Calculus for Economics, Commerce & Management
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Lecture – 18
Application of continuous functions

So, welcome to this lecture, as you recall we have looked at the concept of limit and continuity in the previous lectures and now time has come to look at their some of the applications of these concepts in our topics of economics commerce and management. So, let us me begin with an example.

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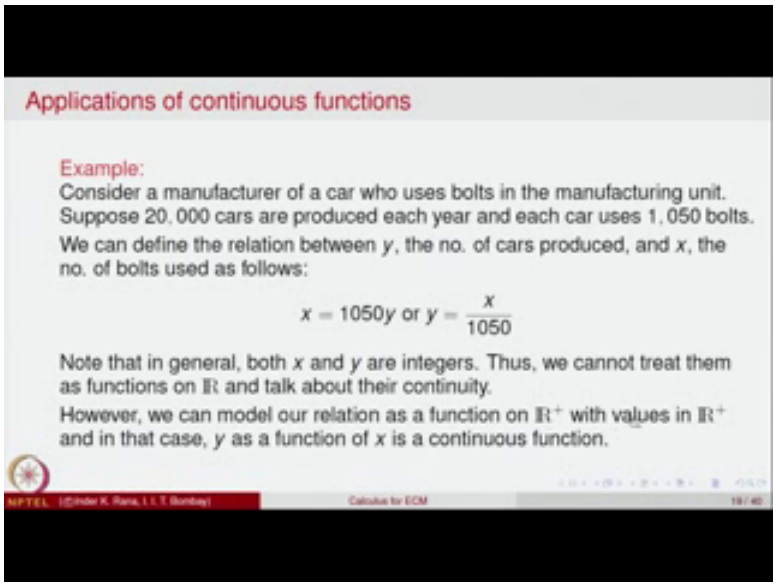

So, consider a manufacturer of a car who uses some bolts in the manufacturing unit and the data given is that 20,000 cars are produced each year right these many cars are produced each year by that manufacturer, and each car uses 1050 bolts for the manufacture of each car. So, if you want to interpret this relation between the 2. So, these many cars are produced and each car uses these many bolts. So, what is the total number of bolts that is being used?

So, we can define the relation between y, the number of cars produced and x the number of bolts used as follows. The total number of bolts that are used is x is equal to 1000 into y where y is the number of cars produced or we can interpret y in terms of x. So, y is equal to x divided by 1050. So, here y is the number of cars being produced and x is the

total number of bolts that are used to produce those many cars. So, each car using this much. So, the of course, this is a linear relation, but the only drawback in this data is the number of cars or the number of bolts are going to be numbers they are integers right they are actually positive integers. So, they are natural numbers. So, we cannot treat y as a function of x, as a function on the real line and talk about continuity of this function.

Of course one can talk about continuity, but that really does not make sense in this scenario. So, for, this goes to the; what is called the modeling of the problem for modeling of the this problem number of cars being produced and the number of cars being produced a number of bolts being used for each one can treat them. So, that is a assumption one makes in modeling that for the variables which are at present discrete, they are taking values which are numbers we make them continuous and say this relation is followed for all practical purposes for mathematical analysis. So, that is one and then one can ask continuity, and then we know that every linear function is continuous of then it will be a y as a function of x will be continuous right.

So, in the modeling relation we can treat it as a function of R plus, 2 R plus y as a function of x and then it is a continuous function. This is very simple illustrations saying how from discrete data one goes to a continuous data right let us look at our next example.

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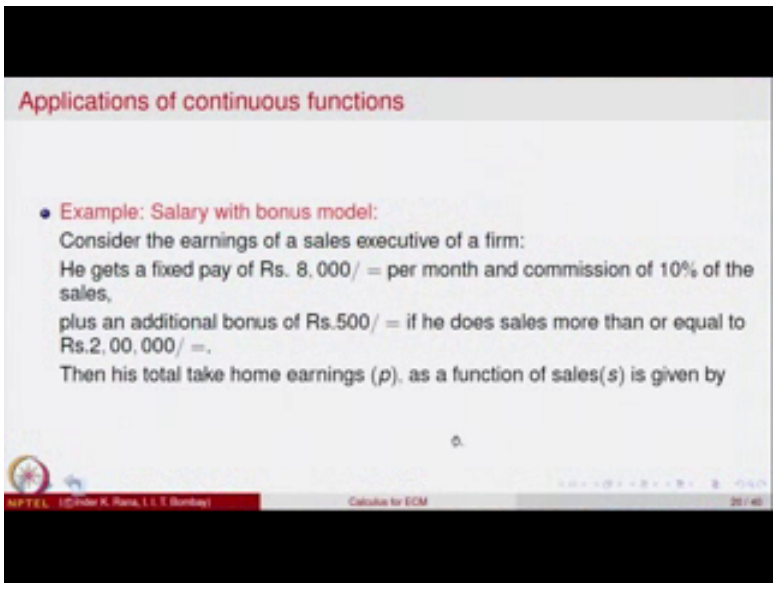

Our next example is model of salary with bonus model that is followed in many forms. So, the model is as follows namely consider the earning of a sales executive of a firm right and as some executive who is in the sales department of a firm. So, his earnings are as follows. He gets a fixed pay of 8000 per month. So, that is a fix pay being on the roles of that firm and commission of 10 percent of the sales.

So, whatever he is able to sell in the in a month he gets 10 percent of that added to his earnings right. So, 8000 plus 10 percent of the sales, and plus an additional bonus of 500 rupees if he does sale more than or equal to 2 lakh rupees in a month. So, there is a incentive bonus given to the executive, that if he sells more than goods more than worth 2 lakh rupees then he will get a 500 rupees of additional bonus for his take away take home earnings. So, 8000 plus 10 percent of the sales plus 5500 is case he makes a sale of at least 2 lakh in that year. So, let us see what is happening.

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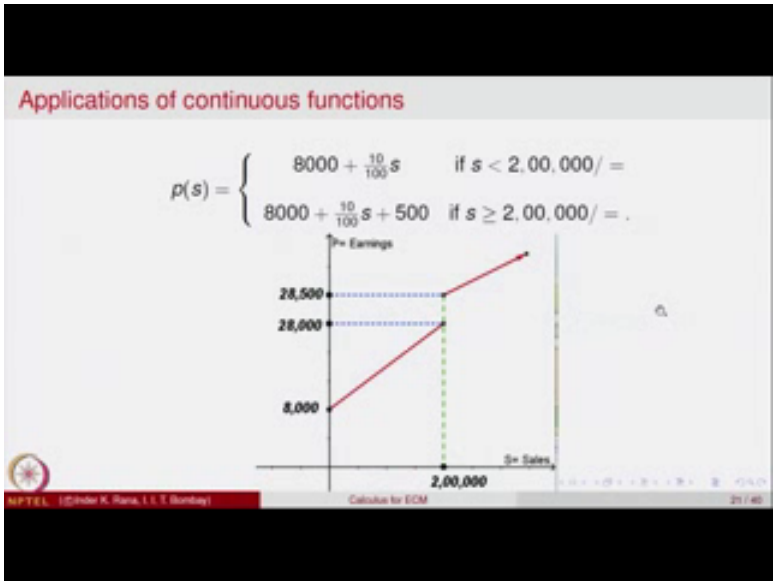

Then his total take home will be if his sale is less than 2 lakh, then 8000 plus 10 percent of the sales let us call sales as s. So, his take home depends on the sales he makes.

So, 8000 plus 10 percent of s that is his take home. If it is less than if it is greater than or equal to as soon as he crosses his sales cross 2 lakh he gets 500 extra. So, that is. So, this is the expression for his take home pay you can call it as p s is his take home pay is dependent on the sales right. So, it is 8000 plus 10 percent of s if the sales is below 2 lakh, and if it is 2 lakh and above then 500 more is added to it. So, let us draw a picture

of this a graph of this. In both cases it is a linear function right p s this is a linear function, a x plus b and this also is a linear function right. So, what is happening is this is the point where the things change sales. So, on the x axis we have put sales and on the y axis is take home earnings. So, if the sales is below 2 lakh if there is no sales at all he anyway gets 8000 rupees right.

And sales up to 2 lakh there is a 10 percent increase. So, this is a 10 percent increase his sales. So, it in keeps on increasing his take home till he reaches 2 lakh. So, at that time his sales will be 28000 right 10 percent increasing 8000 plus 10 percent. So, of 2 lakh that will be. So, 28000 right. Now as soon as he reaches this he gets a jump in his earnings, there is a jump in his earnings right 500 more is added. So, 28000 plus 500 and then again whatever more sales he makes it will keep on increasing right. So, this is the take home graph verses the sales in that scenario, where 8000 is the salary per month is the take home per month that is a fix thing, fix earnings per month and then there is a varying thing which is 10 percent of the sales.

So, there is sales 10 percent of the sales and as soon as the sales reach 2 lakh, he gets jump of 500 more and again proportionately same. So, this linear thing is parallel to this linear thing and if you remember that formula for the slope right slope is 1 over 10. So, its a increasing function. So, this will be parallel to this only there is a jump here. So, in this graph there is a discontinuity at the point 2 lakh in the sales, at that point there is a jump discontinuity for the function right. So, that is a as much as mathematics what is a interpretation of this from the economics point of view. So, let us take the example there are three employees in the company A, B and C and total sales of the month except on the last day the one month is left before the month is over.

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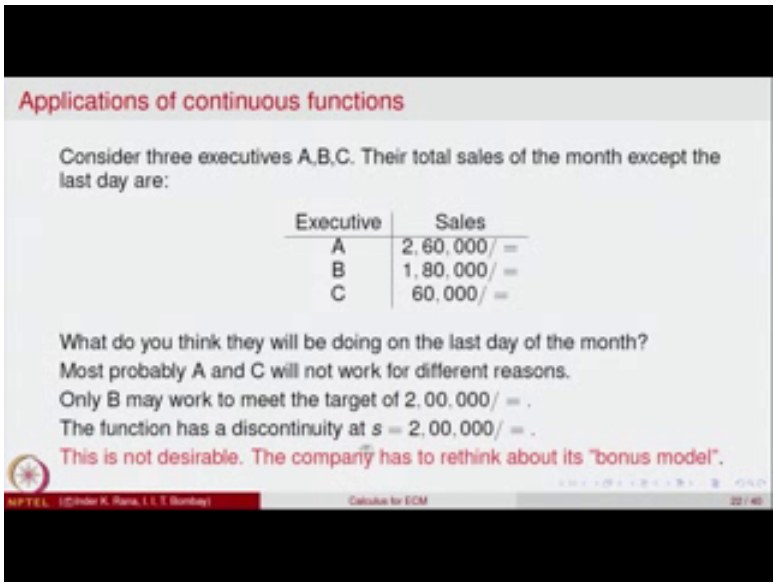

So, one day before the end of the month, this is their data of their sales that they have done. A has made a sale of 2, 60,000, B has made a sale of 1, 80,000 and C has only made a sale of 60,000. So, these are the three executives and that is the data of their sales on the last, but one day of the month the question is what do you think all this three will be doing on the last day? On the last day of their sales on the last day of their working what do you think these three will be doing? Well A has already cross the target of 2 lakh, and he already made 2,60,000 and if he does more sale in on the last day he will get only 10 percent of that more right. He has already achieved the bonus of getting 500 and also. So, most likely he will just relax on the last day because he is not going to add much to his take home salary.

So, he will probably relax the reason is he already achieved the target of 2 lakh and nothing much to be achieved where as if you look at B he is nearing the target he is 1, 80, 000 d he is just 20,000 short of 2 lakh. So, more likely that on the last day he will make a last bit of effort to make a sale of 20,000 at least more. So, that he gets 500 rupees bonus also. So, hopefully B will be out in his territory trying to make the sales of at least 20,000. Whereas, what happens to the sales executive C? H e has made only a sales of 60,000 in that month and even if he is able to make some more sales on the last day he will get 10 percent of that, and hopefully well in a day you over the whole month he is able to sell only 60,000.

And in a day may he may not be able to make 1,40,000 what is the gap to attain the bonus. So, more likely he will also take the life bit more leisurely on the last day and he will also relax because nothing much he is to be gained by toiling and right. So, A and C both will not be working most likely on the last day right only B may be working on the last day. So, this is what is happening to this model of bonus most probably A and C will not work for different reasons and only B may be working right. So, this is happening because the function has a discontinuity at the point 2 lakh when the sales is 2 lakh. So, it is highly suggested to the firm that tweak your bonus formula somehow. So, that this kind of scenario is avoided and everybody works, what they do its they are choice this is not desirable to have a discontinuity in the take home and the sales the company has to rethink the this their bonus model right. So, that is how mathematics helps to make suggestions for economic changes right. So, let us look at another model, let us look at in his social security setup the government decides to give a non employment allowance of 700 rupees per month to all unemployed adults.

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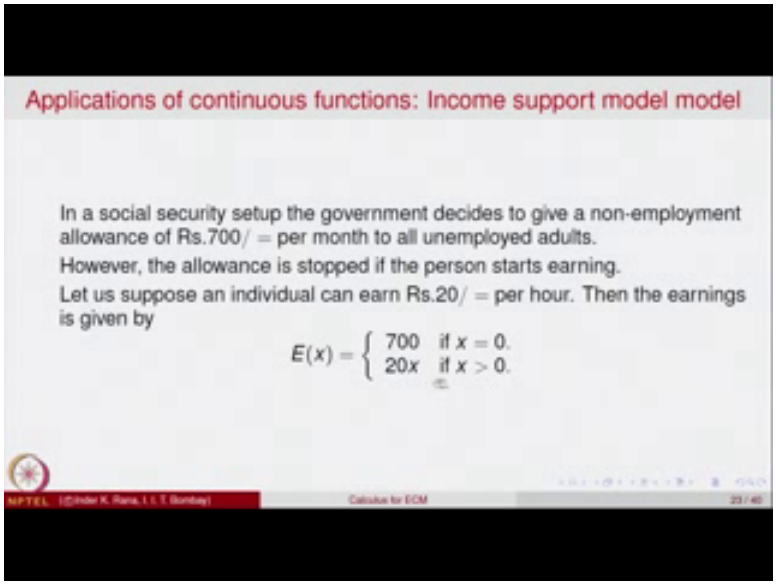

So, the government says that is a populist government which says that everybody who is not employed will get 700 rupees per month as a unemployment allowance right. So, let us see what is happening in this scenario; however, government says if that is the case nobody will be working right says; however, the allowance stop if the persons starts earning right either you just feel happy in your life and you sets 700 rupees right or you

start working and then the allowance will stop right. So, what is happening in this case right?

So, let us suppose an individual can earn rupees 20 rupees per hour right we are just making a assumption say that a individual who can earn with his skills 20 rupees per hour then [earning will be given by the relation between his earnings and so, the relation between his earnings and working per hour. So, E X is the earnings that you can get, 700 if x is equal to z0o; that means, he is not working at all right if he works for x hours x is bigger than 0 then he gets 20 x right. So, let us plot this relation it is a linear relation at 0 the value is 700 or x bigger than a value is 20 x. So, this is a linear function for x bigger than 0 with slop as 20.

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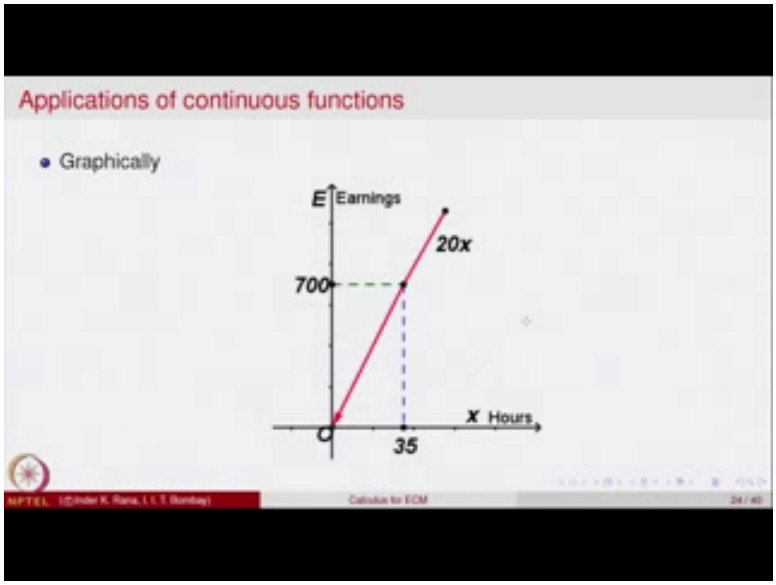

So, when we plot this is what you get right. So, this is when it is 0, this is a earnings 700 right.

So, we have just plotted this part of the thing just to indicate; what is this 20 x. So, at this point (Refer Time: 15:02) starts here right because if he is not working at all he will be getting 700 anyway right. If he is not working at all he will be getting 700. So, if he start working x hours right, then to get 700 rupees right he has to work for 30. So, his earnings. So, if this is the earnings for that person who is working right. If you works for 34 5 hours 20 rupees per hour right then he will get only 30. So, to earn 700 rupees which is a non employment allowance given to that person, he has to work for 35 hours

why he should work at all probably he will just relax 35 hours he has to make put in the work right that is quite a bit of it to get 700 rupees you have 20 rupees rates.

So, most likely he will not work at all.

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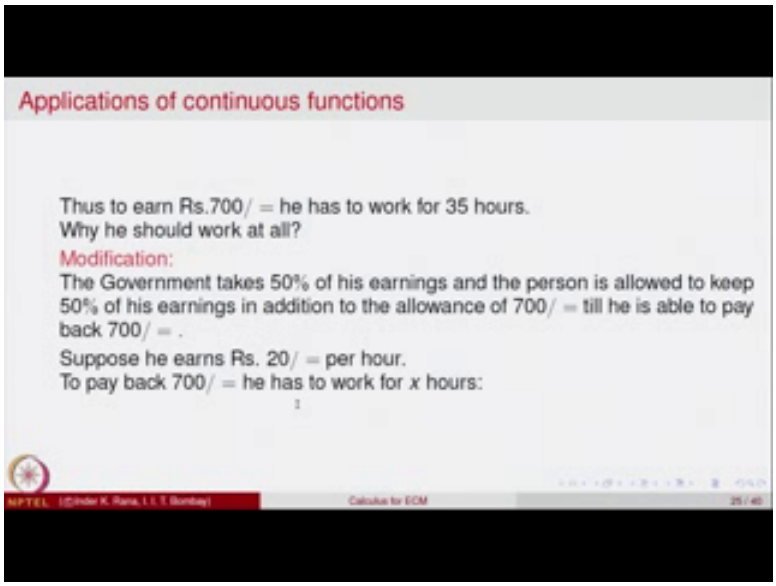

So, this model of the government needs thus to earn 700, he has to work for 35 hours which is not a very good scenario as for as the government is concerned. So, a modification is suggested the government takes 50 percent of his earnings, and the person is allowed to keep 50 percent of the earnings in addition to the allowance of 700 till he is able to pay back 700. So, this is the suggested model as an unemployment allowance is going to get 700 rupees and now he is starts working. So, he works for x hours. So, 20 x he will get right the government will take. So, 700 plus 50 percent of 20 x right that he can keep. So, his earning is 700 plus 20 x divided by 2 right that is earning till the his payback is 700 rupees right.

So, till his earnings become as much as he is able to payback that is 700 rupees, and then he can keep all his earnings right. So, let us plot this model again suppose he earns 20 rupees per hour again same model right. So, then what will happen to way back 700 rupees right how many hours he has to work he has to work for 70 hours right

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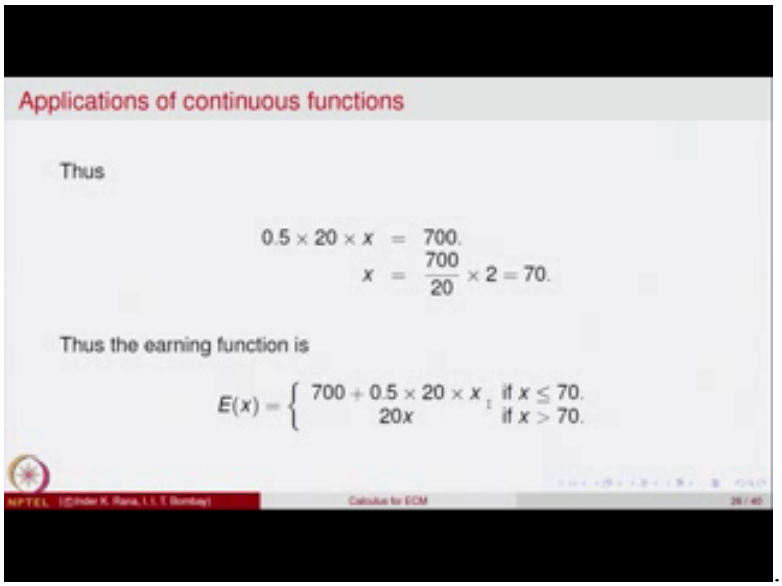

Because only 50 percent he will he is allowed to keep, 50 percent the government has to get back so; that means, we want to find x such that if 20 rupees per hour he is able to earn and the 50 percent of that should be equal to 700. So, that is a equation that we get right so; that means, x is equal to 70 so; that means, if he is able to work 70 hours a day right then he would be paying back the government till then he will be reach the target of paying back the government 700 rupees that is unemployment allowance.

After that if he is able to work more, then he is allowed to keep everything. So, this is the model suggested tweak model of earnings and social security system, 700 rupees he gets anyway whether x is 0 or whether x what x is a number of hours and he gets 50 percent of 20 c. So, that he much he will get if his working less than 70. If he crosses his working hours cross 70 then he gets 20 x; that means, the government is sort of encouraging it the person to work more than 70 hours a week. So, that he can take home everything that is working.

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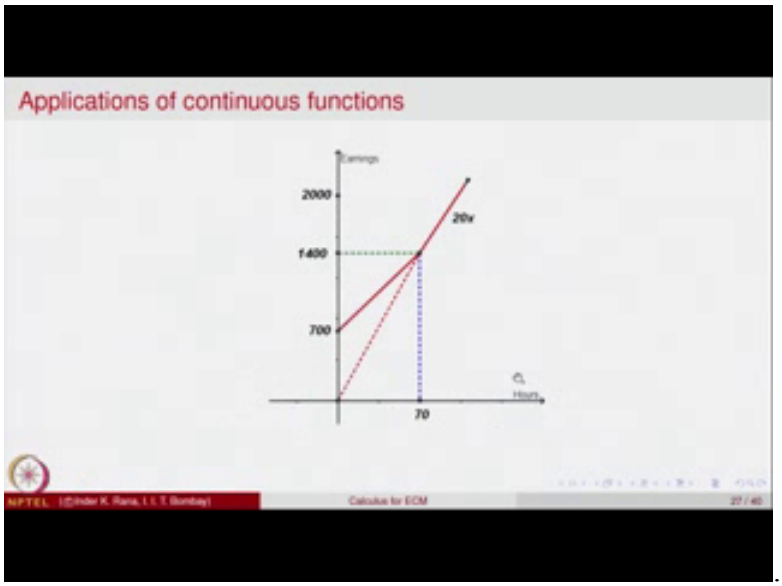

So, in that case what will be the model? So, here is the earlier it was this line right as the earning and so on now what happen? If he does not work he is getting 70. 700 anyway right if he starts working then his take home keeps on increasing till he is able to work at 70 hours per.

So, at 70 hours when he is able to work his earning will become 1400 and after that his increase becomes much better because he is does not have to give back 50 percent is equal to 20 x. So, it is a same old graph that starts here. So, in this model the government is giving an incentive for him to work and reach the target of working 70 hours. So, this is the way one can use continuity to ensure that the people in a mod social security model right. So, there is continuity in the graph let us look at. So, we have seen some applications of continuity, we will see is a more in the lectures to come continuous functions have some very special properties which would like to discuss so that we are able to use them why continuity is important. The first properties called the intermediate value property.

It says if f is a function from a interval a b to r, and it is continuous function if alpha and beta are such that f of x o1 is alpha and f of x 2 is beta.

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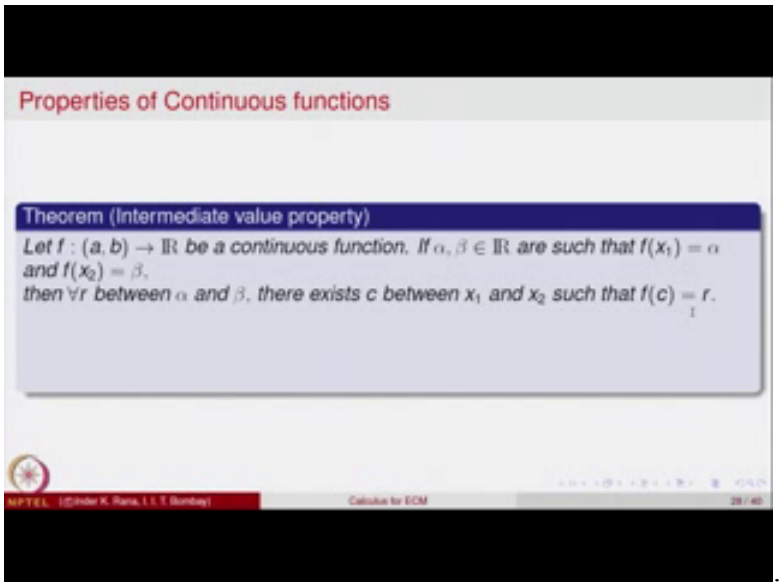

Then for every r between alpha and beta there is a c such that right f of c is equal to r. So, understand this as intermediate value property that if a and b are a f is a function on a interval a b, and the value alpha is taken at some point x 1 and the value beta is taken at some point x 2, then every value r in between alpha and beta is also taken by the function at some point. So, let me draw a graph of this function and try to explain it pictorially what we are saying.

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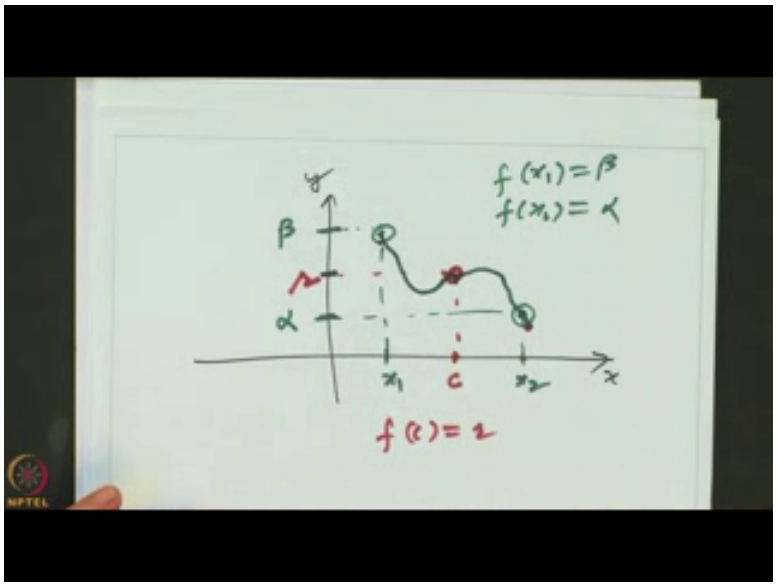

So, this is x this is y right. So, here is a value alpha and here is a value beta this value alpha is taken at some point say x 1 and beta is taken at some point x 2; that means, this point is on the graph and sorry. So, x 1 is beta.

So, let us say x 1 is alpha. So, this point is on the graph. So, f of x 1 is equal to beta f of x 2 is equal to alpha. So, these 2 values are taken by the function and the theorem says that if I take any point in between let us call this point as say r, here r is any point in between then there is a point c such that f of c is equal to r; that means, this is the point; that means, this if this point is on the graph, this point is on the graph, and the function is continuous then there it graph must pass through this point right so; that means, the graph should look like probably this. Another way of saying the same thing would be the following let us say this is the graph of the function ok.

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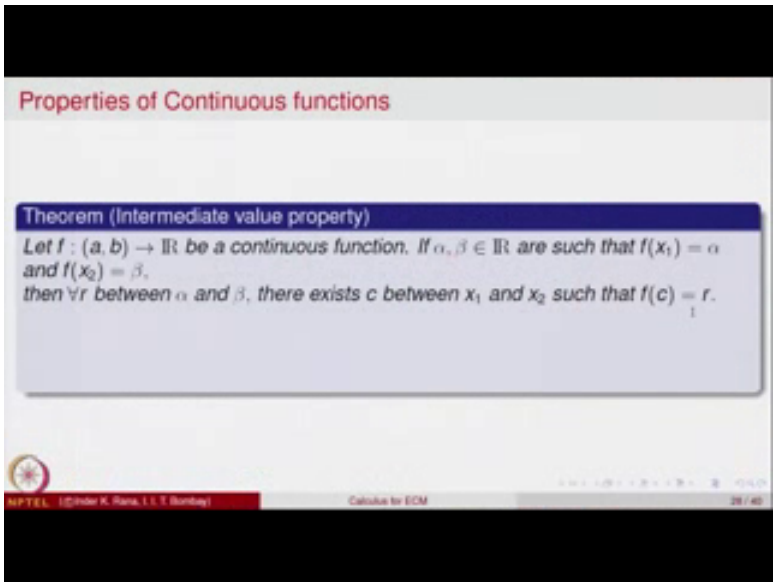

So, another way of understanding this intermediate value property is the following namely let us take the same picture.

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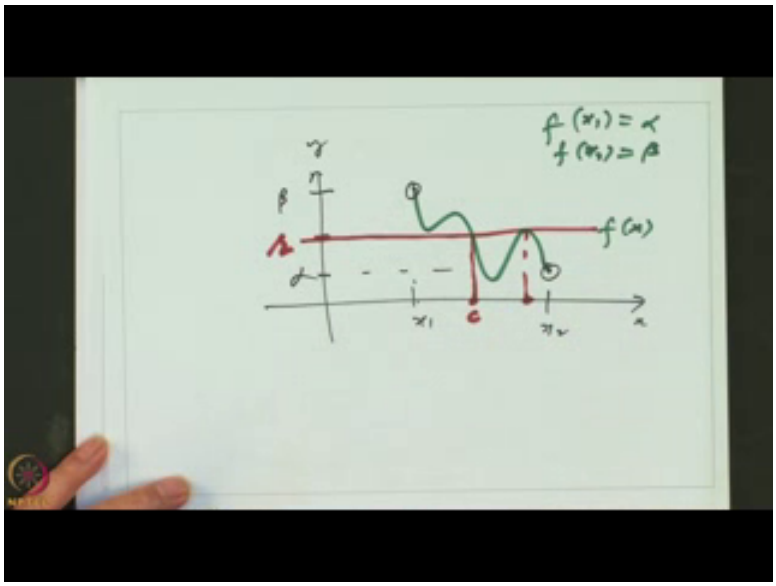

So, this is x, this is y and this is x 1 and this is x 2 this is a value alpha. So, this is point and at x 1 the value is beta, this is alpha, this is a point right. So, these 2 points on the graph, this let us say graph is equal to this this is the graph of the function f of x. So, f of x 1 is equal to alpha and f of x 2 is equal to beta. We want to say that if I take any value r in between alpha and beta then this value is taken by the function at some point geometrically; that means, what; that means, if I draw this line they then it must intersect the graph at some point.

Because then this will be the point c where the value r will be taken (Refer Time: 24:14) the picture, I have drawn there are 2 points this point and this point where this value is taken. So, theorem says that there will be at least one point where the value will be equal to r.

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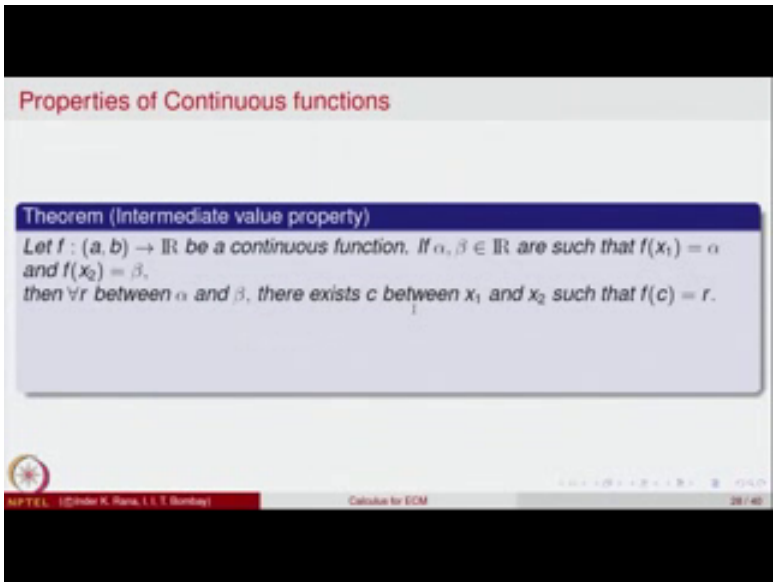

So, this is what is called the intermediate value property, because if this value is not. So, what does that mean? If this value is not taken, let us also look at the negation of this statement if this value is not taken what does that mean? So, let us look at this graph again this is x 1 and this is x 2.

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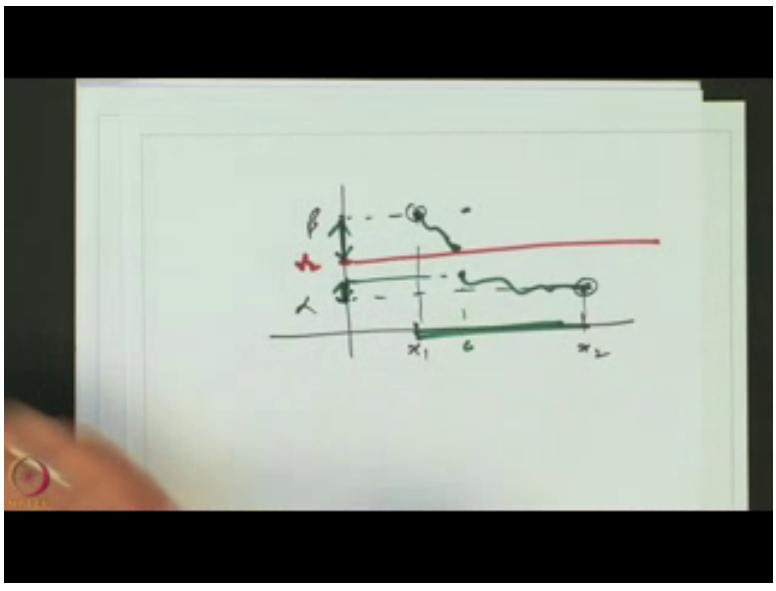

So, this is the point beta and this is the point alpha. So, this is the value right now the graph the function has to start right at x 1 it has start here and go here and let us say there is a point where this value is not taken.

So, this value r suppose it is not taken; that means, what? That means, the graph cannot intersect the line this red line. So, it has to start here and go here. So, it goes somewhere now it cannot cross so; that means, what it can come closer and closer, but it cannot cross it has to jump right. So, maybe it will start somewhere else right. So, it will start somewhere say for example, this point and then go here. So, there will be a jump in the graph of the function. So, that point c right there will be some point in between where it will have to jump over that line. So, if the function starts and it has does not have to cross then it is to jump over and then goes here. So, there will be a discontinuity in the graph of the function. So, for a continuous function it must and this also means that see if this is the case if this is the scenario, then what is the range of the function.

So, the range of the function will be this part and this part, where the domain is this the range is 2 parts; that means, intermediate value property ensures intermediate value property ensures that the images of intervals are intervals or this is the mathematical way of saying that the graph of a continuous function cannot have breaks right. So, this is the mathematical version of the intuitive idea when we say there is a function is continuous, the graph is a continuous curve image. It means if you want to plot that if you start plotting you started one point then you have to reach the other point without lifting your pen. So, there should not be any break in the graph of the function that is mathematically it is because of the intermediate value property. So, we will continue this study of some more properties of continuous functions and their applications in our next lecture.

Thank you.