Calculus for Economics, Commerce & Management
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Lecture – 27
Successive differentiation, second derivative test

Welcome back to the lecture. We had been looking at in the previous lecturers in notion of successive differentiation, derivative of the derivative function. So, we had define the concept of end times differentiability of function. If it is n th derivative n minus 1 th derivative exist in a neighborhood of c. So, derivative of that n minus 1th derivative at the point c is the denoted by fnc. So, that is the n th derivative of function.

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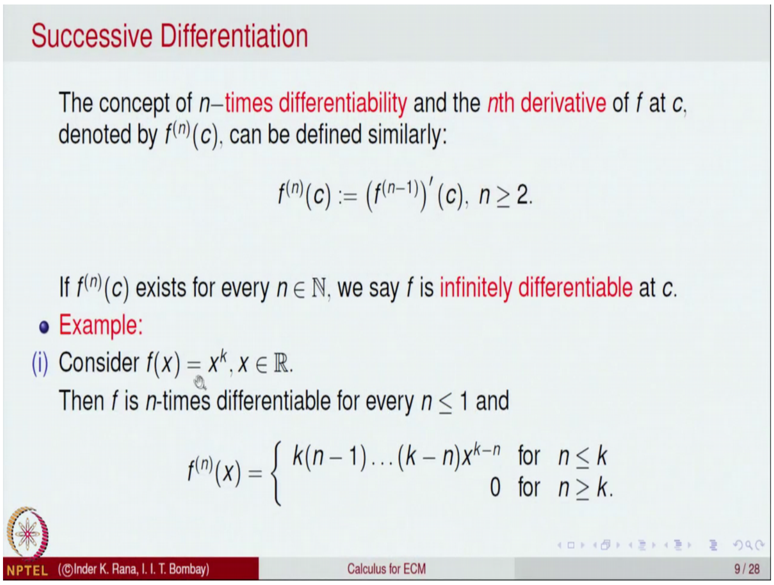

Let us look at the example of f of x is equal to x to the power k for x belonging to R. So, this is our x to the power k where k is a positive integer for example, right. So, this we know that this function is differentiable at every point. So, what will be the first derivative f dash of x will be equal to k times x to the power k minus 1. And when you differentiate again what you will get?

So, that function is differentiable and the second derivative will be equal to f double dash of x will be equal to k into k minus 1 x raise to power k minus 2. So, let me just write this. So, that it becomes clear. So, let us look at the function f of x equal to x to the power k, where k is a number say bigger than or equal to 1, that is fixed.

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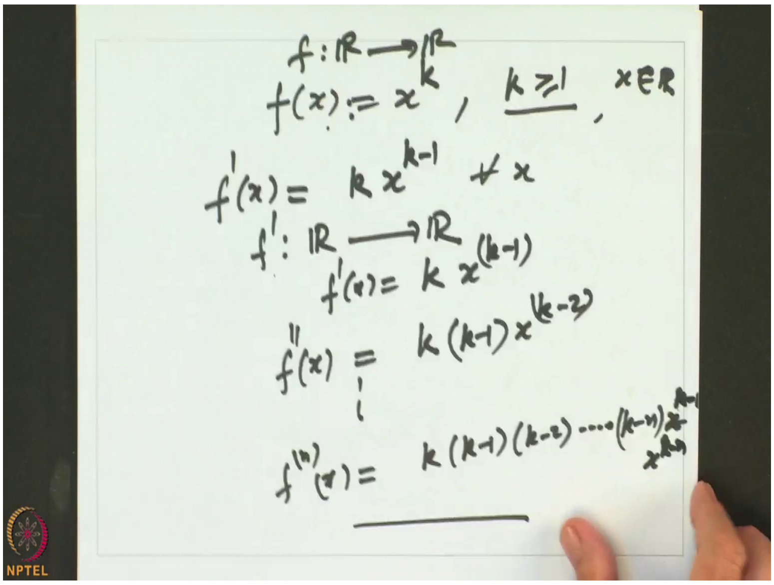

And here x is bigger than x, x is a (Refer Time: 01:47) number. So, we know that from our examples of differentiation f dash of x is equal to k times x to the power k minus 1 for every x. So, this function f dash is a function now again defined from R to R. F was a function from R to R given by f of x is equal to x to the power k. F dash because the function is differentiable everywhere f dash is again a function and f dash of x is equal to k times x to the power k minus 1. Now once again this will be differentiable f k this is bigger than k is bigger than or equal to 1, say for example, this is again difference differentiable and what will be the derivative of this function it will be k into k minus 1 x to the power k minus 2 and so on.

So, this will have derivatives of every order till you can reach f n of x that will exists. So, k into k minus 1 k minus 2 and so on upto k minus n, if you are looking at the n th derivative x raise to power k my k minus n n minus x to the power is k. So, k minus so, this will be x to the power kk x to the power k minus n. So, that will be the n th derivative. So, the derivative so, this is functional have derivative. So, of all order. And in fact, 2 more also when x is equal to k then this will be 0. So, so this is example of a function which has successive derivatives.

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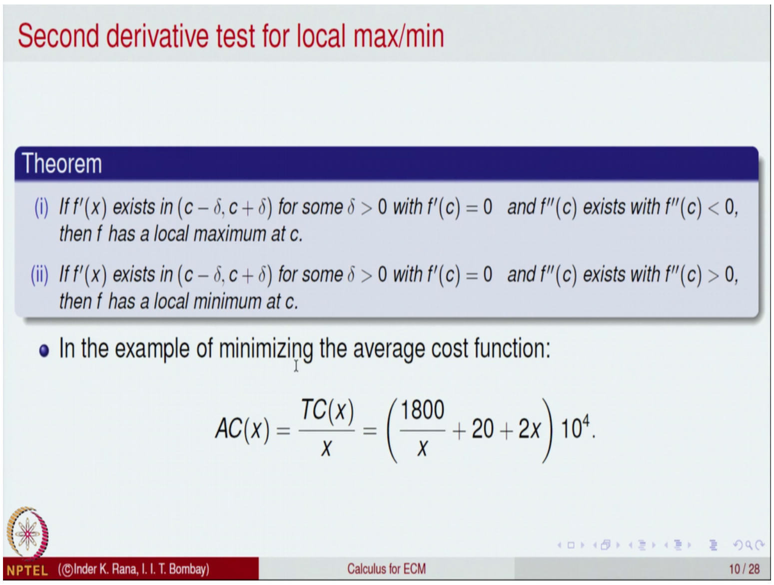

So, in terms of the second order derivative, the conditions says the following. So, it says if f dash of x exists for a function define in a interval, if the second derivative exists in an interval c minus delta 2 c minus delta.

This is the first derivative exists. So, once the first derivative exist in the neighborhood we can ask for second derivative will exists. So, the condition is that the second derivative of the first derivative of the function is 0 at that point, because to analyze local maxima or minima, the necessary condition is any way that f dash of derivative at that point must be equal to 0. So, the first we have to ensure that so, the condition is that the first derivative exists in a neighborhood of the point c. And at the point c the first derivative is 0. In further if the second derivative at the point is less than 0, then f as a local maximum at the point c. So, the condition is in terms of the second derivative, but to define the second derivative at a point we have to define say can something about the first derivative at that point. So, we are saying first derivative exists, in a neighborhood of the point c first derivative is equal to 0 at the point c, if this is satisfied and in addition the second derivative at the point c is less than 0, then if all these conditions are satisfied then f as a local maximum at the point c.

So, similar condition holds for the local minimum, the first derivative exists in a interval around the point c, the first derivative is 0 at the point c, and the second derivative is bigger than 0 at the point c, then the function has a local minimum at the point c. So, this is what is called the second derivative tests, this gives you sufficient conditions to ensure that a function has got local maximum or a local minimum at a point, in terms of the second derivative. So, the first derivative should be equal to 0, and the second derivative at that point should be less than 0, this 2 will ensure that the function as a local maximum at the point c. And similarly, if first derivative is 0 at the point and the second derivative exists and is bigger than 0 strictly bigger than 0 then the function as a local minimum at that point. So, this is what is called the second derivative test. We will not be giving proofs of this, but will use these conditions, the sufficient conditions to analyze local maximum and local minimum of a function at a point.

So, let us look at one example. Recall we had looked in our previous example, while we are trying to analyze minimizing the average cost function of a product. The average cost average cost function is the total cost function divided by the output x. So, is a total cost function and this is the x items being produce. So, this is the average cost of the function. And that in that example came out to be equal to this. So, to analyze now whether it is minimizing or maximizing the average cost to analyze that we looked at the derivative of this and we found that the derivative equal to 0 gives us the value x is equal to 30.

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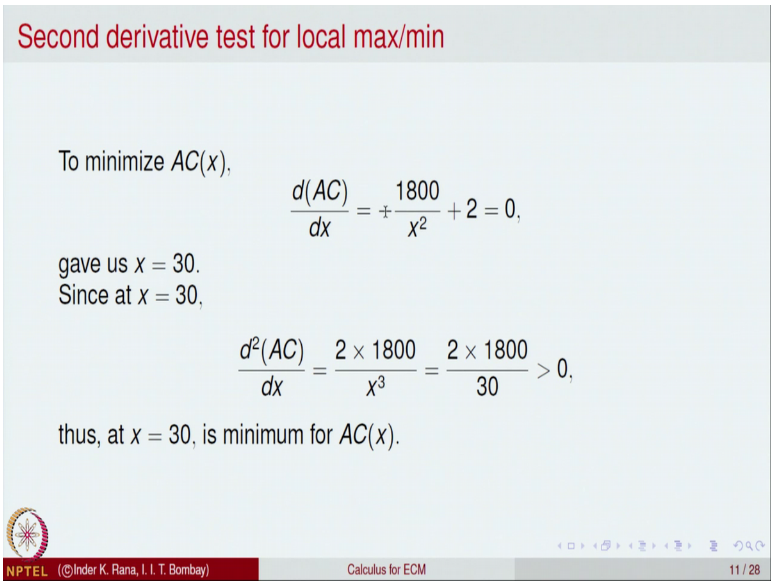

So, at x is equal to 30, the possibility of a local maxima or minima. Now to analyze we can analyze the property of the derivative first derivative itself that is what we have done in the previous example it to analyze whether x is equal to 30 is a point of local maxima or a minima.

But we can analyze because this function is differentiable second derivative exists. So, second derivative for this function at x is equal to 30 turns out to be bigger than 0. So, second derivative for this is minus minus. So, that gives you plus. So, 2 x power goes minus 1. So, that is now x is a minus 3, right. And when you put the value equal to 30, it turns out to be a positive quantity. So, the second derivative of the function average cost function at x is equal to 30 is bigger than 0. So, it must be a point of local minimum for the average cost function. And that is what we had established using only the first derivative test in our previous lectures.

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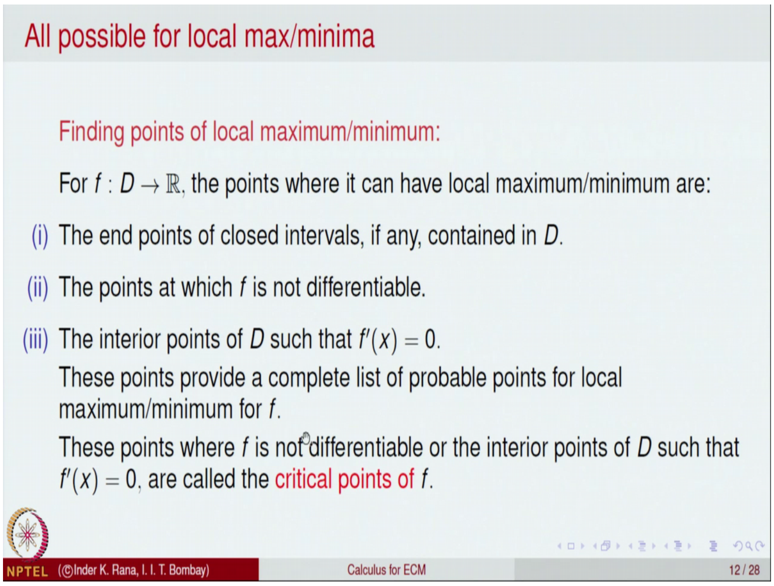

So, let us look at some examples to analyze this points of local maximum minima. So, what is the strategy for analyzing whether a function has got a local maxima minima at some points. First of all we have to locate those points where possibly the function can have a local maxima or minima. So, the points are as follows. First f is a function let us say defined in a domain D. So, it can a local maxima minima at the following points.

The possible points are one end points, in case there are any closed intervals in D. So, end points of those closed interval if there any. Secondly, the next set up points will be the points at which f is not differentiable. There are there maybe points in the domain of the function, where the function is not differentiable. And the function may be having a local maxima or minima at that those points. We are seen in some example in the when we define the notion of local maxima and minima. And the third possibility of this are the interior points, but in interior point our necessary condition what should be derivative should be equal to 0. So, that gives us that if at an interior point of D as a function has to have a local maxima or minima, and the derivative exists at that point then derivative must be equal to 0. So, this is the 3 setup points which need to analyzed for a function to have local maxima or minima. What we are saying is f is defined in a domain d, and if it has to have a local maxima or minima then these are the only possibilities for the function.

So, these points one is the end points of intervals if there are any in the domain D. Second is the points where the function is not differentiable, end points or interior does not matter the points where the function is not differentiable. And the third is the interior points where the function is differentiable and the derivative is equal to 0. So, this points are normally called critical points for the function. So, this gives a complete list of possible points for and local maxima or minima of the function f.

So, this points are called the critical points for the function. Normally some books call the only third as a critical points, and all this point has totally stationary points. So, these are possible candidates for the function to have local maxima or minima. So, strategy for analyzing local maxima minima would be for a function, first of all look at the function look at the domain of the function it may have some end points. So, collate together those endpoints, and look at the points where the function possibly is not differentiable.

So, that is a second bag of points, second collection of points and the third is the integer points you know there derivative exists and the function is equal to 0. So, compute the derivative and put it equal to 0, and solve it that will give you the possible points for the function to have local maxima or minima. So, one will analyze at this free points. So, these are the possible candidates where the function can have local maxima or minima. And we will apply the either the continuity test, or the first derivative test or the second derivative test to analyze weather that point is a point of local maxima or local minima. So, let us look at and an example to analyze this.

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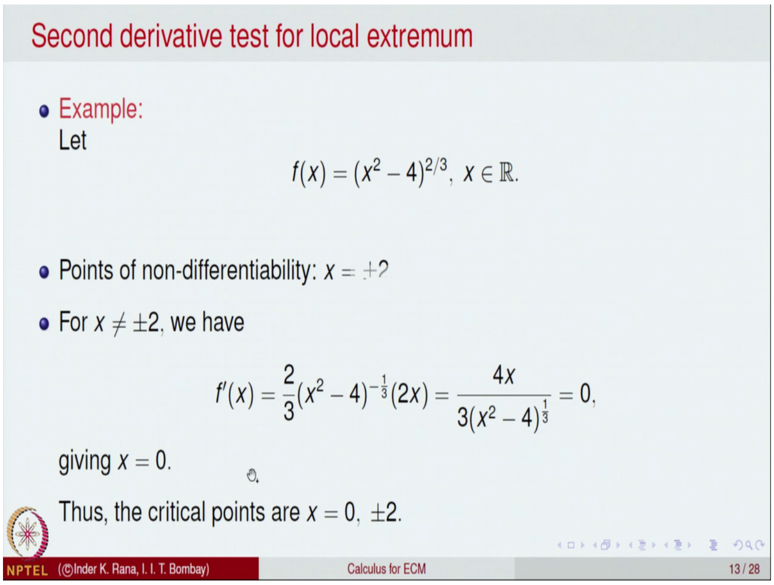

So, let us look at this example f of x is equal to x square minus 4 raise to power 2 by 3, for x belonging to real line. So, this is a function which is defined all points x belonging to the real line.

Now to analyze so, first thing is to look at domain is the whole of real line. So, there are no points where endpoints intervals and such thing. So, first category of points are not valid for this not required. The second is look at the points where the function is not differentiable are there any points where the function is not differentiable. So, this function point of not differentiability are x is equal to plus minus 2. Because when you take x is positive, then x is x square minus 4 is bigger than 0, than the function will behave differently than when it is less than 2.

So, at see x square equal to 4 could be equal to 0 is a values equal to plus minus 2. So, those are the points where the function x square minus 4 will change it is nature. So, accordingly this left derivative of this at this points 2 and minus 2 will change will not be equal. So, this function is not differentiable at the points x is equal to plus 2 and minus 2. If you find it difficult at present to analyze this I would says, that sit down and try to analyze the differentiability of this function at this points plus 2 and minus 2.

Why this is not differentiable? So, try to compute the left derivative try to computer the right derivative at plus 2 as well as at minus 2, and you will see both this points the left derivatives are not equal to the right derivatives. So, for x not equal to plus minus 2, the function is defined. So, let us compute the derivative of this function for the points when x is not equal to plus 2 or minus 2. In that case we can apply our usual formulas of differentiation. So, the derivative is given by the power comes down 2 by 3 x square minus 4 2 by 3 minus 1 into by chain rule derivative 2 x comes.

So, this is the using the power function the differentiation, and the chain rule combine together gives you this derivative. So, the derivative is equal to 4 x divided by 3 times x square minus 4 raise to power 1 by 2 that is equal to 0. So, this gives you at the points which are not equal to plus minus 2; that means, I am looking at the interval minus infinity to 2 in the open interval minus 2 to 2 and in the interval 2 to infinity. The only one possible candidate appears to be for candidate for to be local maxima or minima and that point is x is equal to 0.

So, we got 3 candidates to be analyzed for local maxima minima. One is 2 of them are plus minus 2 another one is 0. So, we get 3 candidates for critical points 0 and plus minus 2. So, let us analyze these points one by one.

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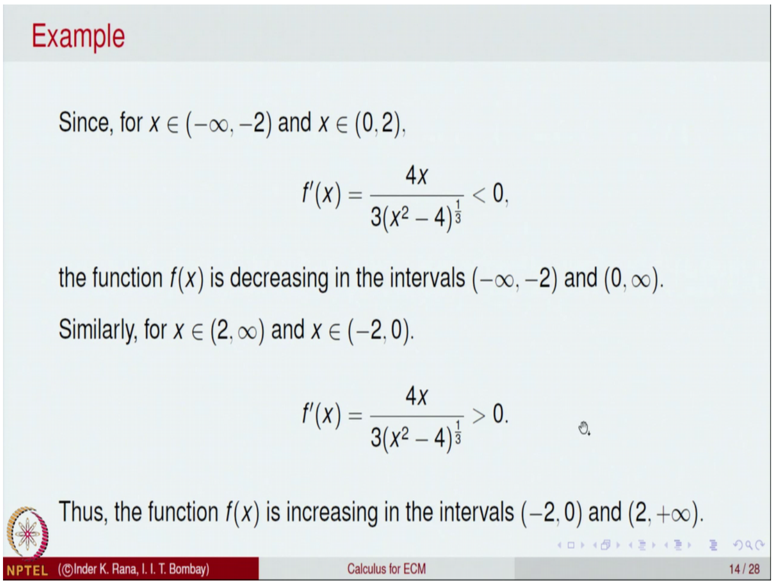

So, in the interval if I look at x belonging to the interval minus infinity to 2, and in the interval 0 to 2. Then f dash of x right if x is here is in this if x is here, than it is negative is fairly will positive. So, this quantity will be negative quantity. So, f dash of x will be negative for x in this portion. And if x is in the portion 0 to 2, then x is less than 2. So, this value again will be a negative numerator is positive, but this value will be negative.

So, f dash of x is less than 0. So, the in the these 2 intervals f x lies in these 2 intervals, then derivative is less than 0. So, that means what? That means, in this interval the function is decreasing and in this interval the function is decreasing.

So, let us right that observation. That the function is decreasing in the interval minus infinity to minus 2, and 0 to 0 to, it should be 0 to 2. And if you look at so, there is a 0 to 2. And if we look at the interval 2 to infinity and minus 2 to 0, then the derivative the sign of this is positive. So, as a result of this we get that the function must be increasing in this portion as well as in this portion. So, these 2 properties can be put together that the function is increasing in this interval minus 2 to. So, the only portion we have not looked at is minus 2 to 2. So, let us look at in that portion.

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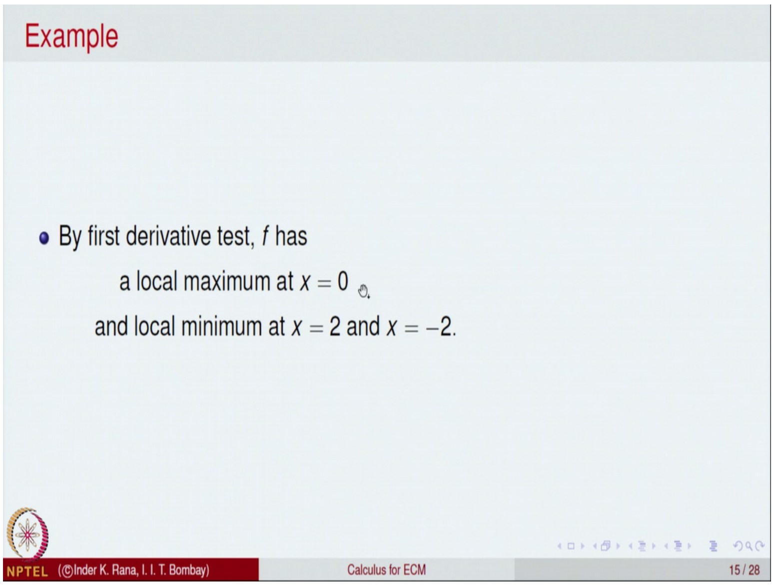

So, the first derivative test applies and we have a local maximum at this point x is equal to 0. And the local minimum at the point x is equal to 2 and x is equal to minus 2.

So, by looking at the first derivative test by the nature of increasing and decreasing, we have just concluded that the function has a local maximum at the point 0, and local minimum at the point 2 and minus 2 both.

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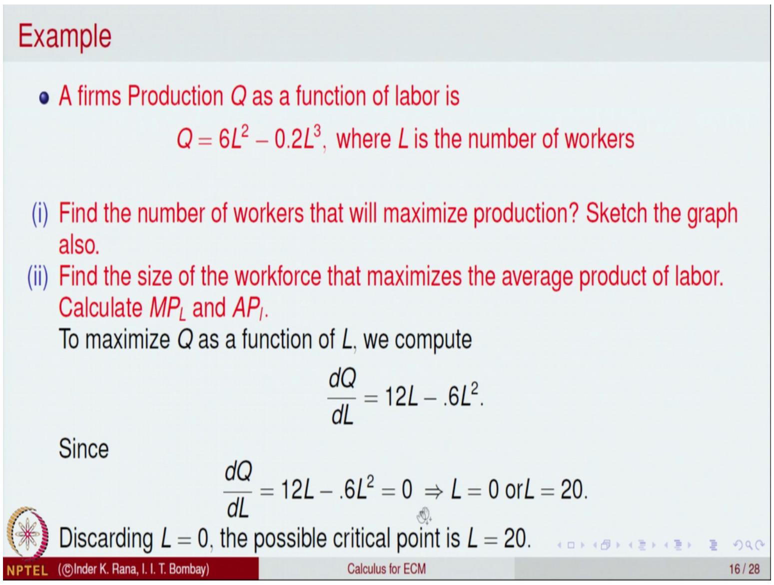

Let us apply these ideas to an example in economics. So, the it is we are given that a firms production Q as a function of the labor is given by Q is equal to 6 L square minus 0.2 L cube, where L is the number of workers and Q is the production. The production depends on how many how much labor is employed in that firm. So, what would like to know is find the number of workers that will maximize the production. So, that is one question. And you want to get sketch the graph also of that. So, find out the number of workers that will maximize the production, and the production is given by dependent on the production is dependent on the labor and this is the order.

So, find the size of the work force how much labor is there for. So, that maximizes the average product of labor. So, you want to know the value of the labor that will give you maximize maximum average product or the labor. And at that point we also want to calculate the marginal product of labor, and the average product of labor at those points. So, that is what you would like to do. So, let us compute that to maximize Q as a function of L the first step is we should analyze is L is the number of workers. So, obviously, L is going to be strictly bigger than 0 right. So now, it is going to be L is going to be integral values. So, will for that sake of mathematics will assume we make a assumption that L is a number which is a positive real number. So, Q as a function of I is given by this, and mathematical problem is to maximize this.

So, for that we will look at the derivative of Q. So, 6 L square the first term the derivative is 2 into 6 12 L minus 0.2, 3 comes down. So, 3 time multiplied by 0.2 into L square. So, that is 0.6. So, derivative dQ by dL is equal to 12 L minus 2 0.6 or 0 0.6 L L square. So, to find out the possible candidates for such that this Q is maximum we have to put the derivative equal to 0. So, find out there is critical points. So, when you put it equal to 0; that means, from here either L is equal to 0, because I will come out you can take a factor L into 12 minus 0.6 L. That gives 2 values L equal to 0 or L equal to 20; obviously, there is if L is equal to 0 there is no production and we are trying to maximize. So, dis we discard the value L equal to 0. So, the only possible critical point is L is equal to 20.

So, at L equal to 20 we would like to know whether this Q is maximum or minimum. So, we can look at dQ by dL. So, that is given by this.

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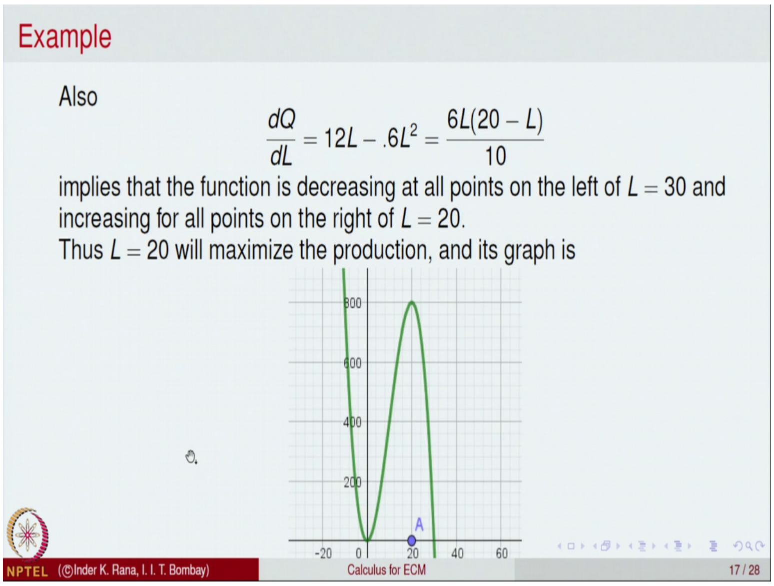

So, we can analyze the nature of the derivative for L on the bigger than 20. So, this quantity when L is bigger than 20 this will be negative. So, this will be decreasing for all points all points on the right of right and increasing. So, this is a written wrongly here. So, when L is less than 20, this value when is less than 20 this value will be positive. So, L is always positive. So, this product is positive. So, on the left side it is positive. So, function is increasing on the left side, and decreasing on the right side when L is bigger than 20 this will become negative. So, decreasing so, this is opposite written this is a typo here the function is increasing, at all points on the left and decreasing at all points on the right of 20.

So, this is the that means, L equal to 20 is a point of maximum. But if you even if you want to analyze at 0, 0 will be point of local minimum if you are negative values of L are allowed. But this is a graph of the function when for L bigger than 0, you can see at the point 20. There is a maximum value. So, this part of the graph is just for the sake of mathematics at negative. You will and this is also interesting to observe that our function, Q as a function of L is a cubic power is 3. So, possible there are possibilities of 2 values graph cutting 2 value. In fact, 3 values so, it has 2 consistent routes L equal to 0, and one other route at some other point. But for another point of view of the problem of economics at L equal to 12 20, when the labors size is 20 there will be maximum production.

So, the production we can see from here the production is increasing, right as the labor increases and it reaches 20 there is a maximum production. And then the graphs drops down starts dropping down; that means, when the labor increase is beyond 20, the profit will sorry the production will start decreasing. So, maximum happens at the point L equal to 20. So, this is how you can interpret the graph also.

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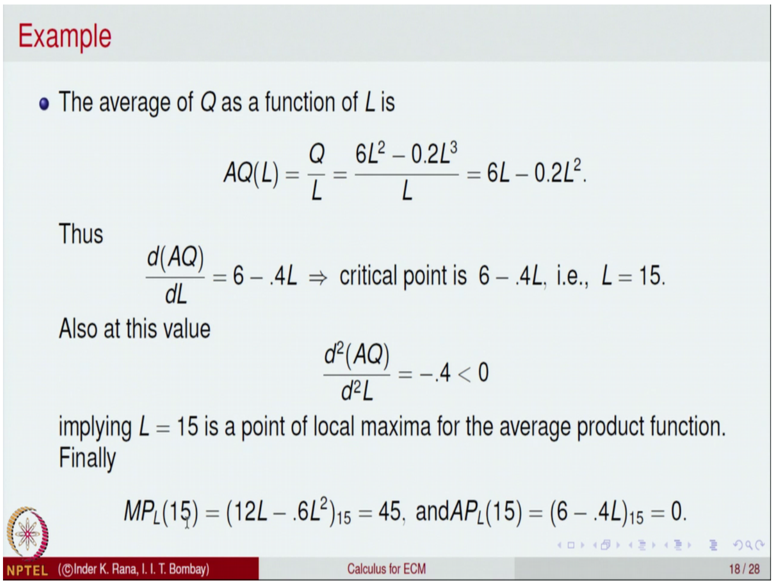

The average of Q is a function of L will be Q by L, right. The average is total divided by that quantity. So, when you divide you get the function 6 L. So, L cancels here. So, 6 L minus 0.20 decimal 2 L square the one value will cancel here. So now, it becomes a quadratic equation is a equation of degree 2. So, when we want to analyze, well what will be the point where it could have a maximum or minimum it is differentiable everywhere we find the derivative.

So, the will basics minus 2 multiplied by point 2 that is 0.4 L. So, the critical point will be when this quantity is equal to 0, this equal to 0 point is this here it is 0 is not written there. So, L equal to 0 that is L equal to 15. So, that gives you the value L equal to 15, then the labor is 15 the average production will be average production of as a function of labor will be will be maximum for L equal to 15. Also, at this value why it will be maximum because at second derivative for this if we will analyze it will be second derivative for this it will be 6 first derivative is 6 minus 0.4 L.

So, when I differentiate this again, this will give you 0, and 0.4 L will give you 0.4. So, the second derivative of average of production with respect to the labor is minus 0.4 which is negative. So, that means, L equal to 15 is a point of local maximum for the average production product function. So, we have found maximizing the product for a labor and the average maximizing for both of them, right.

So now, let us compute the marginal of marginal of production for L, labor when labor is 15. So, you recall the derivative for the production was 12 L minus 0.6 L square that was a derivative. So, that is a marginal we have to evaluate it 15. So, put the value 15 and you get it 45. And similarly, the average of L at 15. So, this is this value this average is 6 minus 0.4 L. So, that evaluated at 15 when you compute that comes out to be 0. So, if you at the point 15, the average cost of production the marginal is 0. Whereas, of the average of the production is equal to 45. So, that is how you calculate the marginal, and we have already seen what does the marginal indicate. So, you can calculate the marginals and apply them an a marginal analysis that we have done earlier can be applied.

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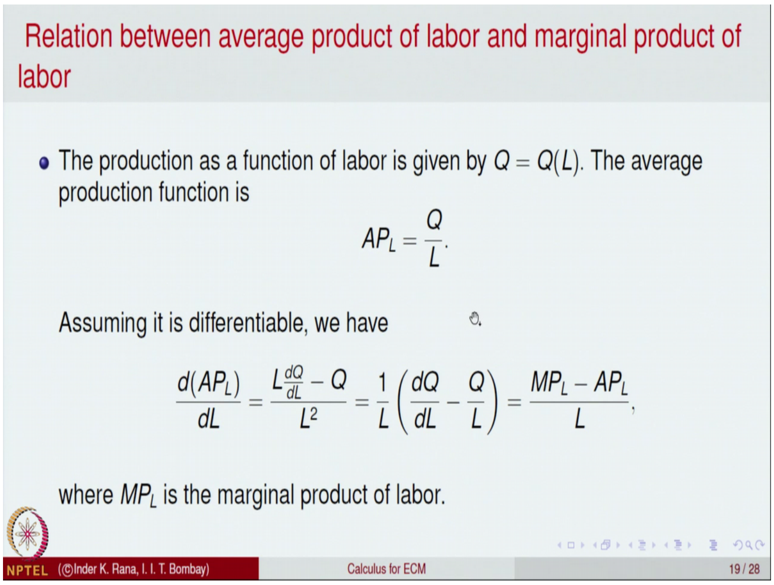

One can also relate this things. So, between average product of labor and marginal product of labor. So, let us look at a production of a function of labor given by Q as a production of L; that is what we had in the previous example. Also, so, the average production will be AP L Q by L. So, in this we are not looking at a particular examples, we are looking at a general function and trying to find a relation between them. So, if we analyze this assuming this all everything is differentiable the derivative of the average production as a function of L will be equal to by the quotient rule, it will be L square L into derivative of Q by L minus Q into derivative of L. So, once you simplify that. So, this is 1 over L dQ by dL minus Q by L. Now dQ by dL is a marginal of production right. So, that is MP L minus AP L divided by L, right.

So, this denominator L is Q by L that is average. So, this is the average the average production is given by this. So, that is AP of L the average production of L. So, let us look at this a bit further where MP L is the marginal of the product of labor that is what we said dQ by dL.

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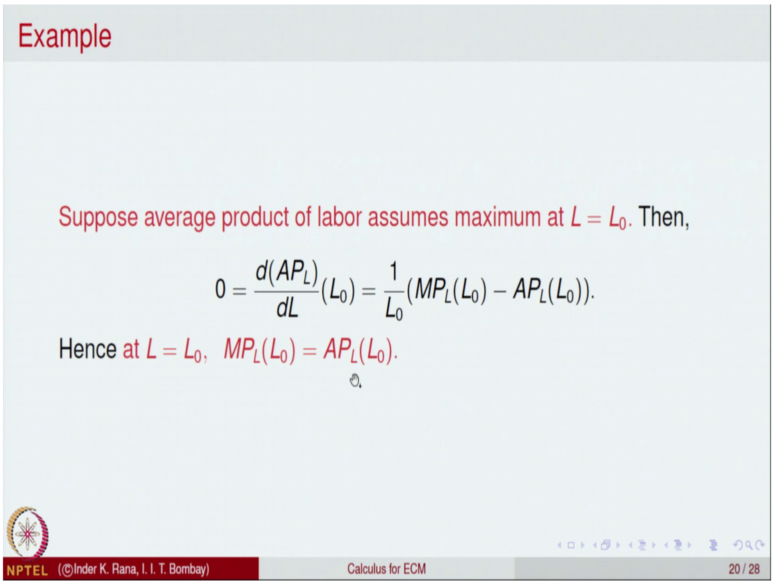

So, once that is obtained, suppose the average product of labor assumes the maximum value at L naught. So, let us assume whatever was the relation, right. And the maximum value is obtained at a value L equal to L naught. In that case report that value; that means, 0 is equal to derivative of AP L divided by derivative of AP L with respect to L at the point L 0, right because it is maximum. So, by our theorem if there is a maximum at a point then derivative must be equal to 0, we get this is equal to 0. And then just now we computed it is equal to this, that average the derivative of AP L with respect to dL was 1 over L naught MP L L naught and AP L at naught; that means, at L equal to L naught the marginal production at L naught is equal to the average production at L naught.

So, these 2 are equal. So, that is the general formula now that one gets.

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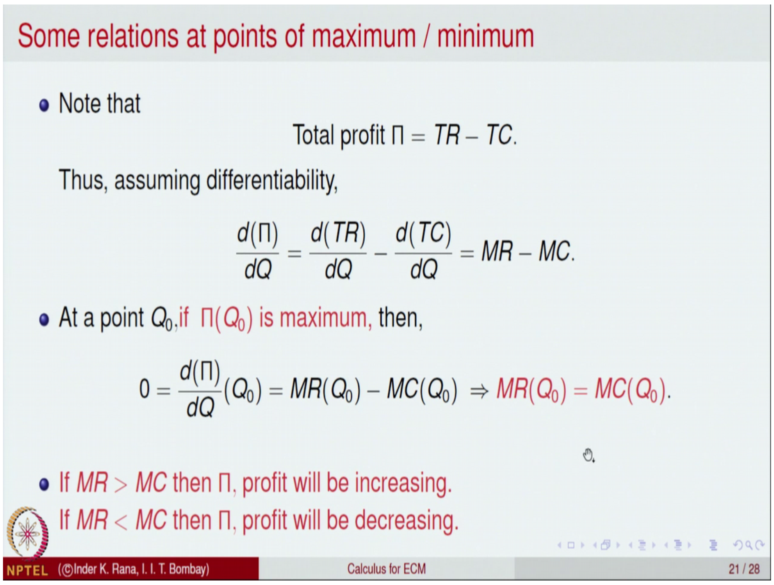

We can also look at the total profit and function at the max points of maxima minima, total profit pi we denoted by pi. This is a Greek letter pi is total revenue minus the total cost. So, if we look at assume that everything is differentiable. So, that will give you derivative of total profit is derivative of total revenue minus derivative of total cost Q, right with respect to Q total cost with respect to Q. So, we are differentiating this assuming each is the function of Q. So, this is TR. So, derivative of TR is the marginal of revenue derivative TC is the marginal of marginal of total cost. So, MR minus MC and at the point of maximum when so, assuming pi assumes the maximum at the point Q 0 we put the value here.

So, when it is maximum you will get 0 is equal to this evaluated at Q equal to Q 0; that means, MR marginal of revenue at the point of maximum is marginal of the cost at the point of maximum Q 0. So, that gives us relation when both are equal at the point of maximum. So, if MR is bigger than MC. So, the here also we can because of from here this formula if MR is bigger than MC; that means, the derivative of total profit with respect to Q is positive. So, pi must be increasing; that means, the profit will be increasing.

So, this is the interpretation if marginal revenue is bigger than of marginal of cost, then the profit will be increasing, that is a conclusion that we get and if it is less than it must be decreasing. So, that is the analysis for the marginal that we get. So, we will stop here today by recalling that we had looked at the first derivative test. We are looked at the second derivative test. And we had applied these things to some examples in mathematics. Purely mathematical examples of given a function, how to find out how to find out local maxima and minima.

So, that possible candidates we said are the points where in the domain where these are endpoints possibly; that means, there are some intervals in the domain. So, endpoints second possibility is the interior points where the function possible is not differentiable, and the interior point third category is the interior points where the derivative is equal to 0. So, this 3 categories of points are the possible candidates for maxima and minima. And to analyze them we apply the test for maxima minima the sufficient conditions, the first one was the continuity test the function is continuous at that point. And the first continuous at that point and increasing on the left decreasing on the right in a neighborhood then local maxima, decreasing on the left and increasing on the right in a neighborhood then it is local minima. So, that was the derivative test.

Correspondingly we had a first derivative test, just increasing decreasing is interpreted in terms of the derivative. So, of the function is continuous at that point; the function is differentiable in a deleted neighborhood of that point c; that means, on the left of c and right of c the function is differentiable on the left of the derivative is less than 0, then the function will be decreasing and on the right if the derivative is bigger than 0. So, that will be increasing. So, if it is decreasing on the left increasing on the right. So, it should be a point of maximum local maximum. So, that is in terms of the first derivative. And finally, we had a second derivative test; that the function has is defined in a neighborhood has a first derivative in a neighborhood of that point c.

First derivative is equal to 0 because that is a necessary condition anyway. So, first derivative is equal to 0, and the second derivative less than 0 implies the function has a local maximum at that point. And similarly, if the first derivative equal to 0 at that point. And the second derivative bigger than 0 implies it has a local minimum at that point. And these tests we applied to some of the examples in our scenario of economics commerce and management. And the last one was about relations between profit and marginal of revenue and marginal of cost. So, we will stop here today, and we will continue this analysis in the next lecture.

Thank you.