Calculus for Economics, Commerce and Management
Prof. Inder K. Rana
Department of Mathematics
Indian Institute of Technology, Bombay

Lecture – 19
Application of continuous functions, marginal of a function

So, welcome to today's lecture. If you recall we had started looking at the notion of continuity for functions of a real variable and its applications in our subject. We looked at continuous functions and we are looking at what is called the intermediate value property for continuous functions which stated as follows.

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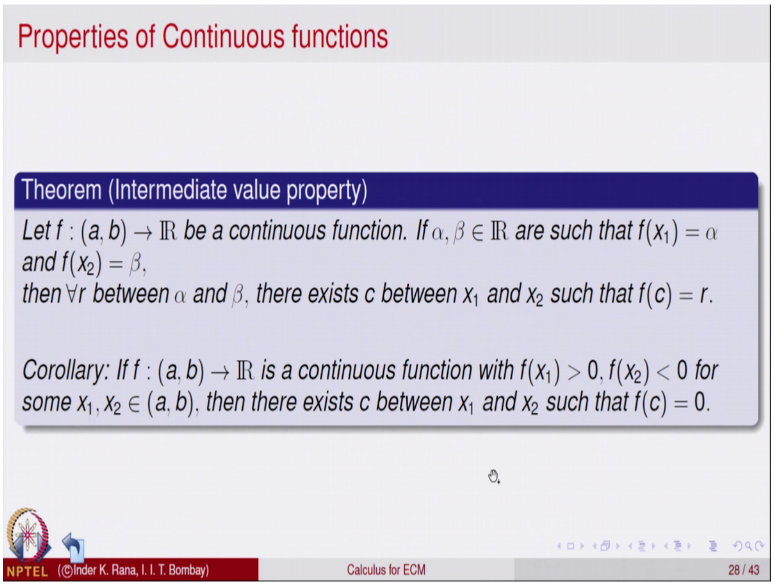

That, let f be a function defined on an interval a b to R which is continuous. So, the function is given to be continuous and alpha and beta are two real numbers which are in the range of the function; that means, that there are points x 1 and x 2 belonging to a b such that f of x 1 is equal to alpha and f of x 2 is equal to beta. So, the function takes the values alpha and beta at some points x 1 and x 2 in the domain.

Then the claim is that for every real number R between alpha and beta there exist a number c such that between x 1 and x 2 such that f of c is equal to r. That means, if two values alpha and beta are taken by a continuous function then every other value in between alpha and beta should also be taken by the function at some point. So, this is the intermediate value because it says that any value in between alpha and beta should be taken at least once by the function if alpha and beta are taken.

So, this is a property of continuous functions and we had seen in the previous lecture what geometrically it means, it means that the break there is no break in the graph of the function. Once you start plotting the graph of the function from the point a comma f of it to b comma f of b once you start plotting drawing the graph you should not lift your pen till you reach from the point a to the point b. So, that is and mathematically this theorem means that the images of intervals are intervals.

So, a very special case of this theorem what be when alpha is a number less than 0 and beta is something bigger than 0. So, the theorem as a corollary of this if f is a continuous function such that f at some point x 1 is bigger than 0 and f at x 2 is less than 0 then for some point in between x 1 and x 2 the function should take the value 0. That means, for a continuous function if at some point the graph is above the x axis and graph is at some other point the graph is below the x axis and if the function is continuous that at some point in between it must cross the x axis. So, there must be a point c where the value is equal to 0. So, this theorem has applications in lot of applications in mathematics.

Let us see one application in our subject of in our subject. So, let us take a demand and supply functions of a commodity are given as the following.

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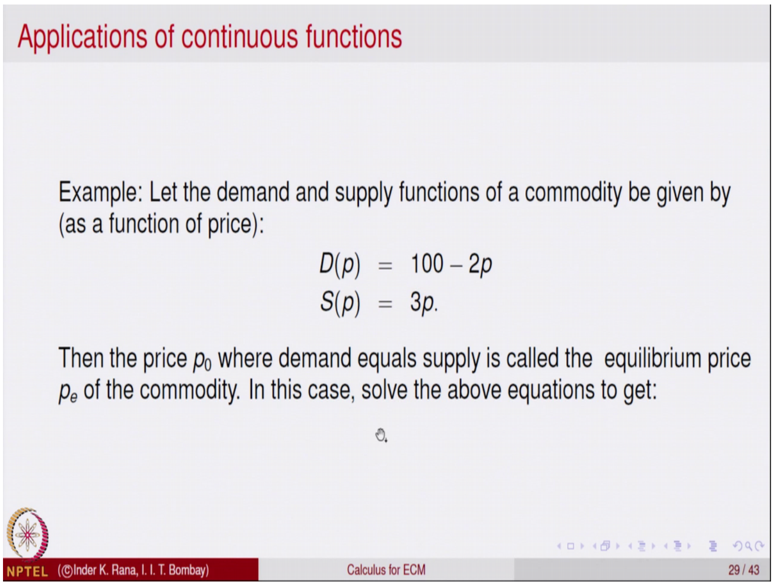

So, the demand function demand is a function of the price. So, demand is equal to 100 minus 2p and the supply is again a function of the price and that is equal to 3p. So, essentially this, this is a system of two linear equations in two variables and one can find what is price where demand will be equal to supply. So, that is what is called the equilibrium price the price p 0 where demand equals the supply is called the equilibrium price and that value is called p e of the commodity. If in this case if you want to solve this equations we can put the value of p from one equation to the other and solve.

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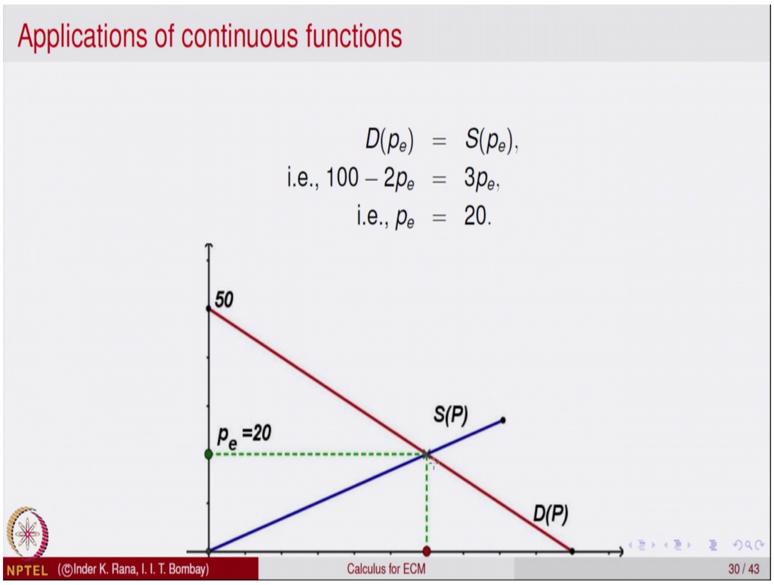

So, D p is equal to S p e demand rate is equal to supply that is another way of doing it and that will give you that this equation demand is equal to 100 minus 2p is equal to 3 times p of e. So, that from here we had p of e equal to 20. So, when the price is equal to 20 demand will be equal to the supply and there will be equilibrium.

And there is I want to illustrate this in other way as follows. So, let us, geometrically this system of two equations when you solve you get this price p e equal to 20. So, this red one is the demand curve that is a linear curve under minus 2p e and the supply is equal to 3p e. So, that is a line linear equation pass in the origin. So, these are the two graphs of these two equations D p e, and the D p e and S p [e]. So, wherever they cross that is a point where that is a point p of e. So, that is common point that is a point of equilibrium when demand equal to the supply. So, that is p of e. So, that is solving it geometrically or solving it by system of linear equations.

Let us look at another way of solving this namely.

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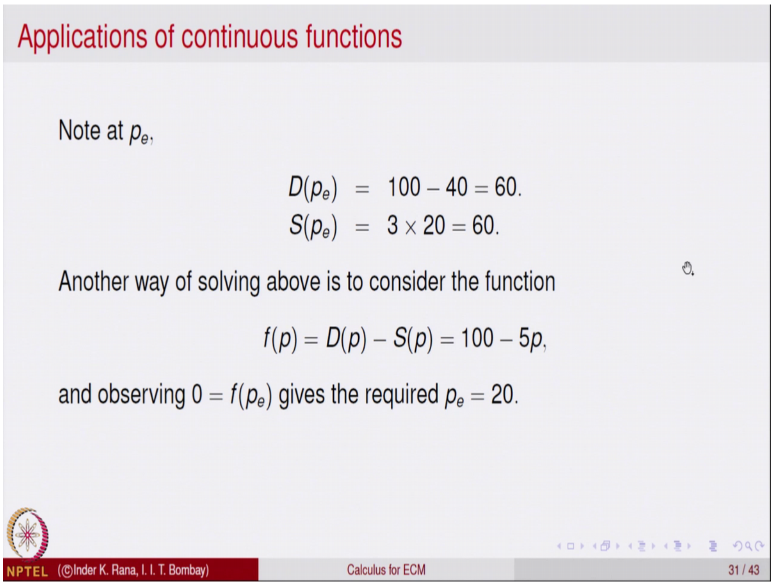

Let us look at the two equations right and at the price when p e is equal to 20. So, you get both to be equal to demand is equal to. So, that is finding at what when the price is what is a demand and what is supply both are equal to 60.

Another way of solving this would be that let us consider the function f of p define a new function which is demand minus the supply. So, demand is equal to 100 minus, so demand function if you recall was 100 minus 2p and supply was equal to 3p equal to 20. So, let us put those values in the two equations. So, let us find f p which is D p minus S p. So, simplify this 100 minus 5 p.

So, for this function if you want D p equal to S of p that is same as saying that we should have a point where f of p is equal to 0. So, what we have to show is and, this is equal to f of p is equal to 0. Now, note that D of p is a linear function. So, that is a continuous function as of p is a linear function that is a continuous function, right. So, and you can observe that from here if p is something say 100 then this value is negative right p is 100 then its value is negative. If p is equal to say 5 then this value is positive. So, there are points where the function takes positive values and there are points where the function takes the negative values and it is a continuous function by intermediate value property there will be a point where will have f p equal to 0, demand will be equal to supply.

But in general finding a point where this f of p is 0 for a continuous function, say intermediate value property just says that there is a point where the function will cross the x axis if it is positive at some point and negative at some point, but it does not tell you what is that point. So, this is purely and an existence theorem which ensures the existence of equilibrium for example, it does not say what is the point of the equilibrium. So, that is one has to find some other ways of locating that point. In our case it is not very difficult because the functions are simple. So, this will be so because at some point so we can find out this points right because linear functions, its f of p, it is also is a linear function. So, you can find out that.

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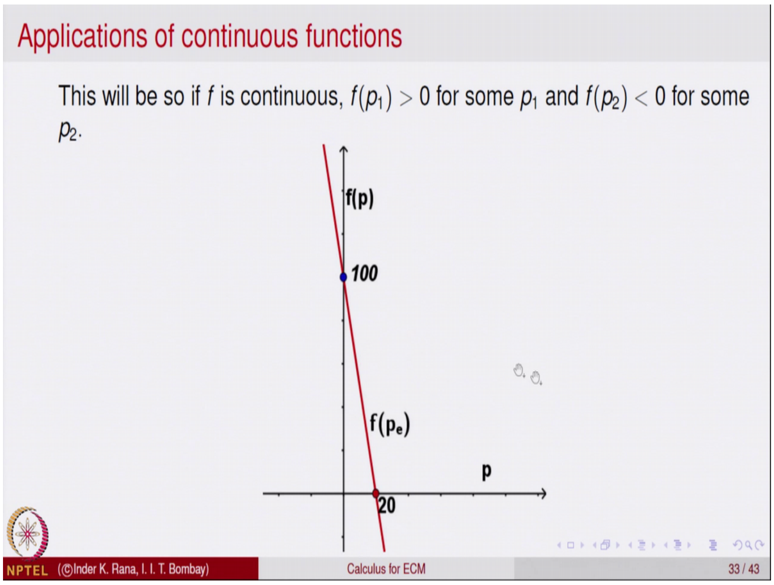

But in general it may not be easy. So, but at least in many complicated problems where you want to show that there will be a point of equilibrium this distance theorem is quite useful. So, intermediate value property of continuous functions comes handy even you want to show the existence of a point where the value is 0.

So, there is we said that the property of intermediate value properties says that the image of a interval, is a interval under continuous functions here is slight strengthening of that result.

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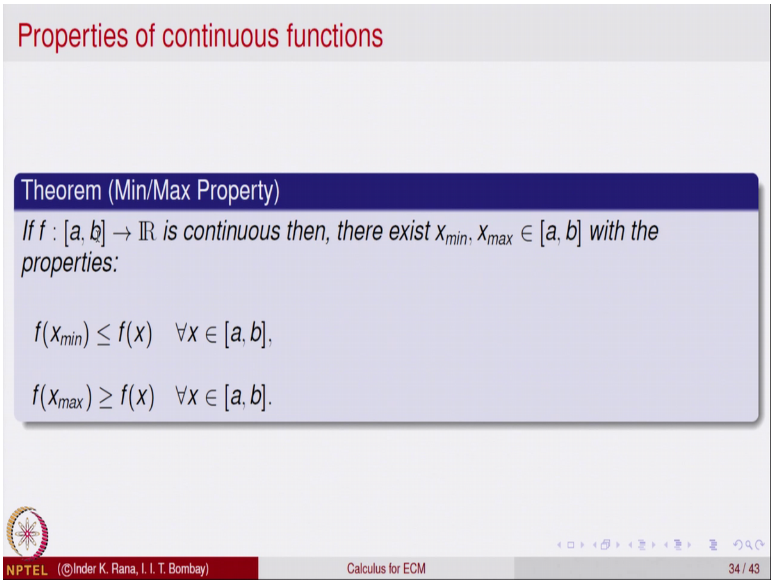

It says that if the domain is a close bounded interval a b. So, note that here is a closed bounded interval. So, f is a continuous function on a close bounded interval then there exist points let us call two points call x min and x max belonging to the domain a b such that f of x the value at every point in between is bigger than f min and is less than or equal to the value f max.

That means this also indicates that for a continuous function there are bounds that the range of the continuous function has to be bounded by the value f x min and f x max. That means, there are two values you can call this as small m say and call this value as a capital M. So, let us just write this property. Let me draw it a picture of this.

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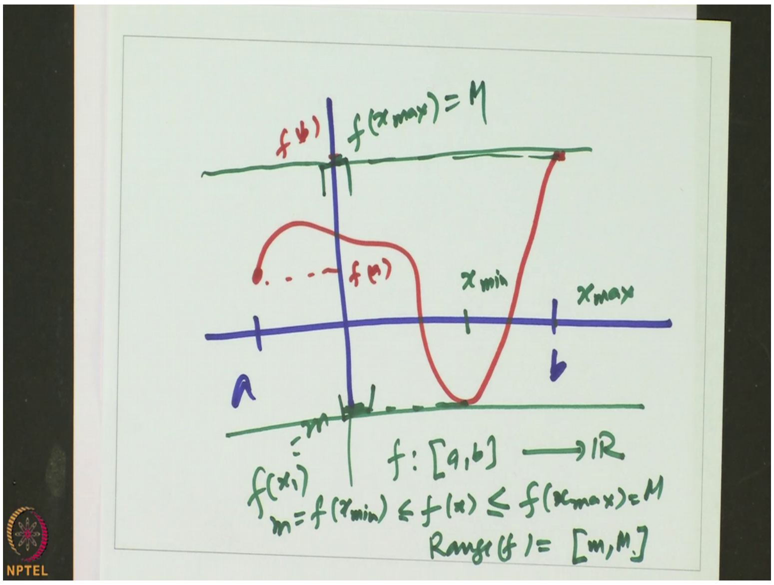

So, let us say this is the function say this is the point a and this is the point b and let us say the graph of the function is something like it goes like this. So, this is a value and this is the value this is the value. So, this is the point f of a, and this is the value f of b.

So, saying that this is a continuous function. So, there is no break in the graph of the function and what we are saying is there is a point where the function takes the minimum value. So, in this case for example, at this point if you call this as x min, this is the value let us call it as m which is nothing but f of x 1 and there is a point obviously, in this case this is this point. So, this is also f of x max. So, this point is also equal to x max right.

So, it says that the graph of the function will lie between these two lines. So, that is what it says. So, if f is continuous function on a close bounded interval a b to R then it says that f of x for every value will be less than or equal to f of x max and will be bigger than or equal to f x min means. So, another also this also in says that the range of f is equal to if I call this as small m, call this as capital M, so small m to capital M. So, this is the value small m and this is the value capital M. So, this theorem says that if f is a continuous function on a close bounded interval right a b then it is bounded; that means, there are two numbers small m and capital M such that the range of f is equal to small m to capital M, that also mean that the smallest value is attained at some point.

So, there will be point x min which will be equal to m small m and x max which will be equal to. So, one way of just understanding this theorem is that for a continuous function on a close bounded interval is bounded and attains it maximum and the minimum values. So, we will see applications of this theorem later on when we want to do optimization problems in economics commerce and management problems. So, this is what is called the min max property of continuous functions right.

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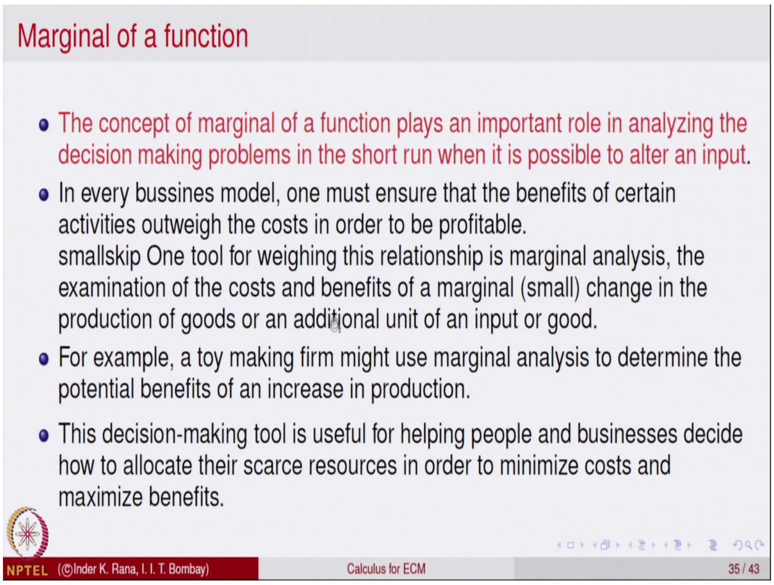

Next we will look at a concept in our subject what is called the marginal of a function. So, what is marginal of a function? This concept of marginal of a function is a important concept and it is applicable is useful in analyzing decision making problems for a short run when it is possible to change an input the input can be changed for a short run.

So, we will try to make it precise mathematically what it means. So, what we are saying is in every business model one wants to ensure that the benefits of certain activities outweigh the costs in order to be profitable right. So, that is what the any business is about. So, one tool for weighing this relationship is called marginal analysis. So, how is it this, what is this analysis let us look at that. The cost and the benefits of a small change, marginal word in English means small which can be very marginal is something which can be neglected in an English, but if you are saying that the cost and the benefits of a marginal change in the production of good for an additional unit of input for a good. So, what is the relationship of this visa be cost and the benefits.

So, let us for example, let us take a toy making firm might be using how they can use marginal analysis to determine the potential benefits of increase in the production some form is making twice and they want to know how this tool what we are saying can be used in analyzing increase in production. So, let us say these are applications of this decision making namely that we can allocate our resources in a better way. So, this is all why these tool is useful marginal analysis.

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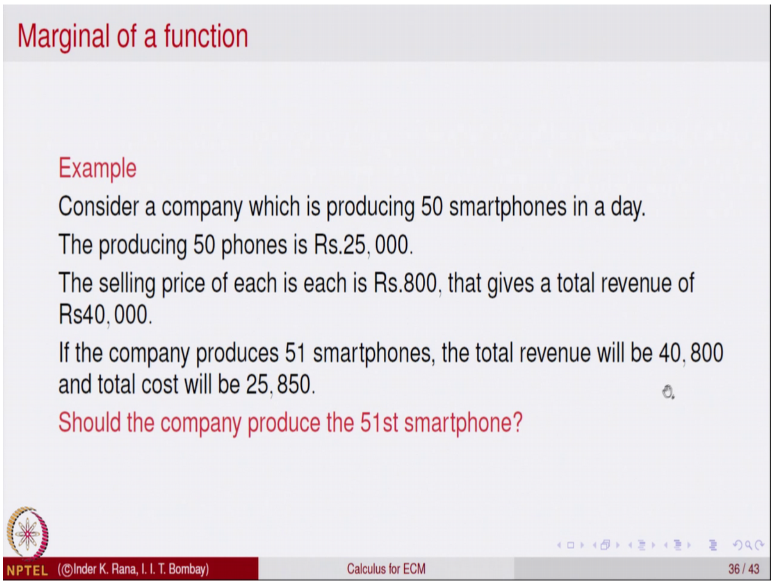

So, let us see what this marginal analysis. Let us look at a company which is producing 50 smartphones in a day. So, the company is producing smartphones and they are their capacity at present they are able to produce 50 smartphones in a day.

The cost of producing 50 phones is rupees 25,000 right. So, that is the cost of producing. So, selling price of each smartphone is rupees 800. So, there are 50 smartphones. So, that gives them a revenue of 40,000 rupees by selling those 50 phones right. So, now, the company, if the company produces 51 smartphones if they increase their production by one smartphone the total revenue would be one more phone is sold earlier it was 40,000 one more phone sold means 800 rupees. So, the revenue will be 40800 rupees. And what will be the cost? The cost of producing the cost, total cost will be 25,850 right because 50 phones. So, when we divide by that cost of producing each. So, total cost will be 25,000 whether we one additional cost. So, should the company be producing one more smartphones or not.

So, this is the cost of producing 51 phones this is a revenues I get. So, what we get is the marginal benefit that. So, marginal means one more phone is produced 51st phone is produced, so 40,800 that is what they are getting and minus 40,000, total will be 800. So, that is a marginal benefit they are getting.

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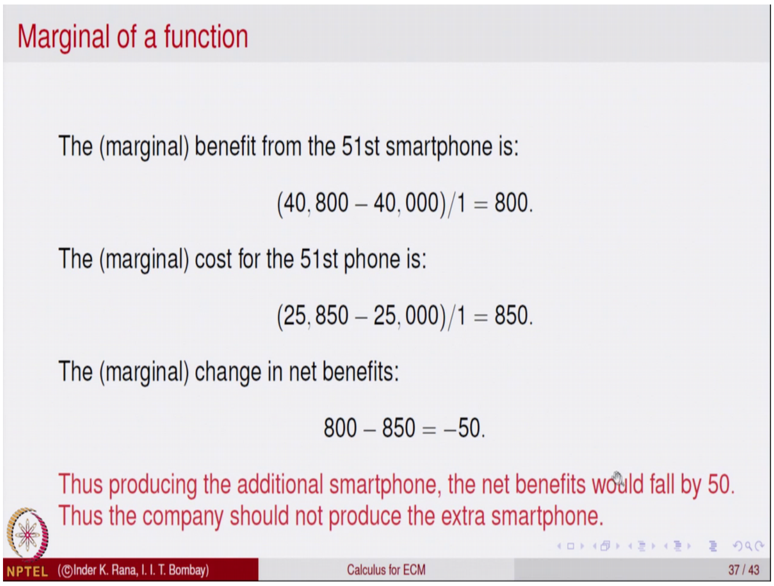

And what is the marginal cost? Cost is 25,850 minus 25,000 right. So, the cost is marginal cost is 850. So, that clearly indicates that there will be at a loss of rupees 50 if they produce one more phone. So, just by analyzing the production of one more keeping the price selling price same leads them to decide that they should not be producing more phones. So, the net benefits fall by 50 rupees. So, they should not be producing more phones.

So, this is how marginal analysis is used in deciding whether you should be doing increasing the production or not. So, let us formally define for a function y equal to f of x.

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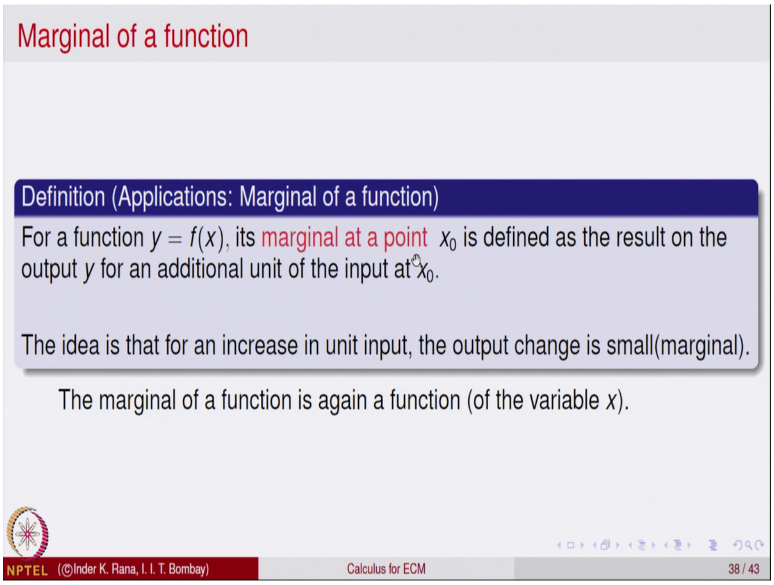

It is marginal at a point x 0 function as a domain x will also some x. So, formally we are defining now what is called the marginal of a function is defined as the result of output right. So, defined as the result on the output by a additional unit of input; that means, at the point x 0 if x 0 changes to x 0 plus 1 what is the change in why that comes right. So, that is called the marginal of the function y. So, note that, what we are saying is increase in a unit input right will output will change marginally. So, note that when x is change to x plus 1 y will change to some value.

So, the marginal is again a function of the variable. So, given a function its marginal is again a function right, a function of the same variable x.

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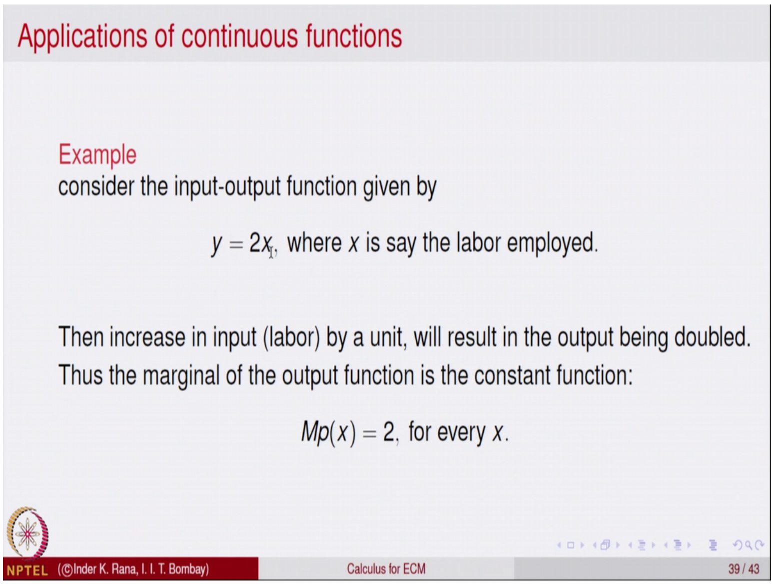

So, let us look at another example to illustrate this. Consider the input output model for a function given by y equal to 2 of x where x is the labor employed. So, this is the input output model input is the labor and y is the output for a labor, so is a linear function y equal to 2 of x. So, it clearly indicates this is if we increase x y is going to increase because the slop for this linear function is positive, so this going to increase. So, increase in output anyway from this formula it is clear that if I increase the input x by 1 then output is doubled it is 2, right. If x is f x goes to x plus 1 right then this will go to 2 x plus 2. So, y will increase by 2 units. So, doubled in a sense that if y was something.

So, increase in one gives a two times increase in y. So, the marginal of this is equal to 2 for every x right. So, marginal is equal to 2 for every x. So, it is quite clear right. So, when x goes to x plus 1 the difference right y will change by 2. So, Mp of x is equal to 2. So, Mp indicates this is the marginal of p, this is what is called marginal of production right output or the production for every x increase, so that is 2. So, is a constant marginal is constant function.

However, note notice that more output is generated even when the large inputs of labor are there even if other are fixed. So, in this model if x keeps on increasing the y keeps on increasing right. So, because the marginal is positive and that is 2. That does not seem to be a very good scenario, but because normally increase in the labor and keeping everything else constant does not increase the production this seems a contradictory to the business model.

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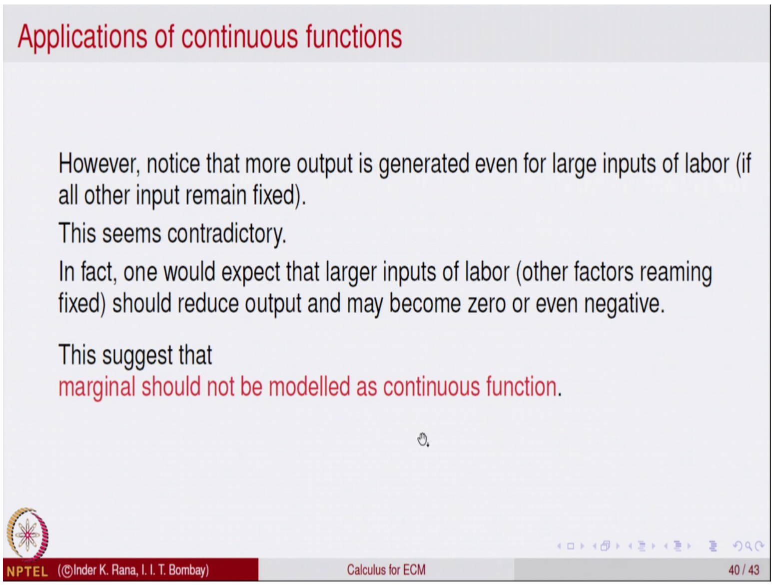

So, what is to be done? So in fact, one would expect that the larges inputs of labor other factors remaining fixed should reduce the output and it may become 0 output may become 0 right.

So, let us look at some illustration. So, marginal should not, idea is that marginal should not be modeled as continuous function. To give a illustration of this let us look at a particular example see the example is that of a thermal power station which has a capacity of 1500 mega watts per hour as and the input is coal. So, it is a thermal power station, input is coal, coal is put in to generate heat and then steam and then gives the electricity run the turbines. So, electricity produce is produce and the capacity of that thermal power plant the maximum it can produces 1500 mega watts per hour right.

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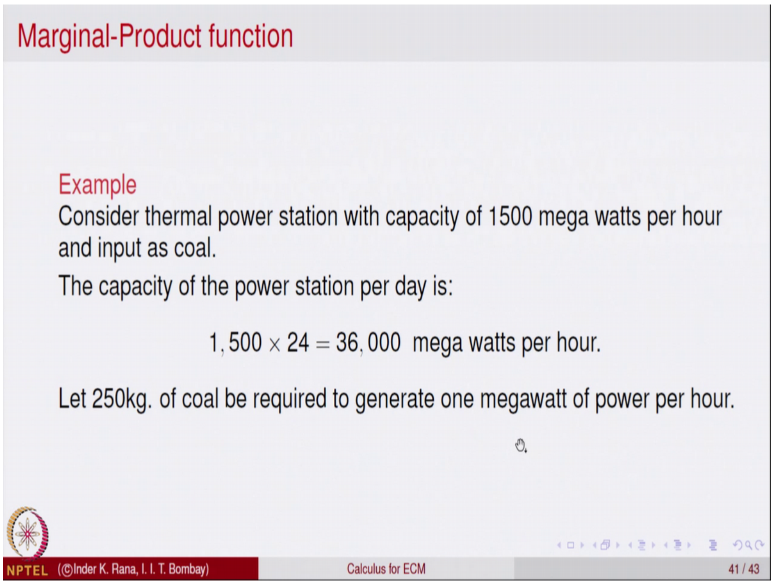

So, in a day, let us calculate how much the power station can produce in a day the maximum it can produce in a day is 1500 per hour, so 24 hours, that gives 36,000 mega watts of. So, this is this should be per day it is not per hour. So, that should be per day right.

So, let us say that 250 kg of coal is required to generate 1 megawatt of power. So, this much of coal is required to generate 1 megawatt of power.

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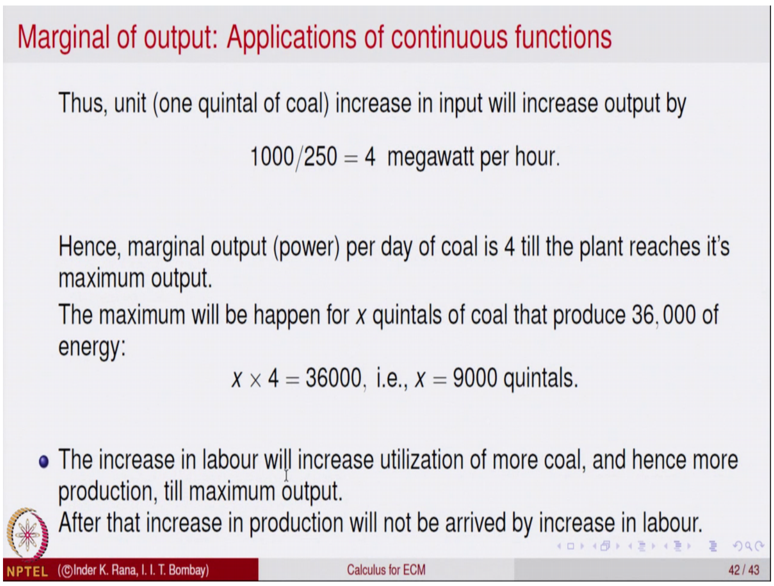

So; that means, the unit of quintal of coal increases in output will increase output by. So, if put a unit, 1 unit of increase; that means, 1 quintal of coal if I we increase then the output will increase by 1000 divided by 250 that is four megawatts per hour right. So, that is clear. So, keep in mind that 250 kg of coal you require to generate 1 megawatt of power right.

So, 1 quintal of coal, we divide by that, so that is 4 megawatt is, so that is. So, if we 1 unit increase will give me 4 megawatt per hour. So, what does that mean? That means, that the marginal output per day for coal is 4 till the plant reaches its maximum output. So, this is the marginal right. So, marginal for the thermal power plant if we increase the input that is coal by 1 unit then it increases by 4 right. So, this is 4 megawatt per hour, so that is till it reaches maximum. So, that when will it reach maximum, it will reach maximum for how many quintals are required to reach the maximum, maximum capacity is 36,000. So, x into 4 is 36,000 that is 900 quintals. So, when x is 900 quintals the plant would have reach this maximum capacity.

That means what? That means, after that even if we bring in more coal increase then capacity is not going to increase the power is not going to increase, the production is not going to increase. So that means, what the increase in labor if you bring in more labor to bring in more coal the increase in labor will increase utilization of more coal and hence more production till the maximum is reached right, and after this increase in production is reached even if there is increase in labor the output is not going to increase because that is a maximum capacity of the plant.

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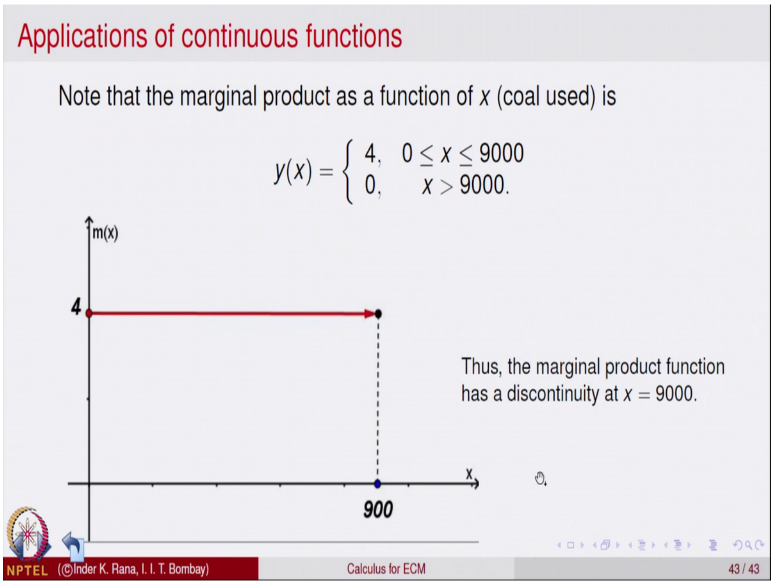

If we look at the y the marginal product, so y x is the marginal. So, marginal remains 4, this is a marginal of the product. So, marginal remains 4 till it should be 9000 here, till 9000 is reach and then suddenly the marginal becomes 0, even if you increase in x there is not going to be any increase in the marginal is not going to change right. So, this is what the marginal curve looks like for this scenario. So, there is a discontinuity for this right. So, there is a discontinuity at the point x. So, that is good enough.

So, normally marginals will be discontinuous. So, in a model for economics one tries to bring in models in economics so that the marginals are not continuous functions. So, that is a conclusion of this analysis.

So, we will continue with our what we have done till now is looked at the notion of limit, notion of continuity of functions and we have looked at some properties of special properties of continuous functions. Namely, one was the intermediate value property and the other was the min max theorem and both of them will be used later in our analysis. And we have tried to define what is called the marginal at present in a very in formal way here. We will come to the marginal when we discuss the notion of derivative a bit later. At present we have what we have looked that is marginals for mainly linear functions they become, they tend out, they become they turn out to be constant functions right. So, we will continue our study of functions and their properties in the next lecture.

Thank you.