 0 and the right limit at the point 0 the left limit at the point 0 will be found by taking a sequence xn converging to 0 from the left side that will give you f of xn as minus xn and xn goes to 0 implied the left limit exists and is equal to 0.

Similarly, if you take the right of one to compute the right limit of the function at the point 0 we will take a sequence coming to 0 from the right side; that means, this is the value of the function. So, we take a sequence xn converging to 0 from the right then f of xn will be equal to 4 xn minus 2 xn square as xn goes to 0 4 xn will go to 0 and 2 x n square will also go to 0 because xn goes to 0.

So, using the fact about limits limit theorem for sequences, we will see that the right limit of this function at the point 0 is also 0. So, this function is continuous in the closed bounded interval minus 1 to 2. So, this will have a absolute maximum and absolute minimum in that interval. So, that is ensured now the question is to find out what is the value which is absolute maximum and the absolute minimum and how do you find that.

So, to do that let us look at the look at the function in the domain minus 1 to 2 we first of all claim that f is not differentiable at x is equal to 0. So, will how do you prove that. So, that it is not differentiable at x is equal to 0. So, let us analyze that by working out the derivative from the left and the derivative from the right.

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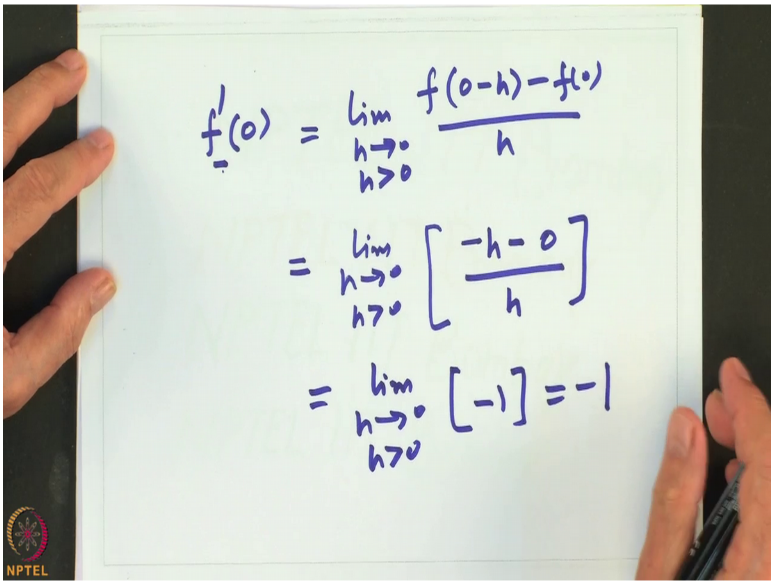

So, let us look at I want to find out derivative at the point 0 from the left side. So, that is equal to limit f of I want to go to the limit. So, let us like 0 minus h minus f at 0 divided by h; h going to 0 and h bigger than 0. So, that will ensure f of 0 minus h.

So; that means, h is positive. So, you on the left side of 0, what is this quantity this is equal to limit h going to 0 h bigger than 0? So, what is f of 0 minus h on the left side of 0 the function is f of x is equal to minus x. So, it is minus h minus f at 0 is 0 divided by the value h. So, this and that is equal to limit h going to 0 and this when you divide this is equal to minus 1. So, this limit equal to minus 1. So, the left hand derivative of the function at the point 0 is equal to minus 1, let us compute the right hand derivative of the function at this point.

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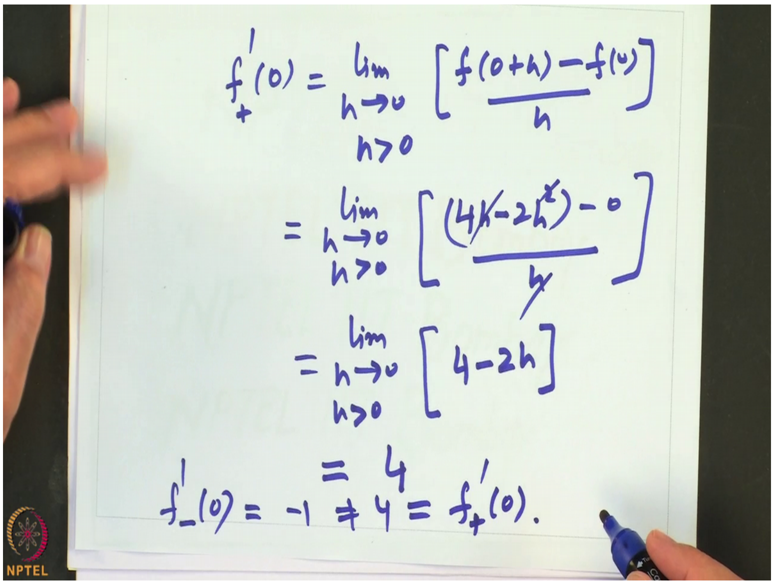

So, let us compute the right hand derivative. So, we want to compute f dash at 0 plus. So, that will be equal to limit h going to 0 of f of 0 plus h minus f at 0 divided by h; h positive.

Again h is positive, but here I have written 0 plus h so; that means, you are on the right side of the point 0. So, let us put the value. So, that is limit h going to 0 h bigger than 0 what is the value on the right hand side the function is defined as f of x is equal to 4 x minus 2 x square. So, this is h minus 2 h square minus the value at 0 is 0 divided by h. So, that is equal to limit h going to 0 of 4 h square minus as. So, this h cancels with this h and one power cancels, with this is 4 minus 2 h; h going to 0 h bigger than 0 of course, and this limit we know h is going to 0. So, that is equal to 4. So, the left hand derivative was equal to the left hand derivative of the function was minus 1 the right hand derivative is equal to 4.

So, f dash minus f 0 is equal to minus 1 which is not equal to 4 which is equal to f dash the right hand derivative at 0. So, implies that the function is not differentiable at the point x is equal to 0. So, that is how you analyze the differentiability of a function of one variable if it is defined differently from the left and differently from the right. So, it is always go to work out these things. So, f is not differentiable at the point x is equal to 0 now at all other interior points, we have to find the derivative and put it equal to 0. So, we saw that this is left hand derivative is minus 1 and the right hand derivative is equal to plus 4. So, that is fine.

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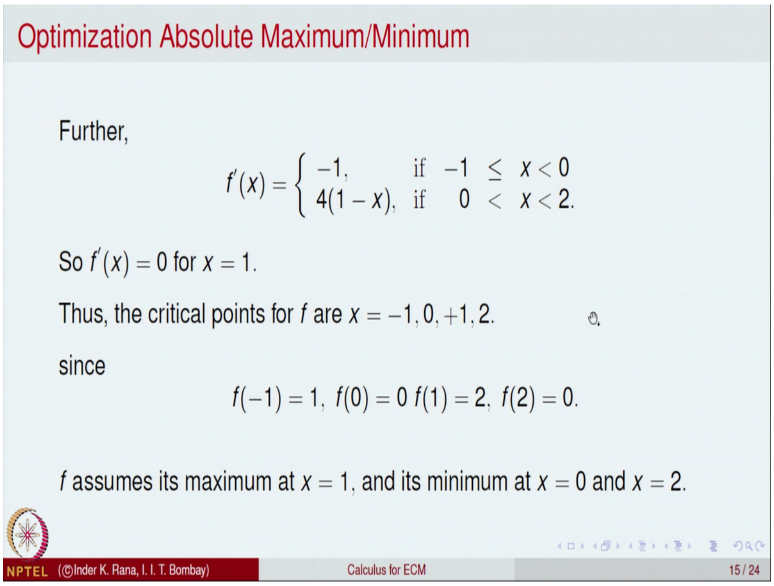

Now, let us look at the derivative the function was defined minus x in the interval minus 1 to 0 at 0, we now we have discarding because that is the point of non differentiability. So, in the open interval minus 1 to 0 the function is minus 1.

So, it is differentiable at the point minus 1 and the derivative is equal to minus 1 and at the point in open interval 0 to 2 the function was defined as let us go back and look 4 x minus 2 x square. So, in the open interval 0 to 2 the function is again differentiable and derivative is 4 minus 4 x. So, let us put these values that. So, the derivative function is minus 1 for minus 1 less than or x less than 0 and it is equal to 4 times one minus x if x is going to m 0 and 2. So, to find out the points where the function can have a maxima or minima, we have to put it equal to 0 and of course, derivative between minus 1 and 0 is minus 1.

So, the function cannot have any point inside that interval only in this interval 4 into 1 minus x is equal to 0 that gives you the value namely x is equal to 1. So, the possible points where the function can have local maxima or minima and absolute maxima or minima are what about what we called as the critical points and these points are the endpoints. So, minus 1 and 2 are the endpoints. So, minus 1 and 2 are the endpoints 0 is the point where the function is not differentiable right interior point and the function is not differentiable and the point where the function derivative exist interior points and derivative is equal to 0. So, these are the 4; 4 points where possibly where the function can have absolute maxima or minima and this be out of this only because of the necessary condition for maxima minima.

So, what we do is find out the values of the function at these 4 points and compare them. So, let us find out the values f at minus 1 minus one the value was minus right at minus 1 the value is equal to minus x. So, the value is equal to one f at 0 is 0 f at 1 is 2 and f of 2 is equal to 0. So, these values are computed by looking at the expression of the function which is given to us. So, let us go back and look at the function. So, function was this. So, at the point minus 1 the value is one at the point 2 you put x is equal to twos in this. So, I compute 4 into 2 is 8, 2 into 2; 4 is 8 that value is equal to 0 in between point was 0 and 1 at the point one the value is minus 1 at the point one the value is 4 minus 2 that is 2 at the point 0 the value is 0. So, let us compare these values.

So, we found out these points and the values at these points for f at minus 1 is one f at 0 is 0 f at 1 is 2 and f of 2 is 0. So, among these 4 possible values the largest value is two. So, the function has absolute maxima, which is equal to 2 and the point of absolute maximum is x is equal to one and the function as absolute minimum that is 0 and the points are 0 and 2. So, at 2 of the points the function as absolute minimum and maximum at the point 1, this is how one analyze is absolute maximum and absolute minimum of a function. So, once again the strategy is first of all given a function a certain that it will have absolute maximum and minimum or and or minimum by looking at the domain of the function and analyzing the continuity of the function.

So, the theorem that a function defined on a closed boundary interval which is at the function being continuous will give absolute maxima or minima will ensure and then we go through the process of as we do it for local maxima and local minimum we look at the points could be endpoints could be the interior points where the function is not differentiable or the points where the function interior points where the function is differentiable and derivative is equal to 0. So, let us look at an example of finding absolute maxima minima as a in a problem in economics. So, consider a linear output.

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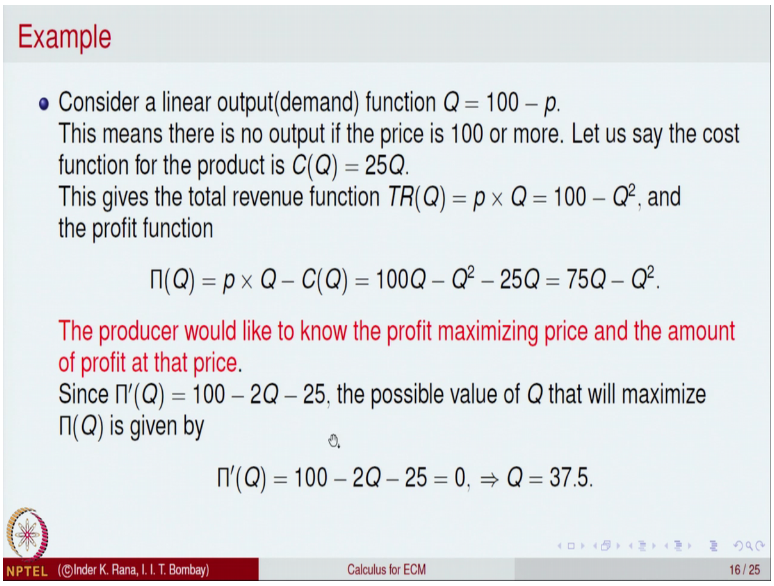

Model given by Q is equal to hundred minus p. So, there here p is output or you can think of it is a demand. So, this means at when p is bigger than hundred this quantity will be negative. So, this does not make sense at all so; that means, there is no product when the price exceeds hundred or more.

So, let us assume that the cost for Q is 25 Q. So, cost of producing Q units is 25 times Q. So, with this let us find the total revenue. So, as we know total revenue is the price in to the quantity produce p into Q. So, 100 minus p multiplied by Q, from here where to find the price of P? So, p is on the taken on that side. So, it is 100 minus Q. So, so the p into Q is equal to 100 minus Q square. So, total revenue will be this is the typo here, it should be 100 Q actually p into Q. So, p is equal to hundred minus Q. So, this is 100 Q minus Q square. So, the total profit is given by total revenue minus the total cost. So, p times Q minus c Q. So, that is the total revenue minus at the total cost. So, that is p is equal to as before 100 minus Q minus Q square and C Q is 25 Q. So, that gives us 100 Q minus 25 Q that gives 75 Q minus Q square.

So, that is the total profit function and to find out the possible points where it can have maxima or minima, first of all we find this is defined for all values of Q bigger than or equal to 0. So, let us analyze what is it is a differentiable function. So, what is the derivative? So, one would like to maximize. So, let us find out the derivative. So, the derivative is equal to 100 minus 2 Q. So, from here or it is 75 minus 2 Q. So, that gives us the value of Q equal to 37.5. So, at the when then number of when the Q the output is 37.5, then the profit will be either maximum or the minimum I we do not know one has to analyze what will be the nature of either apply the first derivative test or the second derivative test one can apply here the first derivative test if you like.

So, it is 75 minus 2 Q. So, second derivative is equal to minus 2. So, the function will have a maximum at the point Q equal to 37.5. Now the question comes is it absolute maximum or not of course, the function is continuous, but we do not know the domain is not a closed bounded interval. So, what we know is surely there is only one critical point. So, we can close the domain of the function and after any point Q 37.5. So, we can imagine our function because we know that there is only one critical point. So, there is only one possible value where the function can have a local maxima or minima or absolute. So, we can imagine the function mathematically to be defined the closed bounded interval say 0 to say 40, if you like and then say it is in the closed bounded interval 0 to 40.

So, it must have absolute maximum or minimum in that portion and says the second derivative at this point second derivative is everywhere is minus 2 minus two. So, this Q equal to 37.5 is a point of maximum value and is the only maximum value the function can have. So, the endpoints are 0 and if we look at one can think of saying at that at the point forty right Q is equal to 40; what is the only that is that the value to be considered for absolute maximum or not. So, one can look at the nature of the derivative see on the left of the derivative on the left side of this the function is derivative is positive. So, it will be increasing on the right side it will be decreasing. So, after 37.5 the function is going to decrease. So, no value bigger than 37.5 is going to occur.

So, we conclude from all this discussion that the second the function as absolute maximum local maximum and hence actually absolute maximum at the point 37.5.

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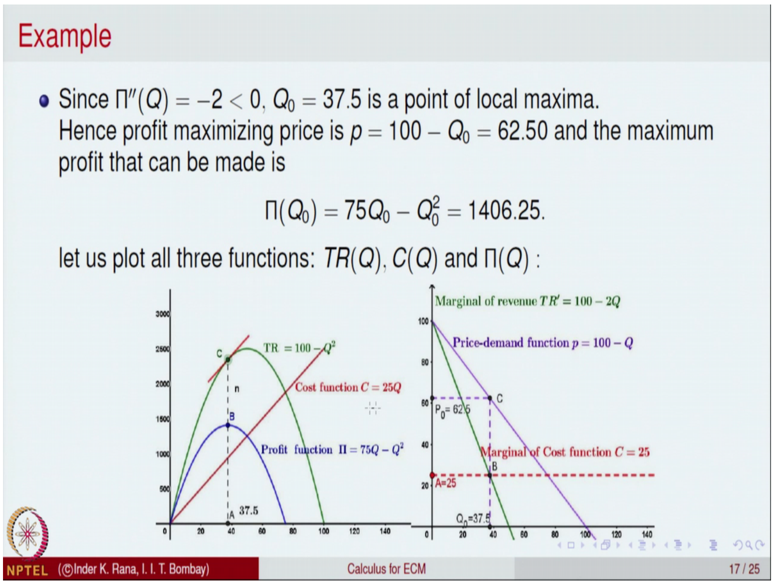

So, what is the profit price maximizing price for the company. So, that we put in the p is equal to 100 minus Q. So, Q is equal to 37.5 that gives you p equal to 62.5 as the maximizing profit right. So, and the maximizing. So, and the maximizing profit is the price at which maximum price the price at which maximum profit can be made is this and the maximum profit will be putting that value Q 0 in pi Q the profit function and that gives you this value. So, this is the graphical representation of all the information that we have done till now. So, look at this blue function that is the profit function 75 Q minus Q square this red line is the cost function c is equal to 25 Q that is a linear functions of graph is a straight line the profit function is quadratic.

So, is a graph is going to be like a parabola and Q the coefficient of Q square being negative it is going to be parabola like this and similarly it is the total revenue is again a quadratic with the negative sign of Q square. So, it is again going to be parabola right. So, this is and if you look at the maximum the quantity Q output where the maximum value occurs. So, maximum profit is at this point and the maximum revenue of course, will be at this point and now here is a observation that we should keep in look at on the in this side of the graph, this is the price demand function. So, there is a blue one this is a price demand function and there is a marginal of the revenue green one is a marginal of the revenue and this marginal cost is this red line that is a marginal cost.

So, that marginal cost meets the marginal revenue at this point b so; that means, the this is the point where the marginal cost will be equal to the marginal of the revenue and that actually is happening because that is a point 37.5. So, one can also see it in the scenario of the graph see this is the cost this is the cost function. So, if you want to say that the revenue at the point where the there is a maximum the marginal is equal to slope of the marginal is equal to slope of this line the total cost; that means, the tangent at this point will be parallel to the line marginal cost because that is linear.

So, that is also indicated here. So, this is how one can interpret graphically the data. So, we are verifying in this example that the marginal cost right marginal of the cost at the point of maximum output is same as the marginal of the revenue at that point. So, that is graphically illustration and we had already checked it by mathematical formulation of that.

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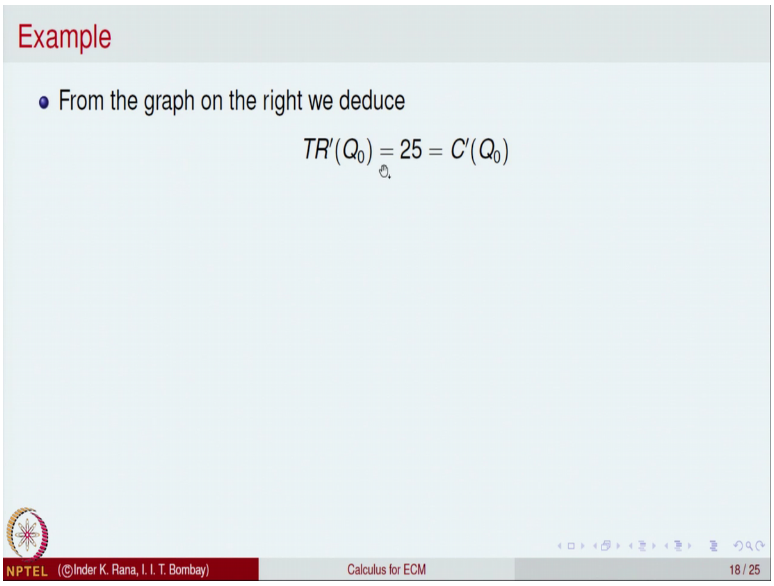

So, this is what we can check on the graph that the marginal of the revenue is same as the marginal of the cost. So, today what we have done in this lecture is try to look at the absolute maximum and absolute minimum how to find absolute maximum and absolute minimum and apply it to some of the examples in economics. So, we will continue this study next time.

Thank you.