Calculus for Economics, Commerce & Management
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Lecture – 26
Local maxima and minima, continuity test, first derivative test, successive differentiation

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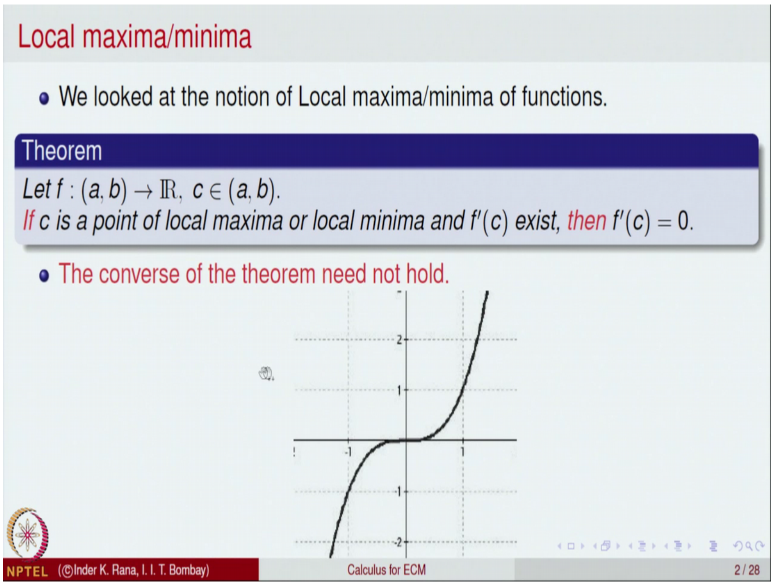

So, let us start recalling our previous lecture. In the previous lecture, we had started looking at the properties of local maxima minima of our functions of one variable in particular we looked at the following theorem namely if f is a function which is defined on a open interval a, b to R and c is a point in that open interval.

If c is this point which we have in selected in a, b is a point of local maxima or a local minima then f dash of c exists and f of f dash of c exists than f dash of c is equal to 0. So, if 2 conditions are satisfied, namely c is a point of local maxima or minima and the derivative at that point exists, then the derivative must be 0, geometrically that mean the tangent must be horizontal and we also pointed out the converse of this theorem need not hold namely the derivative equal to 0 at a point need not imply the function has a local maxima or minima.

We had given some examples here is the graph of f of x is equal to x cube this graph is on the positive side it is positive it becoming flatter and flatter and terms negative value for negative values x cube is negative. So, it turns around. So, the at 0 the x axis is a tangent to this curve y equal to x cube; what it this point you can; obviously, see that there is no maxima or minima because at all points nearby on the left the values are smaller and all the values of the function at points on the positive side the our positive. So, there will be bigger than the value at the function at the point 0. So, this converge of this theorem need not be true.

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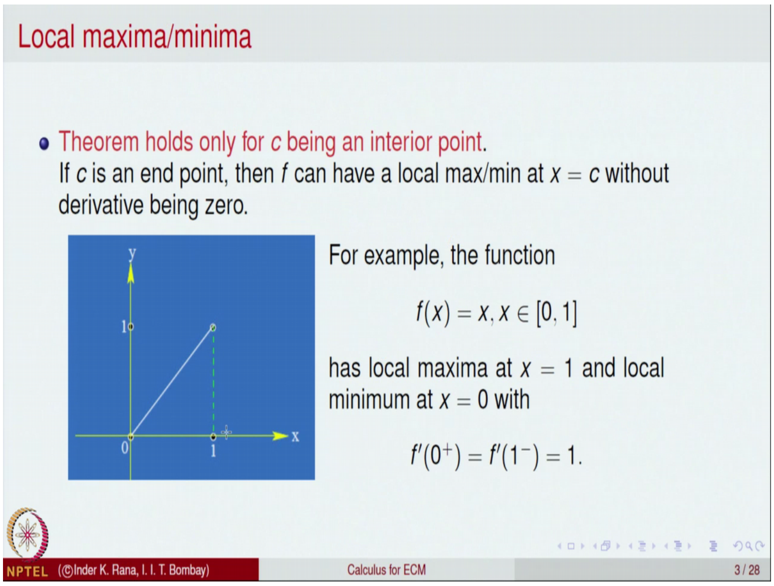

So, that is one observation and secondly the theorem holds only when c is a interior point. So, keep in mind the where the interval is a, b open interval a, b and the function is defined in the open interval a, b. So, this need not hold the conclusion that c is a point of local maxima or minima and the derivative at that point exists.

This need not hold if the interval is having endpoints say for example, let us look at this example f of x is equal to x if x belongs to 0 1. So, this function y is y equal to x and this is the graph of the function. So, the function has got 0 at 0 and is one at one. So, if you look at the neighborhood of 0 in the domain of the function on the right side, the values are positives. So, there bigger than 0. So, 0 is a local minima and one is a point of local maximum, but if I look at the derivative of this function on the left or from the right and the derivative is equal to. So, at this point the only possible is to derivative for the function from the left side or and 0 the only possibility is derivative rate of changes the function from the right side both are equal to 1.

So, the derivative is not equal to 0. So, the if condition of saying that the derivative at l point of local maxima or minima should be 0 is valid when we have a domain the point where we are analyzing is in a open interval.

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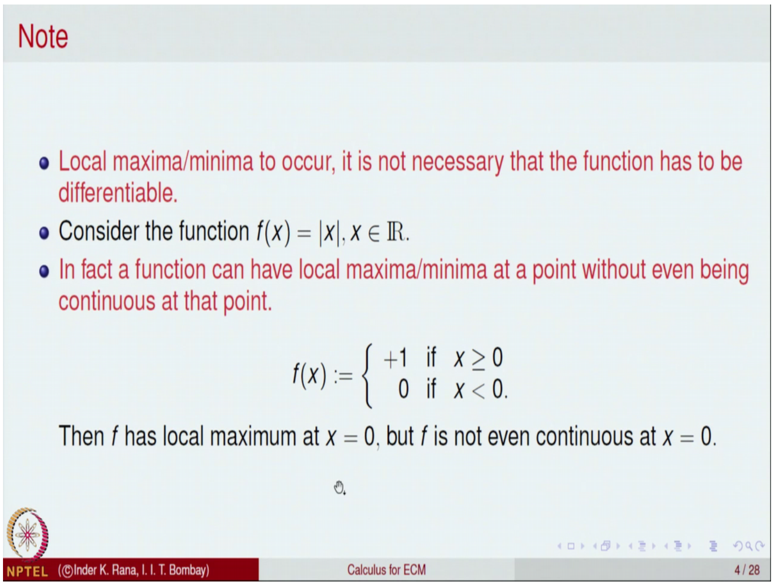

So, that was the second observation, we wanted to point out and also look at that we are saying that this is only a necessary condition; that means, a local maxima or minima to occur it is not necessary the function has to be differentiable we are only this theorem only gives you a necessary condition if the derivative exists at that point, but functions can have local maxima or minima without being differentiable. So, let us look at for example, the f x is equal to mod x the function f x is equal to mod x has got a minimum value at the point x is equal to 0, but the function is not differentiable at that point x is equal to 0, right.

On the left if you look at on the left of this point x is equal to 0 the function is minus x. So, the derivative is minus one. So, left derivative at the point 0 is minus one and the right derivative at the point x is equal to 0 if we are coming from the right side the function is x and the derivative is 1. So, the function is not differentiable at the point 0, but still it has a local minimum at the point x is equal to 0. In fact, something much stronger can be said here this function is continuous at 0, but is not differentiable at 0 and as a local minimum at 0; even there can we can defined a give examples of functions which can have local maxima or minima at a point without even being continuous at that point. So, that is also possible for example, let us look at the function f x is equal to plus 1 at x bigger than or equal to 0 and equal to 0 if x is less than 0 function as a local maximum at the point 0 the value is 1.

So, for all points in a neighborhood of that point x is equal to 0 the value at the point 0 is bigger than or equal to value at every other point. So, the function has a local maximum at the point x is equal to 0, but f is not even continuous at the point 0 the left limit is 0 where the function at that point and the right limit is equal to 1.

So, this function is not continuous at 0, but still the function has a local maximum at the point x is equal to 0. So, the existence of local maximum or a minimum for a function has nothing to do with this property being continuous or property being differentiable; what we are saying is in terms of continuity or differentiability you can give conditions which ensures that a point is a local maxima or minimum.

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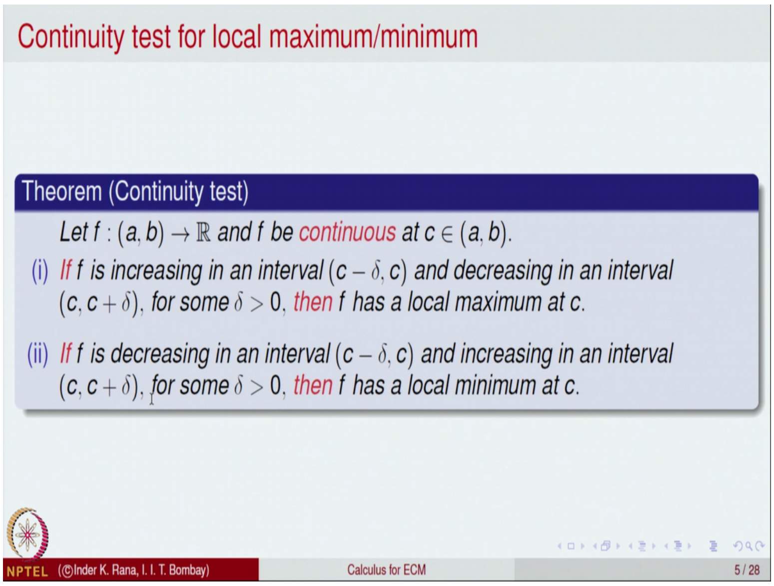

So, our previous theorem give as necessary condition and let us now state some sufficient conditions which ensure that a point which we are analyzing is a point of local maxima or minima.

So, the theorem is called the continuity test. So, this says let us suppose f is a function on a open interval a, b to R and it is given to be continuous at the point c belonging to the open interval a, b. So, if f is increasing in an interval in c minus delta to c; that means, on the left side of c if the function is increasing and it is decreasing and an interval c 2 c plus delta which is a interval on the right side of c, then f should have a local maximum at the point c. So, this is the mathematical way of saying that if the function is rising on the left and falling on the right and it is continuous at the point c, then it must have a local maximum at the point c.

So, the geometric visualization that we have of local maximum is translated into this property namely. So, this is a sufficient condition that if function is continuous at the point c on the left of c it is decrease on the left of c it is increasing and on the right of c, it is decreasing then it has a local maximum at c. So, there is a parallel condition similar condition for local minimum which is says at if f is decreasing on the left side of c. So, that is in interval c minus delta to c and is increasing and internal on the right side of the it; that means, in a interval c 2 c plus delta then it must be local minimum of c.

So, of both this geometric visualizations translated into mathematical concept are true when f is continuous at c right. So, do not make a mistake of an saying that on the left it is increasing on the right it is decreasing. So, it must be a local maximum that happens when you have a condition satisfied that the function is continuous and similarly for decreasing. So, this is what is called the continuity test for increasing and for continuity test for local maximum and local minimum.

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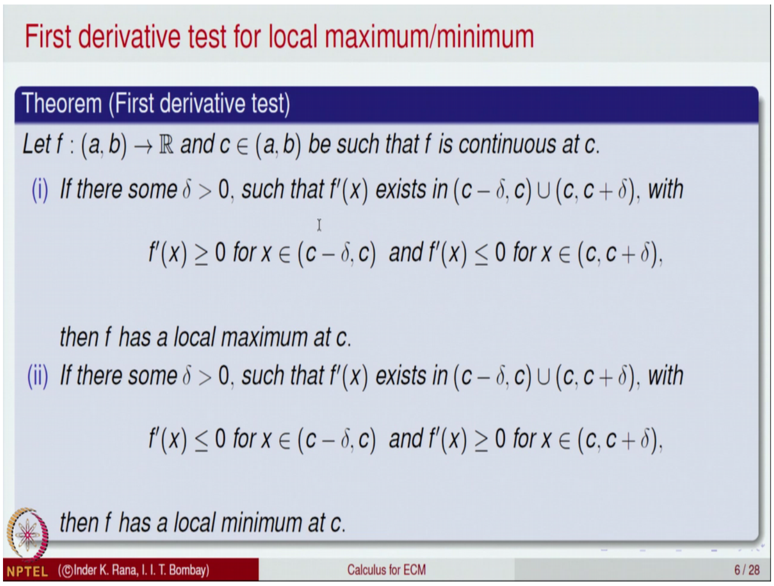

We can also stay we have a theorem in terms of the first derivative what is called the first derivative test if we recall we proved that if the derivative of a function in the previous lecture we had proved that if the derivative of function is bigger than 0, then the function is increasing in an interval and similarly if the derivative of a function is less than 0, then the function must be decreasing in that interval.

So, that property together with the previous theorem gives us what is called the first derivative test suppose f is defined an open interval a, b to R and c is a point in the open interval a, b such that f is continuous at c. So, continuity is always going to be condition. So, suppose there is a; some delta bigger than 0 such that the derivative exists in an interval around c so, but may not be the at point c. So, this is written mathematically as there exists some delta bigger than 0 such that if you look at the intervals open interval c minus delta to c union the open interval c to c plus delta, then the derivative exists in this interval; that means, at every point on the left of c and on the right of c on this interval the derivative exists. So, that is one part of the condition and second thing is if the f dash of x is bigger than 0 on the interval c minus delta to c and f dash of x is less than or equal to 0 in the part c to c plus delta.

So, that will imply by the previous theorems that if this is. So, then f will be increasing on the left and f will be decreasing on the right. So, that will imply that it has a local maximum at the point c. So, this the first derivative test requires 2 things one first of all the derivative was the function must be continuous at the point c and. Secondly, in a neighborhood of the point c may accepted possibly at the point c, the derivative is bigger than or equal to 0 on the left and less than 0 on the right.

Then it has a local maximum at that point and the second condition similar condition is there for the local minimum that there is a delta bigger than 0 say if that the derivative exist in the neighborhood around c accepted the point c such a neighborhood normally is called a related neighborhood because this is a neighborhood around the point c of length c minus delta on the left and c plus delta on the right and only the point c is omitted from it.

So, from a interval c minus delta to c plus delta if we omit the point c new that a union of 2 disjoint open intervals. So, this is called the related neighborhood of the point c. So, the condition says there is in a related neighborhood of the point c the derivative exits and if we have derivative is less than or equal to 0 on the left and bigger than 0 on the right. So, that will imply from the derivative conditions that the function is decreasing on the left and decreasing on the right. So, it must have a local minimum at that point c. So, this is what is called the derivative test first derivative test for analyzing a point to be a point of local maxima or minima. So, these are this you can these are the sufficient conditions which ensure whether a point is local maxima or minima. So, let us repeat once again if you are able to say that the function is continuous at the point c on the left of c, the derivative is positive on the right of c derivative is negative then it has a local maximum.

Then it has. So, if those conditions are satisfied then it has a local maximum. So, that point will be a point of local maximum and similarly if f is continuous and its derivative exists in the related neighborhood with the property that derivative is less than or equal to 0 on the left and bigger than or is equal to 0 on the right, then it has a local minimum at the point c. So, this theorem can be used to analyze whether a point is a point of local maxima or minima for a given function or not. So, one has to check all these conditions and then draw the appropriate conclusion.

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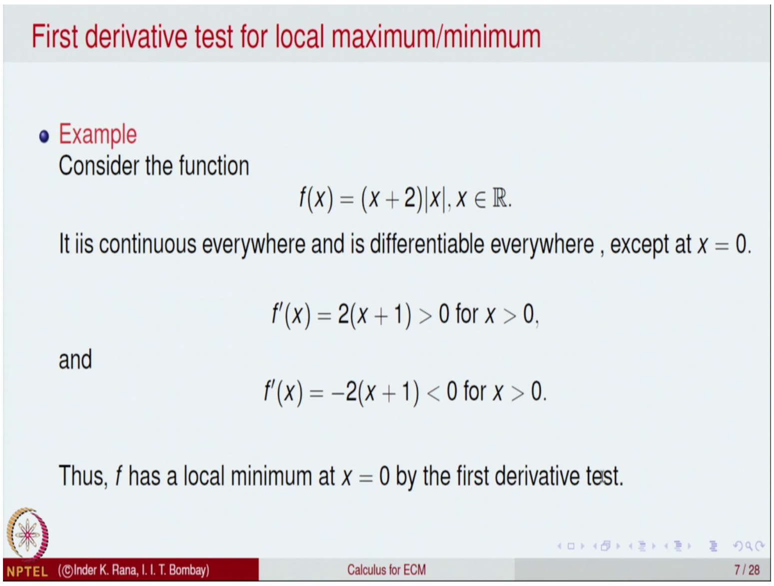

So, let us look at this function f of x is equal to x plus 2 into mod x; x belonging to R now first of all when you look at such a function there is this term mod x coming and we know that modulus of a number x is equal to x if x is bigger than 0 and it is less than x is if x is less than 0.

So, this function is continuous everywhere because it is a product of 2 functions plus 2 which is continuous and mod x is continuous. So, this is a function which is continuous everywhere; let is look at the differentiability of this function it is continuous everywhere except and it is differentiable also everywhere except that the points 0. See if x is not equal to 0, this function is either x plus 2 into x if x is positive and this function is x plus 2 minus x if x is negative and at the point 0, we know mod x is not differentiable. So, this function is not going to be differentiable at mod x that is not the exact reason the reason is if you look at the left derivative and the right derivative for this function; they will turn out to be different. So, to see that let us just analyze because this is the first time we were coming across such an example.

So, let us look at the differentiability of this function at the point.

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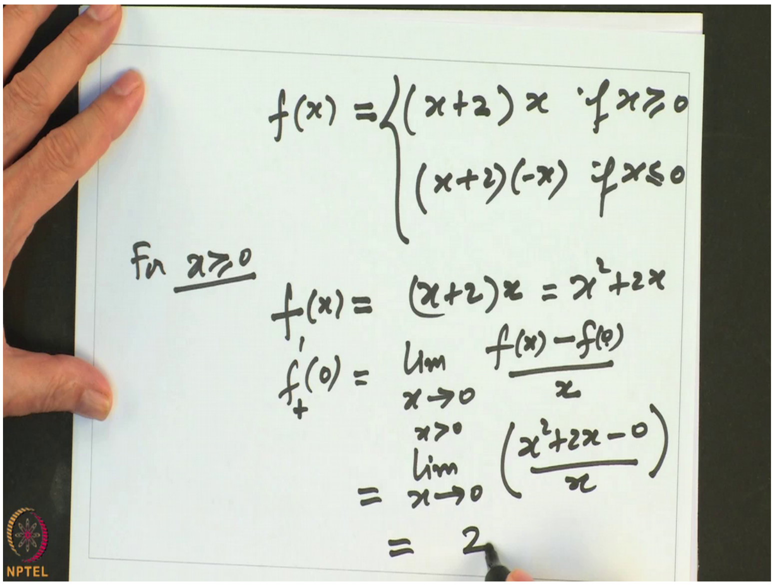

So, f of x the function is equal to x plus 2 into x if x is bigger than or equal to 0 and is equal to x plus 2 into minus x if x is less than or equal to 0. So, for x bigger than or equal to 0 the function is f of x equal to x plus 2 into x, which you can write as x square plus 2 x. So, if I look at the derivative at the point 0. So, what will be the value of the function at 0 is 0. So, what will be the f dash of at 0 and I am coming from the positive sides of plus. So, that will be limit of x going to 0 of f of x minus f of 0 divided by x and x positive.

So, that will be equal to limit x going to 0 and when x is positive the value is x square plus 2 x f of 0 is 0 divided by x. So, that is equal to limit when x cancels. So, it is x plus 2. So, the value is 2. So, the right derivative at the point 0 is 2; let us compute the left derivative of this at the point 0.

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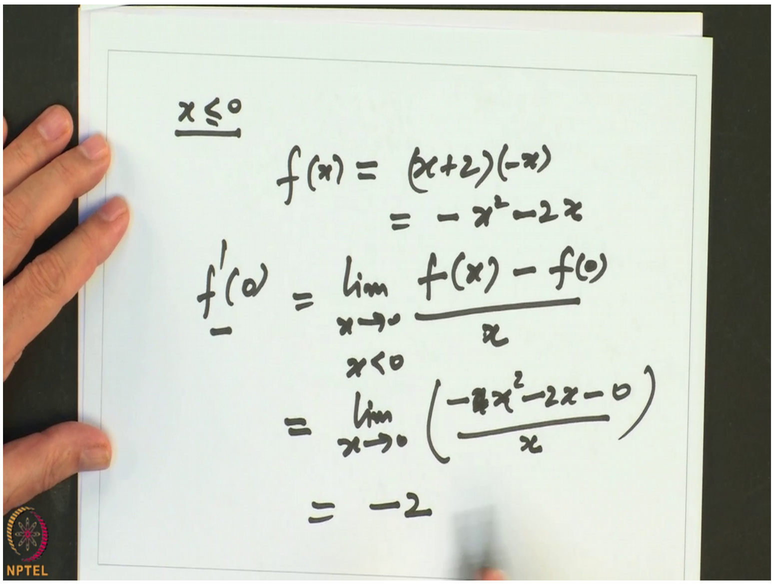

So, for x less than 0; less than or equal to 0 does not matter because the function is uniquely defined at the point 0 f of x is equal to x plus 2 and the mod x is now minus x. So, it is equal to minus x square plus minus 2 x. So, the f dash of 0; from the coming from the right will be equal to limit x going to 0 f of x minus f of 0 divided by x; x less than 0.

So, when you put the values. So, limit x going to 0 of when x is negative. So, it is minus 2 x minus of x square minus 2 x and at 0 the value is 0 divided by x. So, what will be limit of this; once you cancel out x. So, that is equal to x go to 0. So, it is minus 2. So, when you look at the left limit of the function that is minus 2, you will look at the right derivative at the point 0 that is 2. So, this function is not differentiable at the point 0. So, let us come back to the statement that the for the function f of x is equal to x plus 2 into mod x it is continuous everywhere and it is differentiable also everywhere except at the point x is equal to 0.

So, let us look the derivative of this function on the left side and on the right side. So, derivative when the function is big x is bigger than 0 the function is given by x plus 2 into x. So, what will be the derivative for all points on the right side it will be 2 times x plus 1. So, which is bigger than 0 for all x bigger than 0 and similarly, if I look at or the left side of this function then the function is given by mod x will be minus x. So, it is minus x plus 2 into minus x. So, that will be minus of 2 x plus 1 for this is a type of where it should be less than 0. So, this point should be less then right sorry. So, this is for all values x less than 0. So, on the right side the function has positive derivative on the left side the function has negative derivative. So, this implies that f has a local minimum at the point x equal to 0 by the first derivative test.

So, how is the first derivative test applicable at the point x is equal to 0 the function is continuous. So, condition of continuity is satisfied related neighborhood of 0, right, f dash of x is bigger than 0 if x is bigger than 0. So, to the function is rising on the right side and it is less than 0 on the left side. So, it is dropping. So, by the first derivative test the function has a minimum at the point x is equal to 0. So, that is how you will apply the first derivative test by ensuring the continuity of the function at the point that you want to analyze looking at the nature of derivative on the left of that point and on the right of that point at that point the function may not be differentiable that is not required for the first derivative test.

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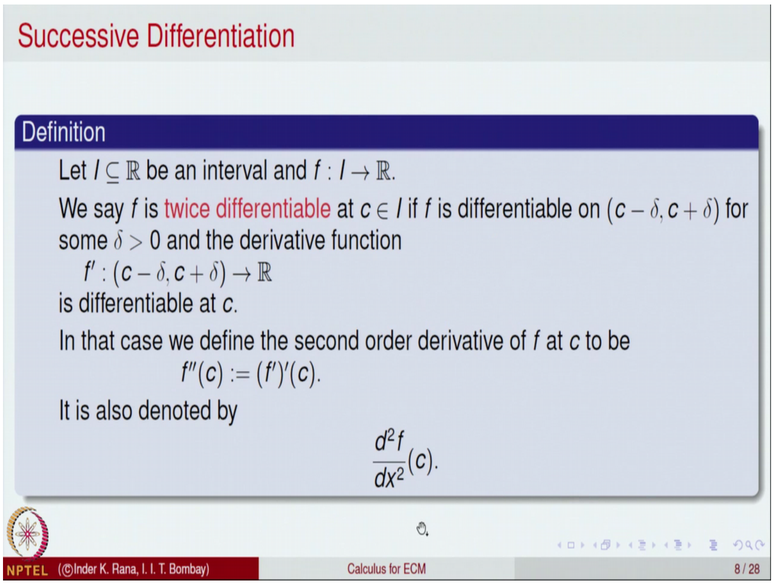

So, let us go a step for that and let us look at what is called second derivative test, but for the second derivative test I is want to recall what is called second derivative of a function and. So, one. So, there is a notion of successive differentiation. So, let us say a function f is defined on an interval I contained and so, as the function domain is a interval and taking real values. So, I is a interval in the real line and f is a function defined on the interval we say f is twice differentiable at a point c. So, it is said to be twice differentiable at a point c which is inside this interval I if f is first of all differentiable in a open interval around the point c in a neighborhood. So, first of all I have this is a condition that f should be differentiable at all points in a neighborhood of the point c. So, what it a neighborhood it is an open interval around the point c of length 2 delta. So, it will be c minus delta to c plus delta.

So, f should be differentiable in a neighborhood of the point c one and. So, once it is differentiable at every point x in this interval will get f dash of x. So, f dash is a function now in the inter which is defined in the interval c minus delta to c plus delta at every point the function is given to be differentiable in the interval c minus delta to c plus delta.

So, f dash is a function and I can ask whether f dash is again differentiable at the point c or not. So, if the func; there is a point if at the point c f is differentiable in an open neighborhood of the point c and the derivative function which is defined in that neighborhood now is again differentiable at the point c then we say that the function f is differentiable twice differentiable at the point c. So, it is some sense the derivative of the derivative function, but keep in mind to define that it is necessary that we should say that f is differentiable in an open neighborhood then only that will make sense.

So, twice differentiability means f is differentiable in a open neighborhood of c and the derivative function is again differentiable at the point c. So, we say f is twice differentiable at the point c and this is second derivative this is called the second derivative of f it c right when f dash is differentiable it will have a derivative this f dash itself will have a derivative at the point c and that normally we denoted as f dash dash, but that is short and as f double dash at c. So, f 2 dash at the top indicates the second derivative one also denotes the second derivative by d square f by dx square at the point c. So, the first derivative as d f by d x and now the second derivative is d square f by dx square. So, there is a notion of the second derivative.

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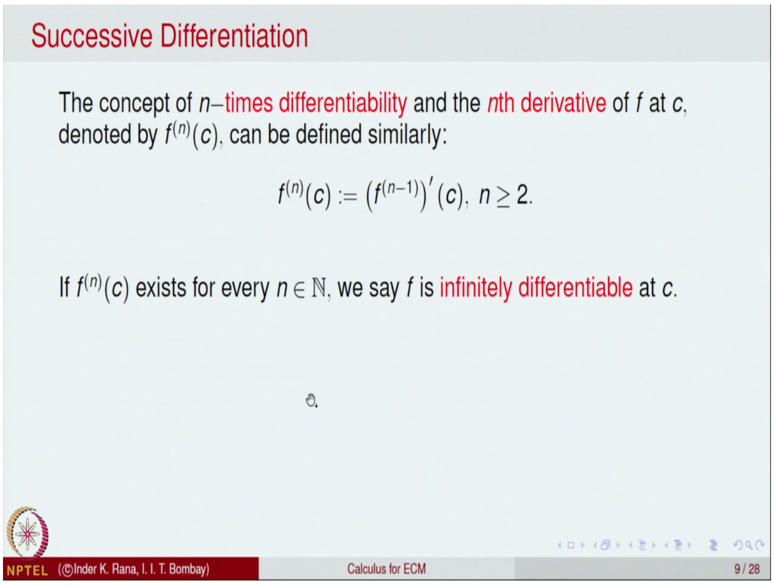

So, one can go on defining provided the higher order derivatives the concept of n times differentiability and the n th derivative of f of c we can define provided. So, to say that f is n times differentiable we should at a point c we should say that the n minus 1 th derivative exists in a neighborhood of the point c and is there differentiable at the point c.

So, that will denoted fnc to be the n th derivative is denoted by this bracket at the top the n th derivative of c to be defined as look at the n minus one th derivative its derivative at the point c when our define will be called as the n times differentiability of the function. So, what will be the second derivative second derivative is the derivative of the derivative of the derivative function. So, we want to have a opportunity to use this, but is interesting to say concept that if every derivative of every order exists for a function then we say function is infinitely differentiable.

If you recall we had said that we will assume that the exponential function is differentiable e raise to power x is differentiable and is derivative is the function itself that is what we had looked at as a assumption and using that we had defined the derivative of the lock function.

So, exponential function derivative is itself. So, automatically that is one example of a function which is infinitely differentiable. So, let us conclude today's lecture by saying we have looked at the notion of differentiability of a function as the condition for analyzing local maxima minima. So, we had the continuity test for analyzing local maxima minima and then we had the first derivative tests for analyzing local maxima minima, we will state in the next lecture what is called the second derivative tests for local maxima and minima of functions sufficient condition in terms of the second derivative.

Thank you.