Calculus for Economics, Commerce & Management
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Lecture – 20
Rate of change, differentiation

Welcome to the next lecture. In the previous lectures we have looked at the notion of concepts of limit and continuity and their applications in this we begin with looking at how the functions change as the independent variable changes how the values of the function, for a function y equal to f of x change. So, we want to analyze this and then we will see its applications in different fields in our field.

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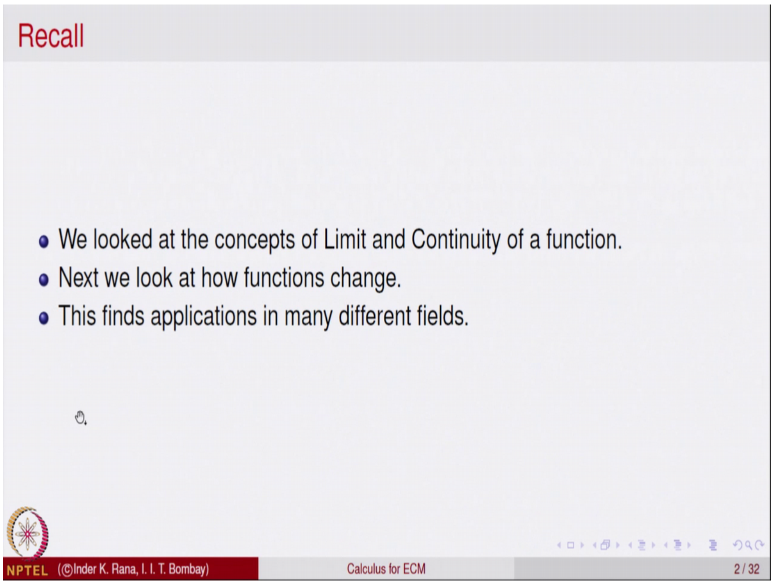

So, this rate of change is applicable in many different fields. I will just give you an illustration before we actually look at this how do I analyze this.

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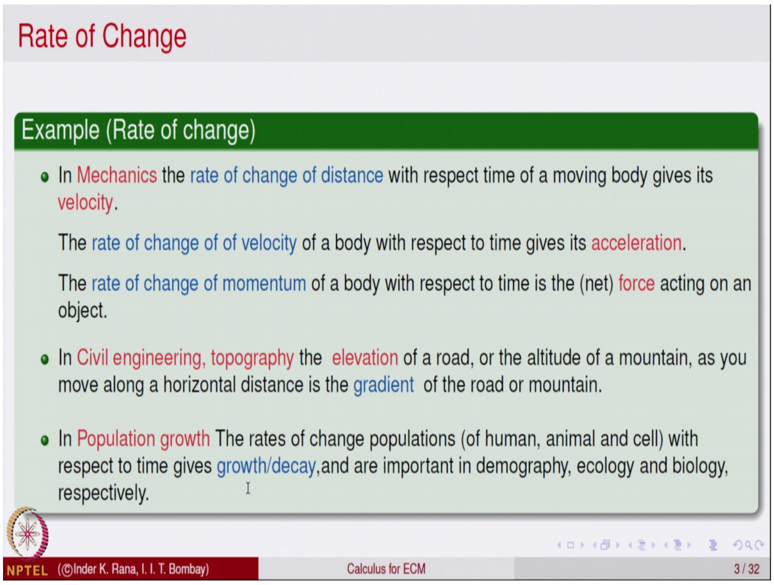

For example in mechanics the rate of change of distance with respect to time of a moving body gives a notion of velocity. This you must have all many of you must have driven a car or you must be setting next to a somebody driving a car and you ask what is a speed and immediately the person looks at a meter and says our speed is 61 kilo meters per hour. That means, at that moment the vehicle is travelling at a speed of 61 kilometer per hour. How does the meter know that this is the speed of the vehicle is so much? We can only find out the average of its speed for example, if at time visible distance. So, if a vehicle travels say at 10 o'clock in the morning if you observe that 10 o'clock to say 10 15, in 15 minutes it has travelled 2 kilometer then we can say the average speed is 2 kilometer divided by the 15 minutes that is average right.

How does that give us the instant speed at a point? So, that is the kind of analysis we want to do. So, in mechanics the rate of change gives rise of distance gives rise to velocity. The rate of change of velocity itself of a body gives rise to the notion of what is called acceleration right you might have also seen that very often that way, you are accelerating your y too fast or slow down and such kind of thing right. So, mathematically that is a rate of change of velocity that gives acceleration.

The rate of change of momentum of a body we will not define what is momentum gives rise to what is called the force acting on that body or in that object. In civil engineering suppose there is the road which is uphill kind of thing going upwards or sloping downwards and then you are moving on that or you are moving towards a mountain the rate of change of elevation gives rise to notion of the gradient in civil engineering or in topography.

Another notion is population or decay of population growth or decay of population the rate of change of population it could be population of humans, it could be population of animal, cells right and many other things the price of a stock right the interest earning of a fixed deposit and so on. They are all come under what is called the growth and decay and the rate of change of the that population with respect to time gives growth and decay these are important things in demography in ecology and biology and so on and in other field also.

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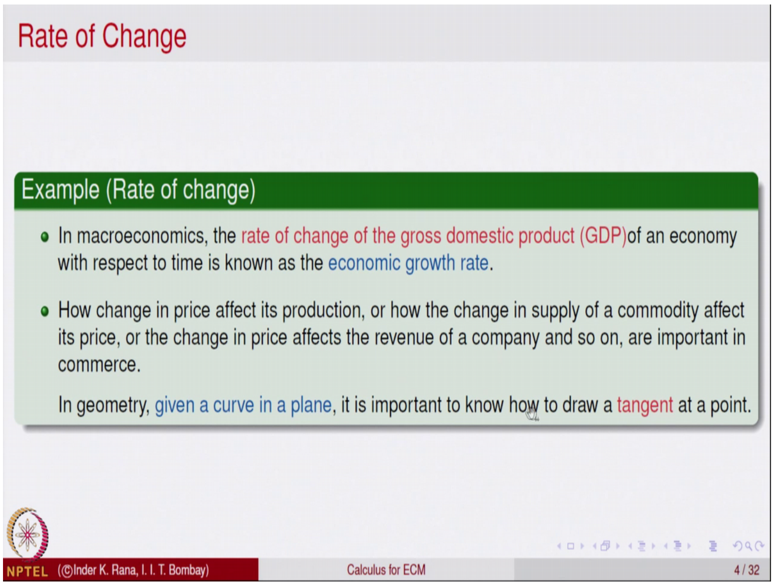

In macroeconomics the rate of change of gross domestic product GDP full form popularly known as depends on is a function of time and the rate of change of GDP is a very important concept in economics and it is hotly debated by the political parties whether the GDP is going up or down, why it is doing that and so on. So, that is economic growth rate or GDP, is rate of change of GDP is called the economic growth rate.

How the prices of, price effects production or how the change of supply of a commodity effects its price, change in price effects the revenue and so on all these are important questions in commerce which one would like to analyze understand and come to a kind of theory which will help or tools which will help one to decide about giving answers to this kind of questions.

In mathematic itself given a curve in a plane it is important to know how to draw a tangent at a point. So, we will look at somehow these things to become to a mathematical tool which will help us to do these kind of things.

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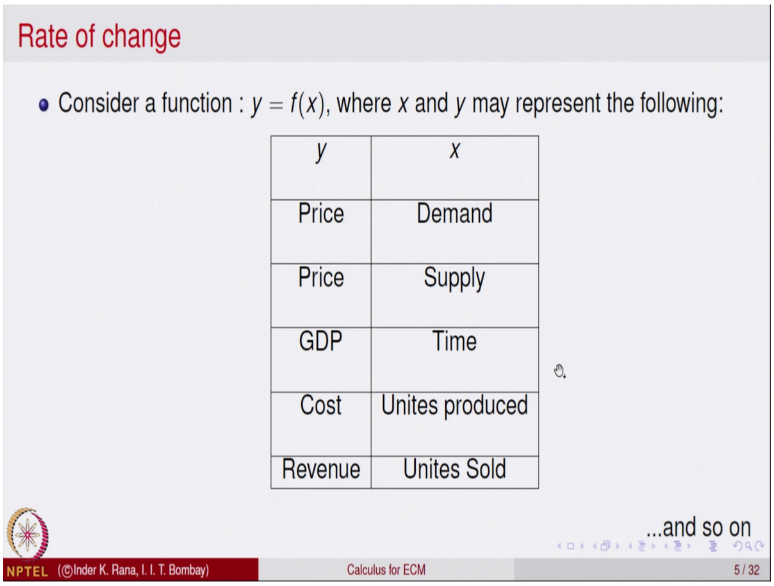

So, let us look at what we are looking at. We are going to look at functions y equal to f of x where y could be price, depending upon the demand price, depending upon the supply, GDP which is a function of time cost right of the number of units produced revenue units sold. So, all these are some functions y function of x where x could be any one of this and y could be any one of this and more also right.

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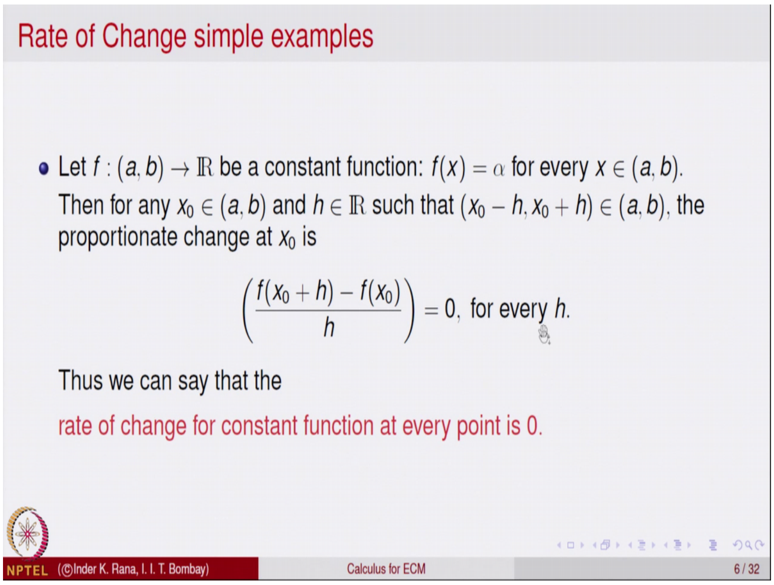

So, we trying heading towards the mathematics. So, let us take a function f of f defined in a interval a b to R and let us say that the we are looking at very simple examples f is a constant function. That means what? f of x is equal to alpha for every alpha belonging to a b. So, it is a constant function at every point it gives the same value. What do you think should be the, what is a change in the function? There is no change because at every point the value is alpha right. So, at any point x naught if I look at x naught minus h the interval x naught minus h to x naught plus h inside a b then what is a proportionate change, change in f of x naught plus h that is a value at a point x naught plus h this is a value at the point x naught divided by the change in x right.

So, x naught plus h the value at x naught there is a change in the value of the function and how much is the change in x coordinate that is x naught plus h minus x naught that is h. For this constant function that is 0 for every h; that means, the rate of change of the constant function is 0 at every points. So, we can safely say for a constant function the rate of change of the values of the constant function is 0. Well, let us look at another simple example the linear function.

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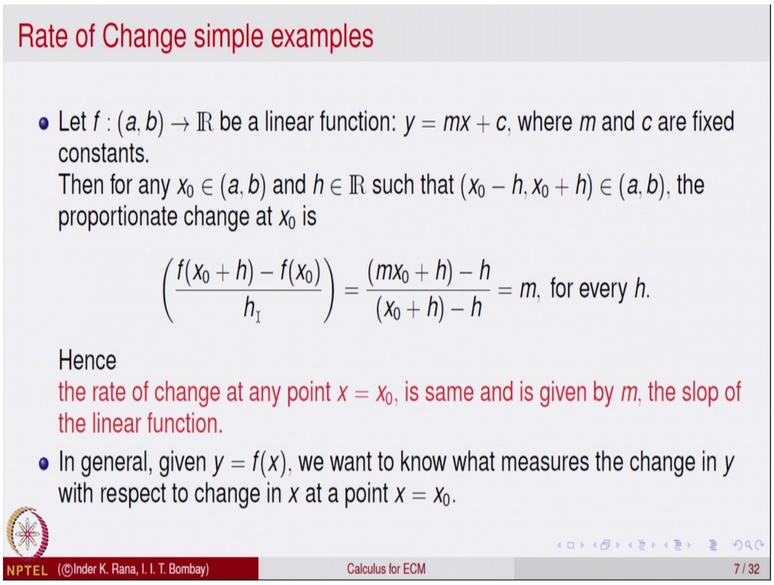

So, let us look at a linear function given by y equal to mx plus c where m and c are fixed constants right in a linear function m if you recall what was m, m was a slop of the line and c is the y intercept, so y equal to mx plus c that is a linear function given to us.

So, let us try to find out what is a proportionate change. So, for a point x naught in a b let us say that we look at a interval x naught plus h and x naught plus h inside a b. So, look at values right. So, let us say h is a, and let us look at the value f of x naught plus h and f of x naught minus divided by h. So, this is a increment is the values of the function and this is a increment in the independent variable x right at the point x naught. So, this is the ratio of the change right, this is a change in y divided by the change in x right. So, that f of x naught plus h is m of x naught plus h from this minus f of x naught and divided by h right.

So, what is the rate of what is the change that is equal to m; f of x naught plus h is m of x naught plus c. Let us look at this slightly more clearly. So, our function is f of, we are looking at the linear function f of x is equal to so that is y equal to m of x plus c.

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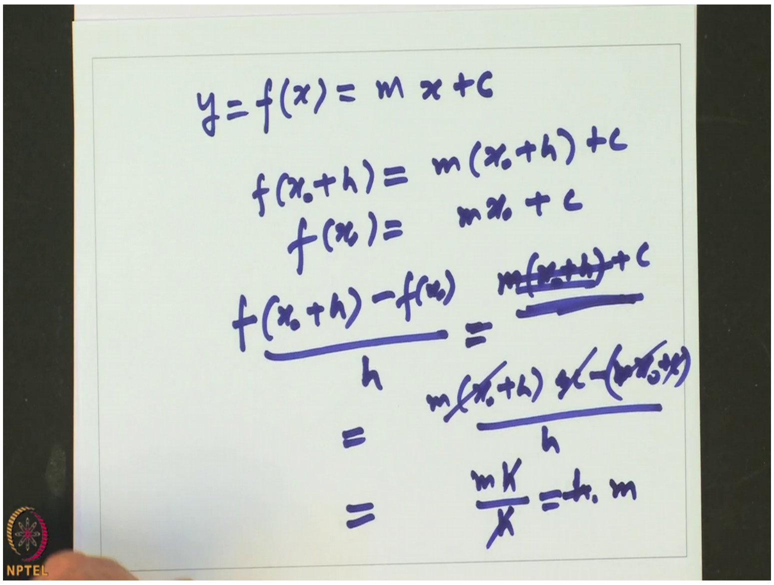

So, at x naught what is f of x naught plus h. So, that is equal to m of x naught plus h plus c. What is f at x naught, that is equal to m of x naught plus c. So, what is the difference m of x naught plus h minus f of x naught divided by h that is equal to m of x naught plus h plus c divided minus. So, let us calculate.

So, that is equal to m of x naught plus h plus c minus m of x naught plus c divided by h. So, what is that equal to? So, m of x naught and m of x naught that will cancel out right and c will cancel out. So, what is left is m of h divided by h that is equal to h right. So, for a linear function the rate of change is same. So, this is equal to h cancels out and that is equal to m. So, rate of change is equal to m for every value of x. So, again the rate of change is a constant function. So, the rate of change at every point x is equal to x naught is a same constant m the slop of the linear equation right. So, these are simple examples which illustrate the use of the calculation of rate of change of a function at a point.

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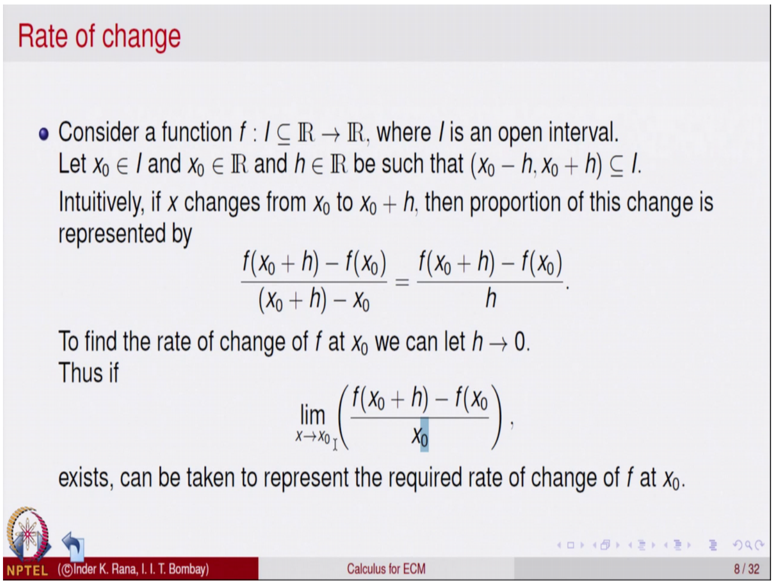

What happens if we have a general function? So, for a general function to find the rate of change at a point let us look at a point x naught and let us look at a value h. So, that x naught plus x naught minus h and x naught plus h both are inside the interval I. And then look at the change that occurs when you move from the point x naught to x naught plus h. So, in that case this proportionate change is represented by f of x naught plus h minus f of x naught divided by x naught plus h minus x naught, x naught cancels. So, this is what left out x naught is fixed. So, this is a function of h.

And we want to know what is. So, this is a proportionate change for h. So, we want to know what happens to this proportionate change at the point x naught so; that means, what we should make this h smaller and smaller. So, let us try to calculate, one will try to calculate when this quantity, this is now a function of h x naught is fixed right. So, this whole thing this ratio is a function of h and one can try to look at what happens to this as h goes to 0. So, this says we should be looking at the quantity namely limit of this quantity right f of x naught plus h minus f x naught this should be h here this should be h and limit x going to x h going to 0. So, this whole expression needs to be rewritten. So, this essentially says we should be looking at we should be looking at the limit of this proportion. So, we should be looking at limit; h going to 0 of f of x naught plus h minus f of x naught divided by h. So, so this is the proportionate change for h and limit of this, this is what is important to be looked at.

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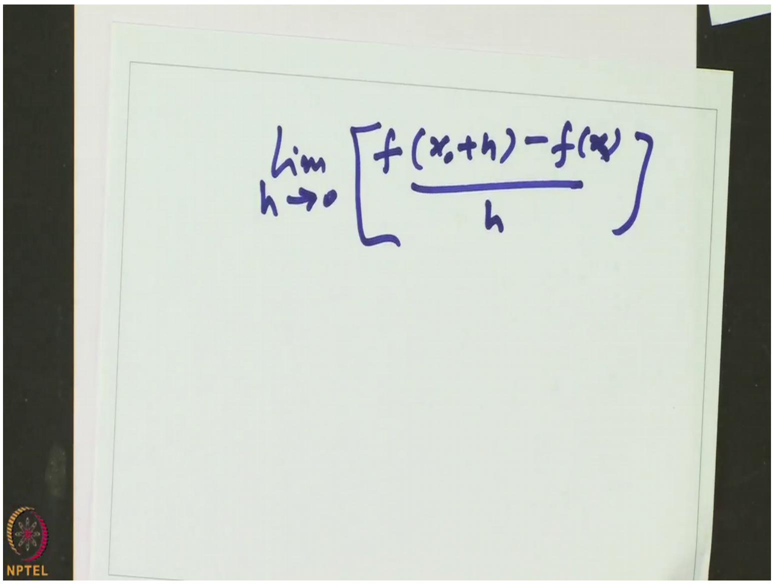

So, this we can think of representing this quantity if it exist that to be representing the rate of change of the function at the point x naught.

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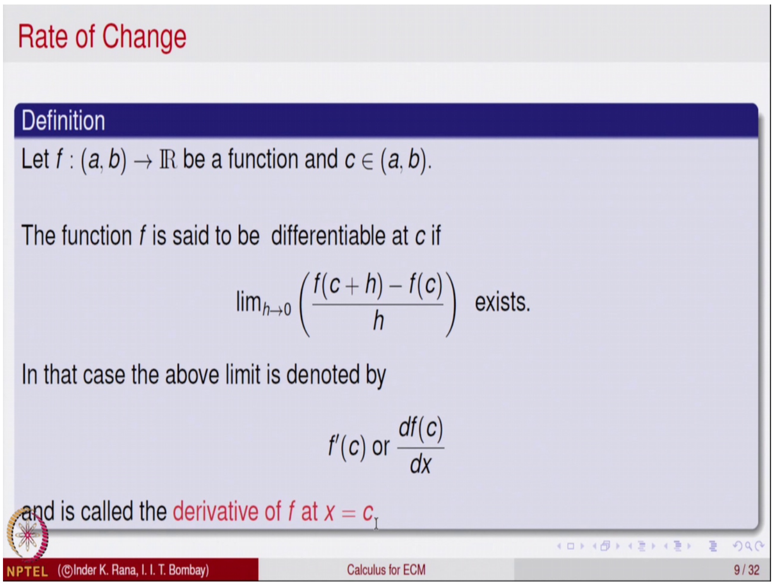

So, to make it more precise let us make the definition. So, let f of f be a function from a b to R such that and c is a point inside a b. We have purposefully taken a open interval and a point c inside this inside the open interval because once this property of open interval if you take c plus h right whether h is positive or negative that will also belong to a b. So, we look at the change f of c plus h minus f of c divided by h so that is a change when you go from the point c plus h to c; c plus h could be on the positive side or it could be on the negative side depending on h is positive or negative. So, this is a change for a change h units in x. So, one would like to know the limit of this as h goes to 0. So, if this quantity exist in that case we say that the function has a derivative at the point x is equal to c and it is denoted by f dash of c or it is also, it is a d f c by d of x.

Let us try to see how does this help in giving us the various concepts that we have been thinking about.

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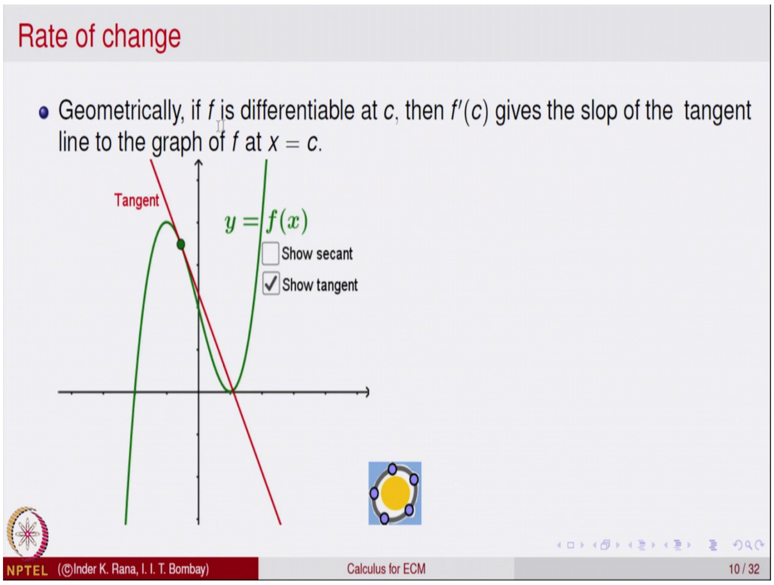

Geometrically saying that a function is differentiable at a point c that limit exist is essentially saying that f dash of c gives the slop of the tangent to the graph of f of c. So, to illustrate this idea a bit more let me draw a picture and see how does this help.

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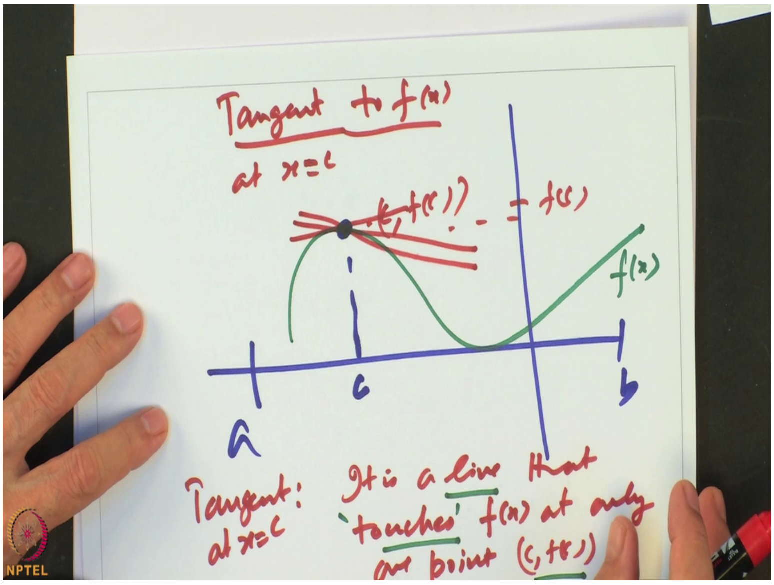

So, let us look at, this is some axis and we have got a function which is say like this that is my f of x, and let us take a point say let us take a point c here and that is my a and that is my b. So, at this point I want to know what is I want to draw a tangent to f of x.

This is a geometric problem at x is equal to f of x at the point x. That means what? This is the point c comma f of c right that is in the graph of a function. So, this is f of c and that is c. So, here I want to draw a tangent. So, what is a tangent? Tangent is a line. So, normally we take tangent what is a definition of a tangent is a line that touches f of x of the graph of a of x at only one point, drop of tangent let us say at x is equal to c is touches at the point c comma f of c. Now in this if you look at it is a line. So, we know what is a equation of a line what is a meaning of this word touches right. So, we know that this line has to pass through this point c comma f of c and it is a straight line

So, what is that qualifies which line qualifies to be the tangent at this point. So, can we say that can we say that this is the tangent line or this is a tangent line or this is the tangent line, which line is the tangent line to the graph of the function? To understand that what we do is the following. So, let me draw the graph once again and see what we are trying to do. So, this is, let me draw a slightly more involved graph. So, let us say this is my graph.

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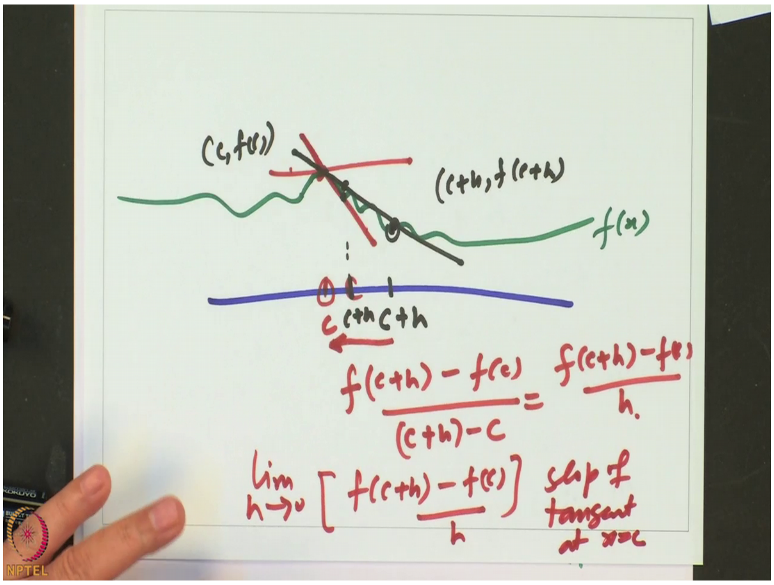

So, this is my graph of f of x and we are trying to look at this is the point c and this is a point where I am trying to look for a tangent, which line I should take a like a tangent and the line should have the property that it touches the graph only at one point.

So, what we do is we try to approximate this. How do I approximate this? We take a point nearby. So, let us take a point c plus h here say for example. So, that gives me a point here right. So, this point is c plus h comma f of c plus h and this is the point which is c comma f of c. So, given this let us look at the line joining these two. So, let us look at this line which passes through these two points right. Now, what is happening is this line is cutting the graph at many points right so obviously, this is not qualify to be a tangent, but what we do what we observe is if I take this point closer if I take c plus h here then what is the value. So, then this is a point that I get and what will be my line now. So, now, the line has become like this right. So, it has avoided many other points in between.

So, n to draw a line we know want to what is a information that is required to draw a line, we have to should have a point and its slop. So, what is the slop of this line which passes through c and c plus h? That is if you recall the equation of straight line y 2 minus y 1. So, that is c plus h minus f of c divided by c plus h minus c right. So, that is equal to f of c plus h minus f of c divided by h. So, that is a slop of this line and as this point comes closer and closer to c this line will merge all the points to one point. So, that will become a tangent to the tangent to the function f of f of x at the point c. So, what we are saying is limit of this h going to 0 of f of c plus h minus f of c divided by h gives the slop of tangent at x is equal to c. So, geometrically this gives the slop of the tangent at that point.

Since we are looking at these things let us one can also look at it a bit geometrically in a dynamic geometry setup. So, let me show you what I am trying to do in a more sophisticated way. So, I am opening this picture in GeoGebra which I have mentioned earlier we can try to use that. So, it has loaded. So, this is the picture of this function.

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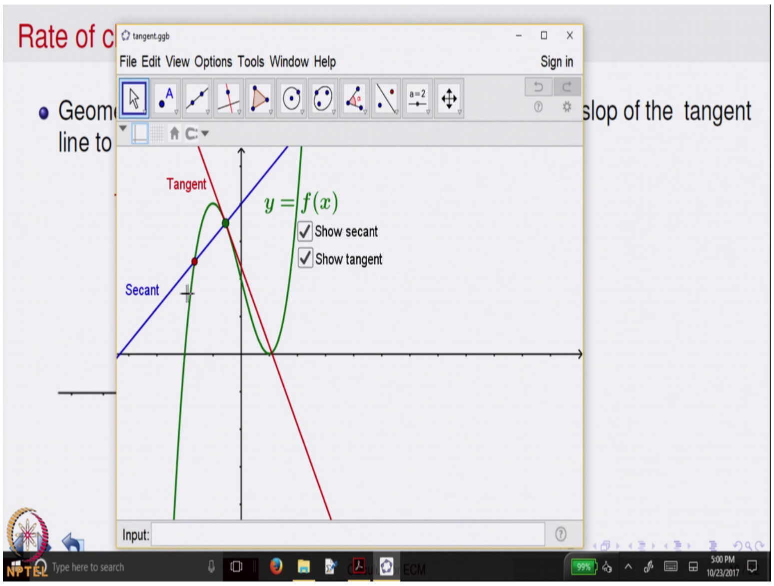

So, let me just make the window a bit larger. So, that you are able to see. So, we are saying that this is a tangent right. So, this is the curve this green is the curve y equal to f of x at this point I am trying to draw a tangent and we are claiming that this red line is the tangent. So, I said that to ensure that this is a tangent let us take a point nearby join that right, so that line at a nearby point is called a secant. So, this point could be on the left side of c or it could be on the right side of c right. So, I take a point here. So, this is the secant line that is the point, this is the point on the left side we have taken and taken the line joining it. So, this is normally called a secant line. So, the point is as I come closer and closer to this point c this line should become the tangent.

So, let us move this. So, I am moving this point closer and closer, closer and closer and you see that when I approach that point that becomes the red line becomes the blue line.

So, secant becomes tangent as I approach from, this is I approaching the point c from the left I can go on the right side. So, it is this point. So, this is the point this is the point nearby this is the line joining. So, such a line is called a secant line. So, the slop of the secant line becomes the slop of the tangent line as this approaches this.

So, the point on the left or on the right this approaches right. So, that is what dynamic geometric and show you that means, right. So, geometrically if f is differentiable at a point then f dash of c gives the slop of the tangent to the line to the graph of the function at this point c we already know that the point is here to draw the line we have to only know the slop. So, slop is given by the derivative of that function at that point if it exist.

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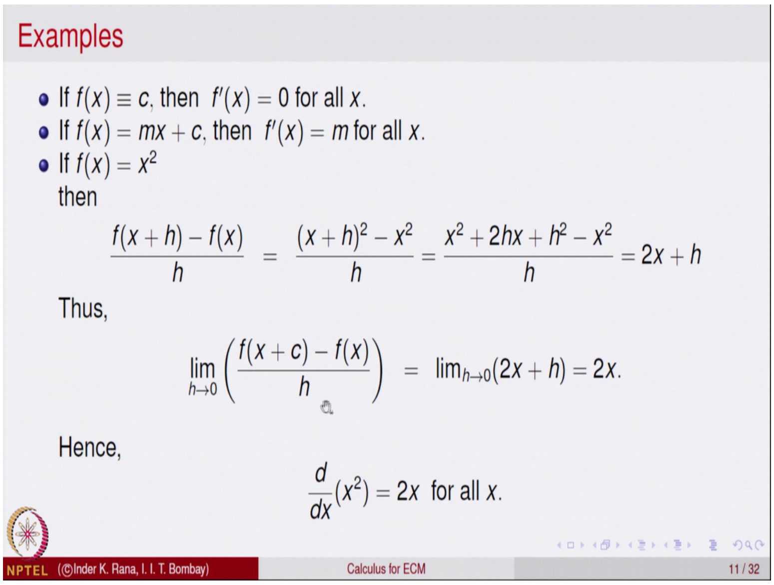

So, let us look at as we observe d earlier for a constant function derivative at every point is equal to 0 we have seen that. Let us look at the linear function f of x is equal to mx plus c and we saw that the derivative of this is equal to m which is the slop of the line. So, derivative of a linear function is nothing but the slop of the line at that point right and the slope is same everywhere. So that means, derivative of a linear function is a constant function f dash of x is equal to m for every x.

Let us look at f of x is equal to x square when x is a real number. So, to find it derivative at a point what we have to do we have to look at the f of x plus h minus f of x divided by h. So, what is f of x plus h that is x plus h whole square minus f of x that is x square divided by x plus h minus x. So, that is equal to h. So, we have to find limit of this quantity as h goes to 0. So, let us simplify this say width. So, this gives us open the algebra square. So, this gives you x square plus 2 h x plus x square h square. So, that is algebra identity a plus b whole square equal to x square plus 2 h x plus h square minus x square. So, x square cancels out and what you are left with is 2 h x plus x square divided by h, h cancels out, so it is 2 x plus h limit h going to 0. Now, all the theorems that we had proved for the limits of a function of one variable limit of this is nothing, but equal to h going to 0 is 2 x.

So, for the function f x is equal to x square its derivative is equal to 2 x. So, this function is f of x is equal to x square is differentiable everywhere with derivative equal to 2 x. So, this power essentially is coming down as 2 and this power becomes less 1. So, that becomes x to the power 1. So, that becomes equal to 2. So, d of one writes d by d x of x square is equal to 2 x.

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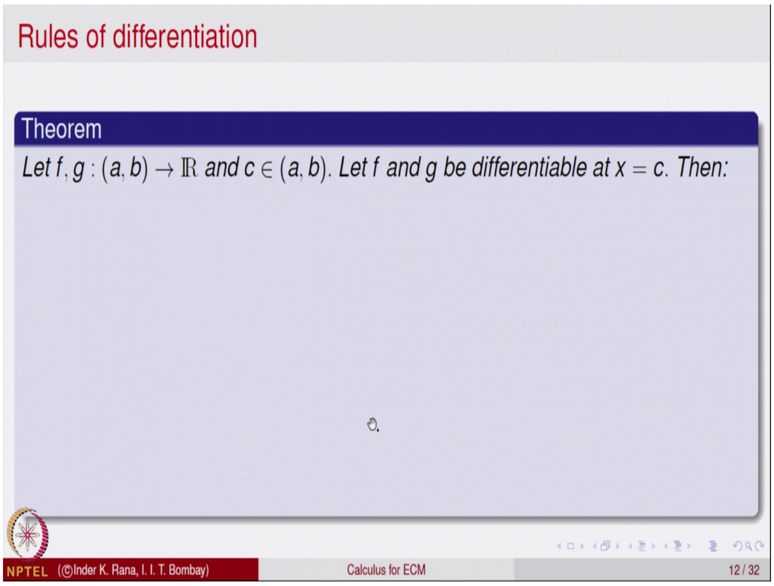

So, in the next lecture we will describe what are the rules of differentiation. These are basically theorems that help us to find derivatives of some complicated functions. So, we will do it in the next lecture.

Thank you.