Calculus for Economics, Commerce & Management
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Lecture – 28
Average and marginal product, marginal of revenue and cost, absolute maximum and minimum

Welcome back to the study of calculus and it is applications. When the previous lectures we had looked at the various concepts about the derivative of a function, and how derivative is applied in analyzing local maxima local minima. We looked at the various conditions which tell us how to check whether a critical point is a local maxima or local minima. We looked at the continuity test, then we looked at the first derivative test. And then we looked at the second derivative test for analyzing that a point of critical point for a function is a point of local maxima or minima. So, I will start with the last example that we had been looking at in the previous lecture.

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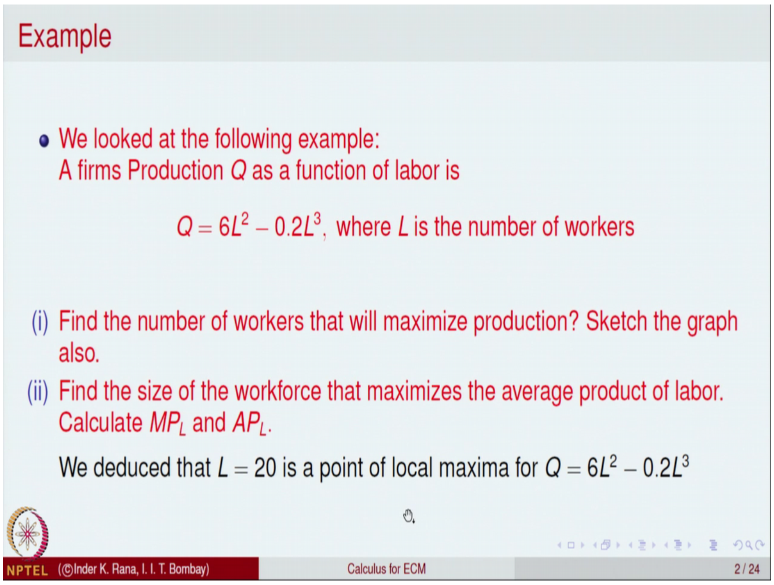

So, if you look at there are example namely, a firms production Q as a function of labor. So, the production is q, and depends on the labor the relationship is given by Q is equal to 6 L square minus point 2 L cube, where L is the number of workers. So, what as one is interested in find the number of workers that will maximize production, and sketch the graph. Find the size of the work force that maximize is the average product of labor and calculate the marginal product of labor and the average product of labor at that point. So, way did that first part, and we checked that at L equal to 20 is a point of local maxima for the function Q is equal to L square minus 2 L cube.

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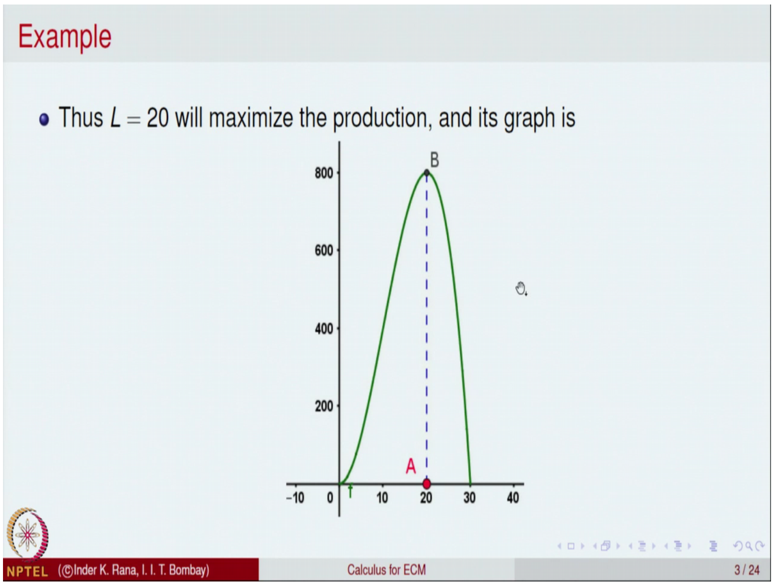

So, we are done that in the previous example, and this is the graph that we got for Q as a function of L.

So, at 20 there is a maximum value right.

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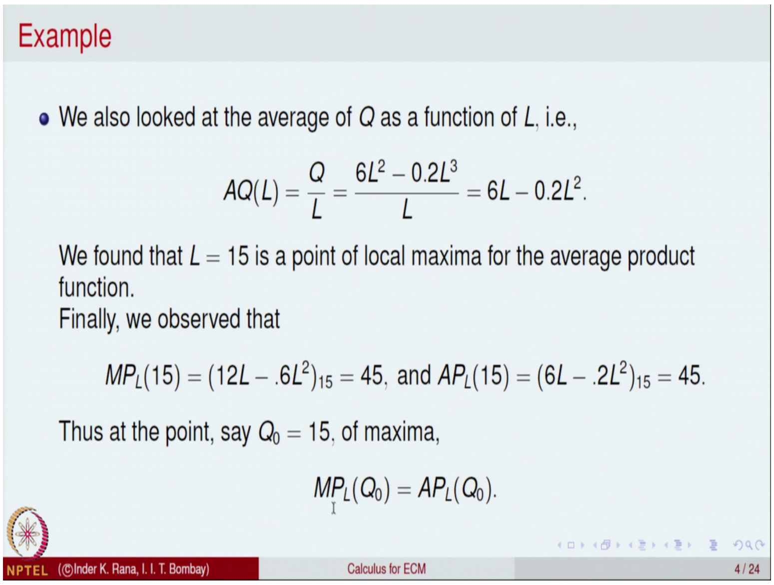

So, let us go a step further. And then we looked at what is called the average production on the labor. So, the average as a function of L of labor is production divided by L, total production divided by L. Q as a function of L is this divided by L that comes out to be 6 L square minus 0.2. L square L 6 L minus 0.2 L square. So, note this is a quadratic power is 2. So, this is example of a quadratic function. So, when we maximize this by looking at the derivative and putting it equal to 0, found the critical points and analyze them. We found that L equal to 15 is the point of local maxima for the average product function. So, this function L equal to 15 is the local maxima point. So, at L equal to 15, we can calculate what is the marginal of production.

So, marginal of production is the derivative of the production. So, that is 12 L minus point 6 L square evaluated at the value L equal to 15. And you simply why it comes out to be 45. And similarly, the average production that we had in the previous slide evaluated at L equal to 15, if you calculate that that also comes out to be 45. So, that says that at Q equal to 15 which is a point of maxima for the product function as a function of labor the MP L is the marginal of production as the function of labor at Q 0 is marginal of labor at Q 0.

So, these this equality is there for this function.

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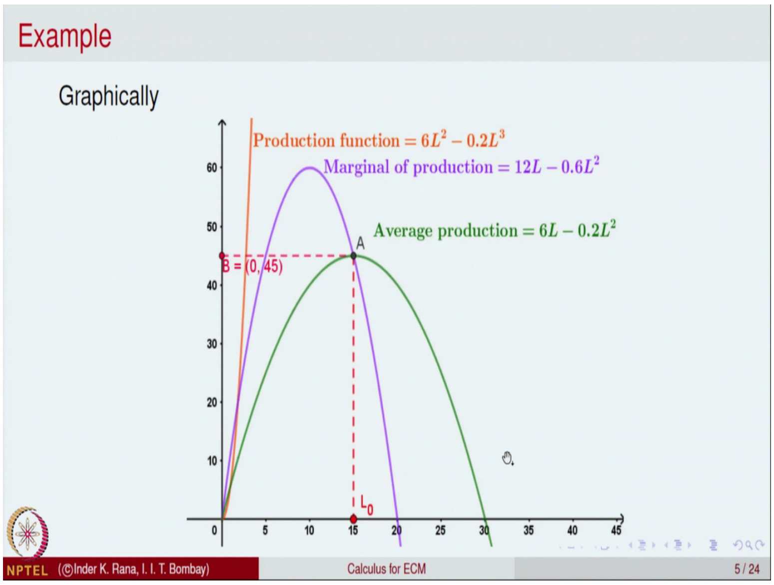

Graphically this can be shown as follows. So, this green graph is the average production that is 6 L minus point 2 L square. And this blue violet graph is a marginal of production that is 12 L minus 0.6 L square, and 6 L square minus point 2 L cube, that is a production function that is red that is not completely shown here. And all the graphs are bigger than 0, because number labor is always going to be positive, right.

As it at least going to be some labor there is no labor no workers no production will be there. And these 2 graphs namely the marginal of production and average of production, these 2 intersect at the point A. So, that is the point A where L 0 is 15, and the value is 45. So, for so, this is the point where both of them agree. So, at the and this is a point L 0 is the is a maximum point of maxima for the average.

So, at the average the point where the average production takes maximum value, the marginal production and the average production they are equal to each other. So, that is what graphically this is saying right.

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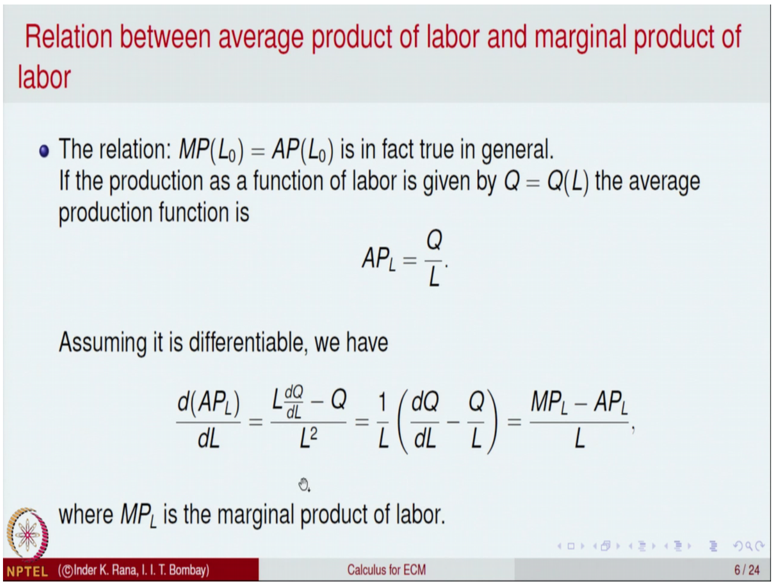

So, let us go a bit further. Let us find relation between the average product of labor and the marginal product of labor. So, the relation which we just now studied for the particular example that marginal production of labor is equal to average production of labor is A. In fact, equality which is true in general not necessarily only for this example. So, let us look at that scenario. So, let us look a production function production as a function of labor is Q is equal to Q L. So, production depends on the labor L. So, the average of this production will be Q by L.

So now let us find out the assuming it is differentiable we can differentiate this function is A. So, to differentiate this Q is function of L, L itself is in the denominator. So, to find the derivative of this we need to use the quotient rule. And of course, we are assuming L is bigger than 0, here because the labor is bigger than 0. So, what does the quotient rule give? Is the one function divided by the second function?

So, second function square second function as it is derivative of the first function minus the second function into derivative of the first function. So, L square L into dQ by dL. So, dQ by dL minus Q and derivative of L is 1. So, this is the derivative. So, if you take 1 over L out common what we get inside is dQ by dL minus Q by L. So, that gives dQ by dL is marginal of production and right and Q by L is the average. So, this MP L divided by MP L divided by L.

So, at the point of maximum, we know that an MP L is the marginal product of labor, this is equal to 0.

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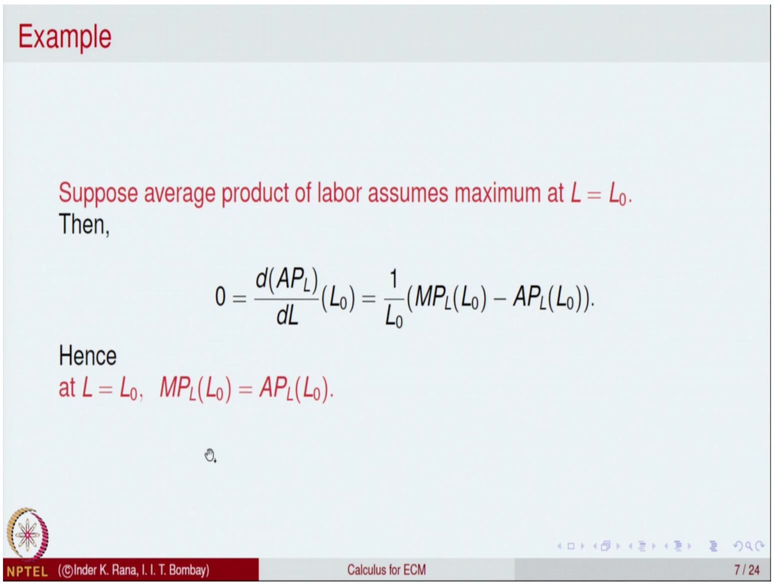

So, when there is a maximum there is necessary condition says L should be equal to L 0. So, in that case the 0 is equal to this. So, that gives you at L 0 MP L is equal to AP L at L 0. So, this gives us a way of computing ascertain that for any product function as a product of labor, if you look at the average production of labor, and then maximize it then the value at the maximum point L 0 is same as the marginal of the production at the point L equal to L 0. So, this is general formula, and we verified in the previous example.

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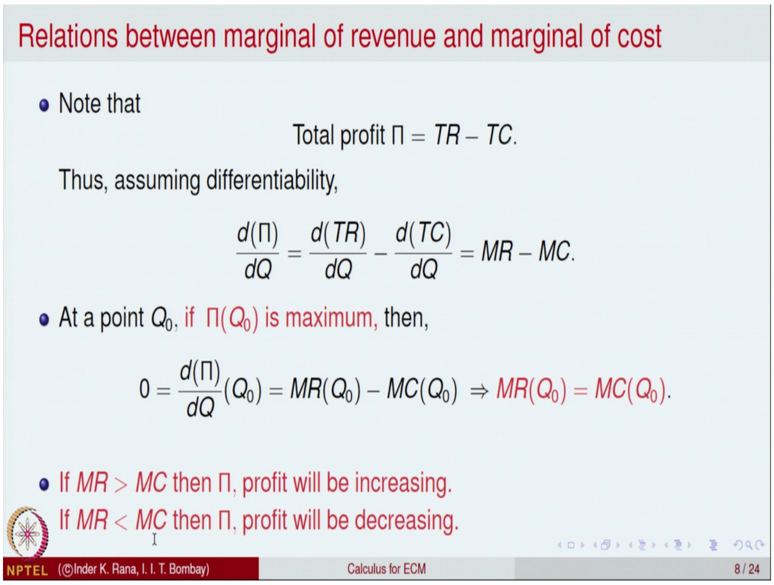

Let us do a similar analysis for the profit function. For total profit function pi is defined as the total revenue minus the total cost. So, if we so, assuming differentiability, if you assume everything is differentiable, then the marginal of marginal of the profit. So, that is a derivative, derivative for the profit will be equal to derivative of total revenue minus total derivative of the total cost.

So, that says the derivative d pi by d Q is derivative of TR with respect to Q minus the derivative of total consumption, total cost with respect to Q. So, this is where we are using the derivative formula the derivative of the sum and difference is the corresponding sum or difference of the derivative. Now derivative of TR is noted by MR.

So, that is a marginal of revenue, and derivative of total cost is denoted by m of c, that is a marginal of cost. So, and if Q is a point of maximum, let us call that is a Q 0. Then pi Q 0 is a maximum then by the necessary condition we know that dp d pi by dQ evaluated at the point Q 0 must be equal to 0. So, that gives us the relation that MR minus MC evaluated at Q 0 must be 0; that means, MR at Q 0 must be equal to MC at Q 0. So, this gives us the relation, that the marginal of revenue is always equal to marginal of the cost at the point of local maximum of the function that is the total revenue.

So, marginal of revenue is equal to marginal of cost. The cost at the point of maximum or the minimum. In fact, but here it will be only maximum because the production is there and profit is there. So, this gives us a consequence if MR is bigger than MC, see if MR so, this is from this relation if MR is bigger than MC in some interval, that will mean that d pi by dQ is positive. So, the derivative of for the profit function is strictly bigger than 0. So, by our result on calculus that implies is the function must be strictly increasing.

So, MR bigger than MC implies profit will be increasing. And similarly, if MR is less than MC, then the derivative is less than 0. So, the theorem on calculus will tell us that if the derivative of a function is strictly less than 0, then the function is decreasing. So, that says that if marginal of revenue is less than the marginal of the cost in some interval. So, these not at a point now, this is at a interval because we are applying it in a interval. So, if in a where Q is varying if in a interval marginal of revenue at every point in that interval is less than the marginal of the marginal cost at every point, then the profit must be decreasing.

So, these are properties about MR and MC in an interval. So, if MR is bigger than MC in an interval where Q o is varying then profit will be increasing. And similarly, if MR is less than MC then the profit will be decreasing. So, let us now look at another scenario where the demand function for a good is given by P plus Q equal to 30.

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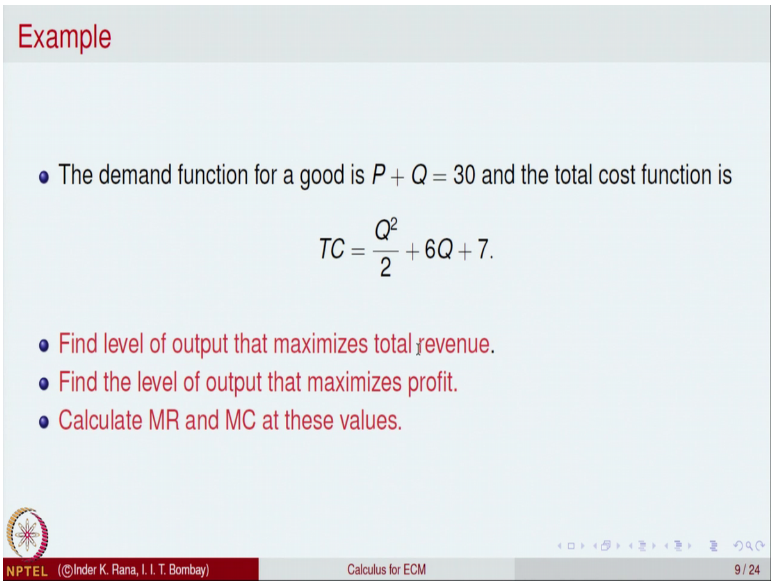

And the total cost function is given by T of c, that is a total cost function as a function of Q is Q square by 2 plus 6 Q plus 7. So, this relation P plus Q equal to 30, helps us to write P in terms of Q or Q in terms of P. As and when we require to be. So, total cost as a function of Q, the quantity produced is given by this. So, as a consequence of this, we want to find out the level of output that maximizes the total revenue. So, to maximizes a total revenue, first we have to calculate what is the total revenue function and then maximizes it. So, total cost is given right and relation between P and Q is given. So, this will help us to write what is the total revenue and the maximizes. And we also want to find out the level of output that maximizes the profit. And then look at calculate MR and MC at these value.

So, all these we want to do to. So, the let us try to solve the first one to write the revenue.

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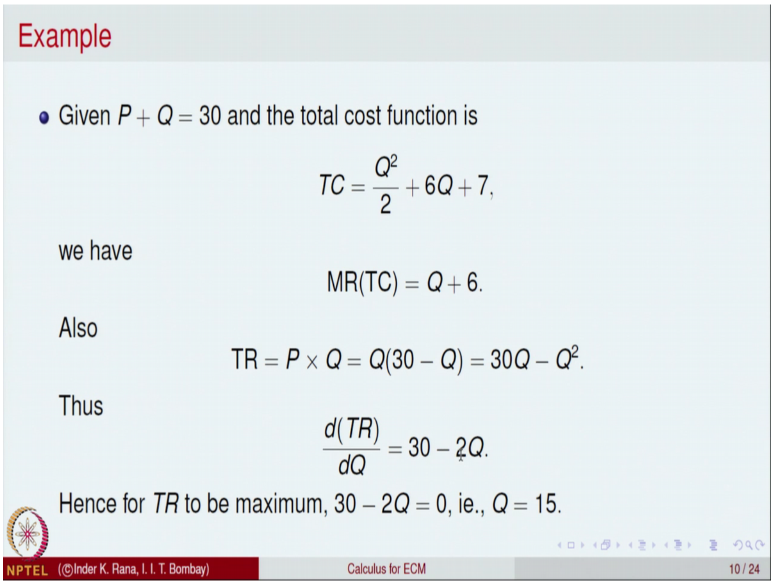

So, since the total cost is given by this and P equal plus Q is equal to 30. So, from here at least we can straight away get that marginal of the cost is differentiate this. So, that is the derivative of TC. So, 2 times Q by 2 so, 2 cancels. So, that is Q 6 Q gives the derivative 6. So, marginal of revenue is equal to Q plus 6 that comes directly from here by differentiation. Whereas, calculate the total revenue the total revenue is the quantities produce and the price. So, P into Q and the value of P is got from here 3, 30 minus Q. So, when you multi put this value and multiply total revenue depends on Q by the relation 30 Q minus Q square. So, we want to maximizes this total profit. So, what is the process for maximizing? The calculus says first find out the derivative of this. Put the value of the derivative equal to 0, and find out the points where this possibly can happen.

So, let us do that derivative of TR. So, 30 Q that gives you 30 and minus Q square gives you minus 2 Q.

So, I am using the theorem the derivative of the difference is the difference of the derivative. We find the derivative of the revenue is 30 minus 2 Q. And the point where possibly it can have a maximum value is Q is equal to 15 because 30 minus 2 Q equal to 0, gives you Q equal to 15. Can we say that Q equal to 15 is the is the point of maximum? Well, there are 2 ways of deciding that. One can decide by the first derivative itself look at 30 minus 2 Q right. So, on the left of it; that means, on the left of the point Q equal to 15, 2 Q will be less than 15. So, this will be positive. So, derivative is positive on the left. So, the function will be increasing on the left side, and it will be negative. So, it will be decreasing. So, the functions will be going up, and then going down on the left to the right.

So, Q equal to 15 by the first derivative tests itself tells us that Q equal to 15 is a point of local maximum. You can also apply the second derivative tests here. Because this function is differentiable the second derivative exists, and the second derivative for this is equal to minus 2. Because 30 will give you derivative 0, minus 2 Q will give you the value minus 2. So, the derivative at every point is negative. So, Q equal to 15 by the second derivative test must be a point of local the maximum.

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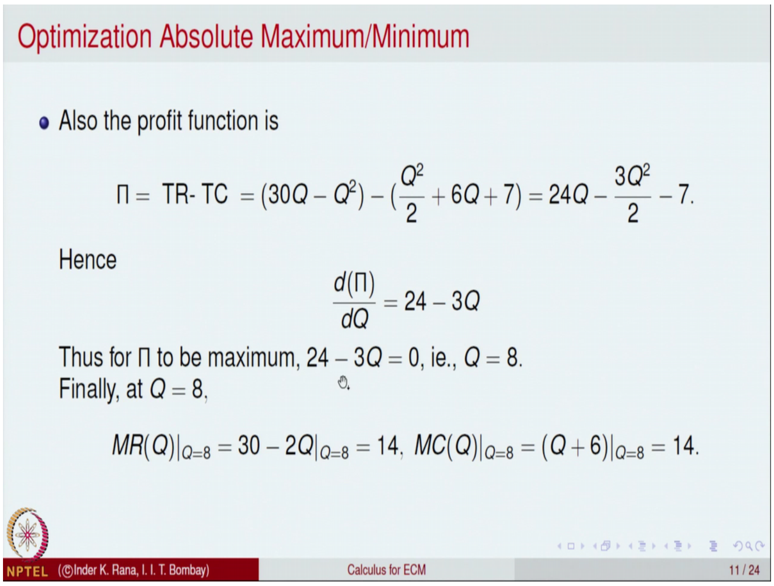

So, let us find out the total profit function for this, because we want to maximize the total profit function. So, the total profit is total revenue minus total cost, total revenue just now we have found this and minus the total cost. So, we put those values and simplify that comes out to be equal to 24 Q minus 3 Q square by 2 minus 7. So, to maximizes profit as a function of Q we look at the derivative.

So, derivative is 24 minus this 2 comes down that cancels it that is 3 Q. So, derivative of the profit function with respect to Q is 24 minus 3 Q. So, the property will be maximum when 24 implies 24 minus 3 Q must be equal to 0. So, if that is the case that is the necessary condition. So, that gives Q equal to 8. So, when Q is equal to 8, there will be maximum profit. But what is the price at Q equal to 8 that we have to look at the price and Q relation.

But anyway, we can also calculate the marginal of revenue at Q equal to 8. So, this is the marginal of revenue with that we found as 30 minus 2 Q in the previous slide. So, that is it Q equal to it is 14. And if we calculate the marginal of the cost at Q equal to 8 that also we have to found out was Q plus 6. So, where your rate 8 is equal to 14. So, marginal of revenue at the point of when the revenue total profit is maximum that is Q equal to 8 is equal to the marginal of the cost at Q equal to 8.

Now so, this was the profit maximize when Q is equal to 8, right. So, if you want to find out the profit at that point, we can use that relation P plus Q is equal to so, let us just go back and see what was the relation between P and Q. So, that we can find out the maximum for the profit also. So, we had P plus Q equal to 30, because maxim happens when Q is equal to 8. So, when you take it that is why into P is equal to 22. So, that gives you the maximum profit. So, this is how calculus is apply to the various scenarios in economics commerce and management problems. Let us now look at what is called the absolute maximum and absolute minimum function.

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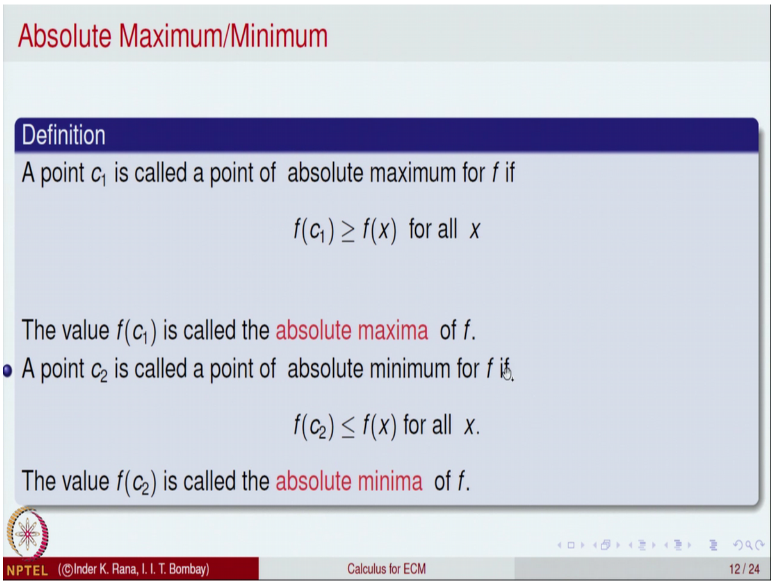

So, till now mainly we have been looking at what are called the local maximum and minimum; that means, in a neighborhood of that point, the value is the maximum value is the largest value or in a neighborhood of that point the value is the smallest value. We would like to know given a function, in all it is domain we want to know what is the largest value the function can take.

So, we want to know what is the largest value the function you can take it is domain. So, let us define a point cone is called a point of absolute maximum for a function, whatever it is domain maybe. So, if f of c 1 the value at the point c 1 is bigger than f of x for all x in the domain of the function, whatever be the domain of the function, right. And similarly, and the value f of c 1 is called the absolute maxima of the function. So, the point is called the point of absolute maximum. And the value that the function takes the largest value of the function is called the absolute maxima or the function. Similarly, we can defined what is called the absolute minimum of the function in the domain of the function to be the a point c 2 is called the point of absolute minimum, if f of c 2 is less than or equal to f of x for all x in the domain of the function.

So, this value above c 2, this value of the function at that point of absolute minimum is called the absolute minima of the function at that point right. So, is to quite clear that absolute maxima of the function is the value which is the largest in the domain of the function. And at the point where it takes that value is called the absolute point of absolute maximum. And similarly point of absolute minimum, and the absolute minima of the function. So, the problem we want to discuss is given a function. How do you locate the possible points where the absolute maximum or the minimum can occur? And how to decide what is absolute maxima and what is the absolute minima of the function.

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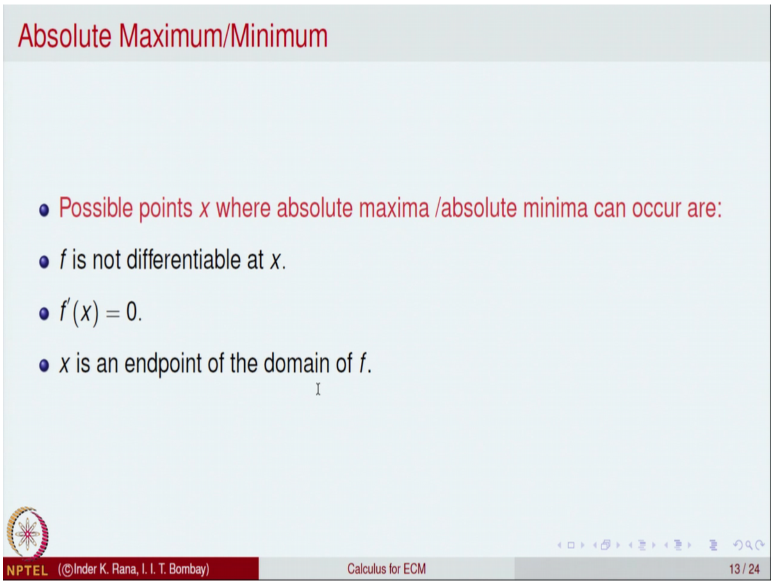

So, for that let us observe that if a point x is absolute maxima or absolute minima, then it is also a local maxima or minima, right. So, that says that the possible points where which we should be looking at which you should be analyzing to locate the absolute maxima and minima are the points where local maxima minima can occur. Collect this points find out the values of the function at this points and see which is the largest which is the smallest, right.

So, let us look at this values. So, this is the first are the points in the domain where the function is not differentiable, these are possibly some points where the function is not differentiable, or where the first derivative is equal to 0, where the function is differentiable at interior points and the derivative is equal to 0. Or the points endpoints and the domain of the function could be some interval and the endpoints of those intervals are the possible points. So, these are the 3 set up points which we analyzed looked at for local maxima minima also. So, those are the same set up points, where we will where possibly the function can have absolute maximum or minimum. Because the reason being absolute maximum is also a local maximum. But one has to ensure that this absolute maximum or minimum exists. So, a theorem that way the maxima minima theorem that we stated when we looked at the continuity of the functions in a domain said that if f is a function defined on a closed bounded interval. And if this function is continuous, then it is bounded and attains is maxima and minima.

So, that theorem is very useful in ascertaining that a function has a local has a absolute maxima or a minima in the domain. So, mostly we will look at the domains which are closed bounded, intervals and the functions which are continuous. So, ones that is ensured you look at the points this setup points where f is differentiable, f is not differentiable, derivative exists and is equal to 0 or the set-up points which are endpoints. Compute the values of the function at this points, and compare and see what is the maximum or what is the minimum value of the function. So, we will continue our lecture in this discussion of absolute maxima minima in the next lecture.

Thank you.