Calculus for Economics, Commerce & Management
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Lecture – 17
Computing limits, continuous functions

And let us begin todays lecture by recalling, that we have been looking at the concept of limit of function at a point. We had seen what the definitions of limit of a function at a point, and we have seen the concept of left limit. And the right limit and we have seen examples of functions and techniques of computing limits. Today we will start with looking at some more theorems in calculus, which are useful in computing the concept of limit. We will not prove these theorems because they require more of analysis, but we will treat them as tools for computing limits. So, let me state the first one which is called the algebra of limits. Algebra of limits for functions is very much similar to the algebra of a limits for sequences.

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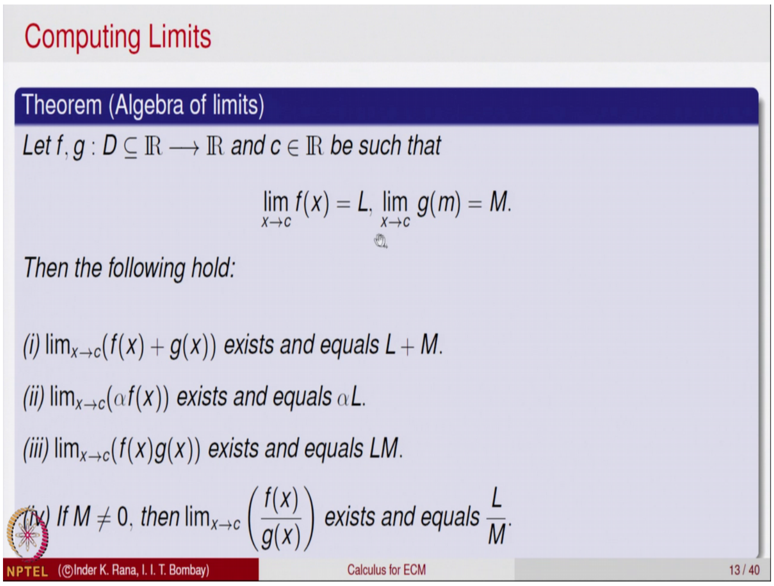

So, let us look at that suppose f n g are 2 functions, defined in a domain D in R and c is a point in R such that the limit of f x X going to c of f exist and is L and limit of g, x going to c this should be x is equal to m. So, limit of f x is equal to capital L and limit of g x, x going to c is capital M. So, limit of both f and g exists, as x goes to c one is L and other is m. So, this is given to us. So, then the following hold the first result says if I add the 2 functions if I look at f x plus g x that is a sum of the 2 functions and then take its limit, that also exist and is equal to L plus M. So, the sum of the limits is equal to limits of the sum. The second result says that if I multiply a function by alpha. So, if I look at alpha times x, that also exist and the limit is equal to alpha times x.

So, a scalar multiple of the function it is limit exist, and is equal to the scalar same scalar multiple of the corresponding limit. So, as a consequence of this if I take alpha equal to minus 1, then the limit x going to c of minus f x will be equal to minus L, and that can be combined with a earlier result to say that limit of f x minus g x will be equal to L minus L, L minus m right and the third thing is we looking at the product of the 2 functions product of f with g. So, it is defined as f x into g x limit of the product also exist, and is equal to the product of the limits that is L M So, limit of the sum is equal to sum of the limits, limits of the scalar multiple is equal to scalar multiple of the limit, and limit of the product of the functions is equal to product of the limits and finally, if this M is not 0 then I can divide by that.

So, it says that if m is not equal to 0 then f x by g x is defined for all points sufficiently close to c. So, you can find the limit, limit x going to c of x by g x exist and also equals to the quotient of the corresponding. So, limits of the quotient whenever limit of the quotient is not 0, is equal to the quotient of the limits. So, these results are very much similar to the results the; for algebra of sequences, once again algebra means I am just adding scalar multiple multiplying or dividing so on. So, these results are very useful in computing limits of slightly more complicated functions.

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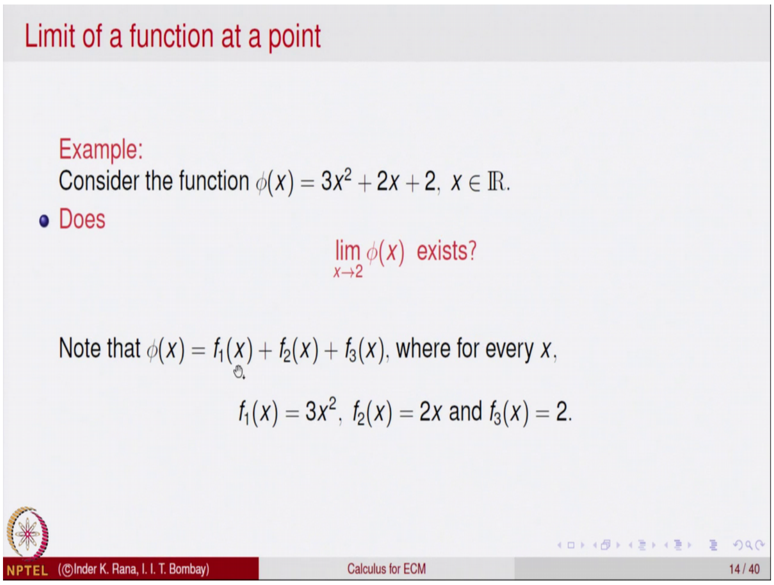

So, maybe we should look at an example for example, if I look at the function 3X square plus 2X plus 2 for x belonging to R. So, this is a function it is denoted by this Greek letter phi. So, phi is a Greek letter. So, phi is a function defined on whole of real line and phi of x is equal to 3 times x square plus 2 x plus 2.

So, we can think of as a sum of 3 functions. So, one can define we want to know what is the limit of this as x goes to 2 does it exits or not. So, to do that, let us break this function in phi into 3 parts first part will be f f 1 which is this, second part is 2 x and third part is constant function 2. So, we can write phi x equal to f 1 x plus f 2 x plus f 3 x for every x where what is f 1? F 1 of x is equal to 3 x square. So, this part is called f 1, f 2 is equal to 2 x and f 3 is equal to the constant function 2. So, we can write then function phi. So, phi is equal to phi f 1 plus f 2 plus f 3 where. So, how we will use that algebra of limits because phi is the sum of these 3 functions, and amount to find out what is the limit of phi x as x goes to 2. So, we will find out the limit of f 1 as x goes to 2 limit of f 2 as x goes to 2, limit of f 3 as x goes to 2, and using the algebra of limits we will add up those limits.

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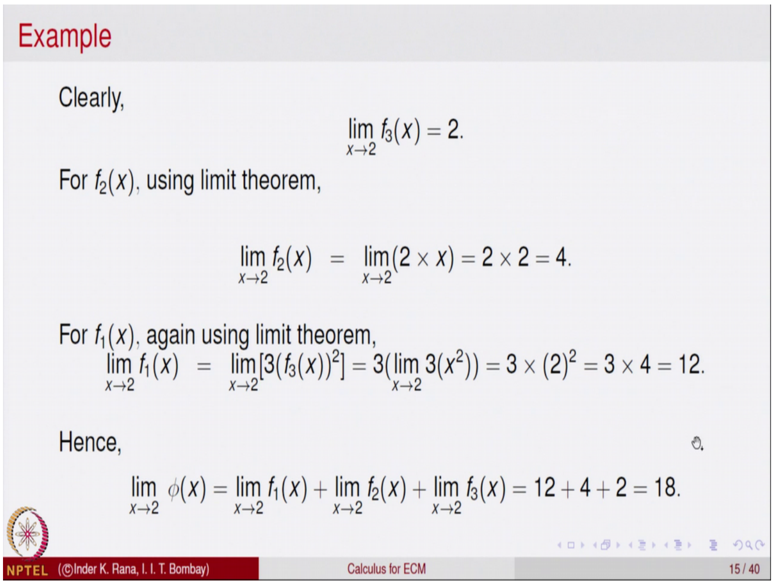

So, let us do that. So, limit of f 3 as x goes to 2 is equal to 2, because f 3 is the constant function right. So, every value is equal to 2. So, the limit also will be equal to 2 right. For a constant function the limit at every point exist, and is equal to the same constant right. Because if I put x n then f of f 3 of x n is equal to 2 constant sequence which will converge to 2. So, the limit of f 3 as x goes to 2 exist. What is a limit of f 2 as x goes to 2? Limit of f 2 of x is limit of 2 x now using the algebra of limits treat this as alpha times f. So, 2 comes out. So, it is 2 times limit of x as x goes to 2. So, that is equal to 4 and similarly for the f 1, limit of f 1 as x goes to 2 is limit of 3 x square right as x goes to 2. So, once again alpha f. So, alpha 3 comes out. So, it is 3 times limit of x square, but 3 times limit of x square is equal to 3 times limit of x, the whole thing raise to power 2.

So, that is equal to 4. So, that is equal to 12. So, let us write this using limit theorems limit of f 2 x is equal to 2 times X. So, that is equal to 4 and limit of f 1 x is limit of x going to 2, f of x square is not 3 there is 1 so that is 3 times limit of 3 x square and that is equal to 4 times. So, that is equal to 12 there is a type again there this 3 is not require. So, that is equal to 12. So, basically there is how algebra of limits is applied. So, limit of phi x going to 2 is limit of f 1 plus limit of f 2 plus limit of f 3, and that is equal to 18 that is how we will apply the algebra.

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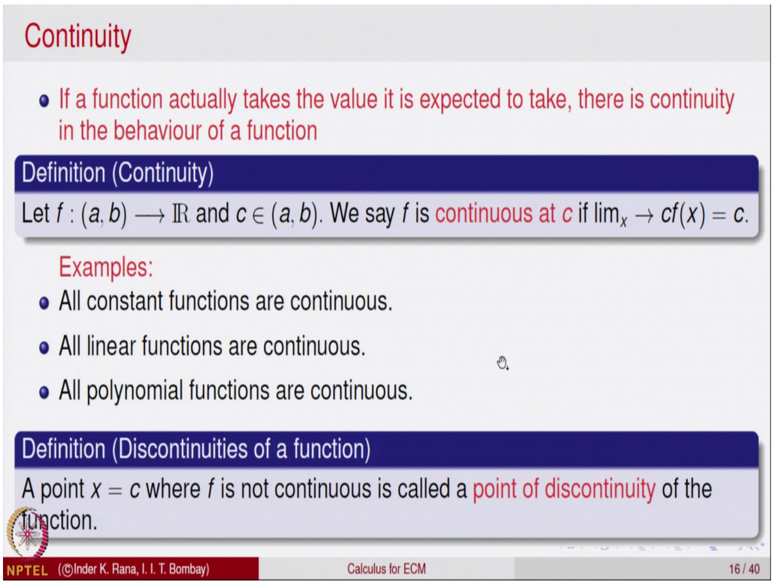

So, once we have understood the concept of limit of a function at a point, recall if a function actually takes the; what was the concept of limit? Limit was the value of the function, that the function is expected to take by looking at its properties at nearby points.

So, if a function actually takes the value it is expected to take, then we will say there is a continuity in the behavior of the function. So, this motivates our next concept. So, we define what is called continuity. Let f be a function from interval a b to r and c is a point in a b. So, for continuity the function should be defined at that point, and we say it is continuous at c if limit x going to c of f of x is equal to f of c. So, this is the again a type of here. So, let me write this definition because it is definition let me write it very clearly. So, what is continuity? So, f is a function define on a interval a b to r c is a point in a b.

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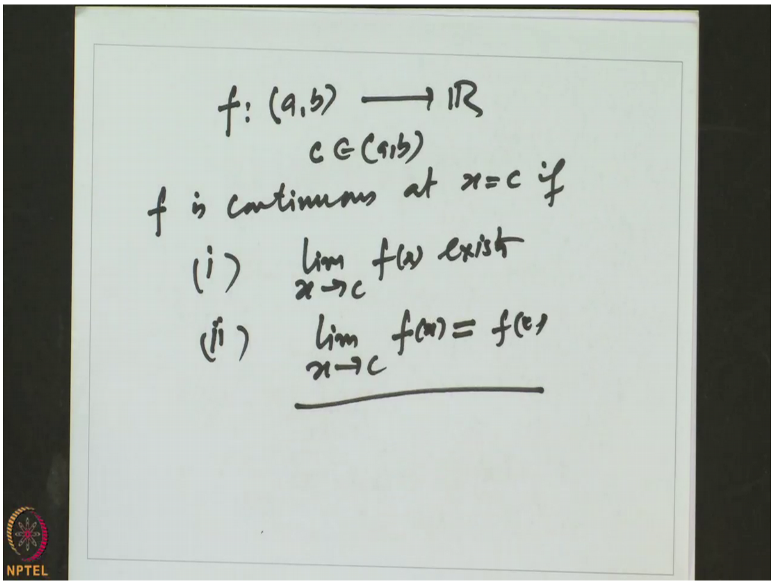

So, we say f is continuous at x is equal to c if 1 limit x going to c f of x exist, and second that the limit x going to c f of x this limit should be equal to the value. So, that is f of c that should be f of c.

So, this is the concept of limit of a function, continuity of a function at a point. So, the typo here is that limit of x going to c of f of x should be. So, this formula should be written properly yes limit of f of x, x going to c is equal to f of c. So, let us look at some examples to illustrate this point. So, what we are saying is continuity means the function should be defined at that point, the limit at that point should exist and the limit should be equal to the value of the function at that point. That is what is called continuity; there is a continuity in the behavior what you expect the function to do its actually doing it you predict looking at the values nearby it should be equal to some value, and that should value should be the taken by the function. All constant functions are continuous because the values do not change.

So, there is no problem, all linear functions are continuous. So, let us see why all linear functions are continuous.

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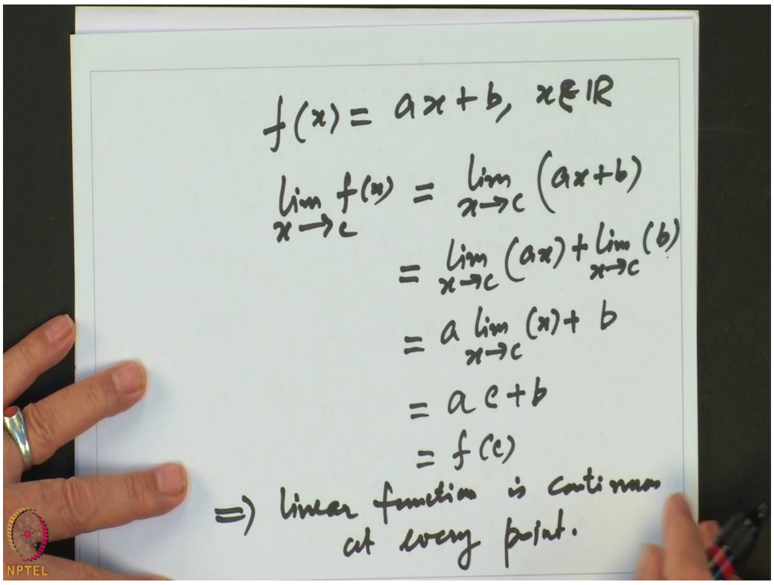

So, let us take a linear function. So, what is a linear function? So, f of x is equal to say a x plus b, x belonging to R. So, that is what we called as a linear function right. So, domain is the whole real line and the formula is. So, what is limit of f x, x going to c. So, that will be equal to limit x going to c of a x plus b, and that is equal to by the algebra of limits, it is limit x going to c of a x plus limit x going to c of b, and that we know by algebra of limits that is a times limit x going to c of x, plus there is no x here is a constant function. So, that is equal to b. So, it is a time x going to c. So, that is a c plus b and a c plus b is precisely equal to f at c when we put x is equal to c that is c plus b.

So, that implies that the linear function is continuous at every point. In fact, we can by the same logic we can go a step further and say that every for example, a quadratic is a continuous function. So, f of x.

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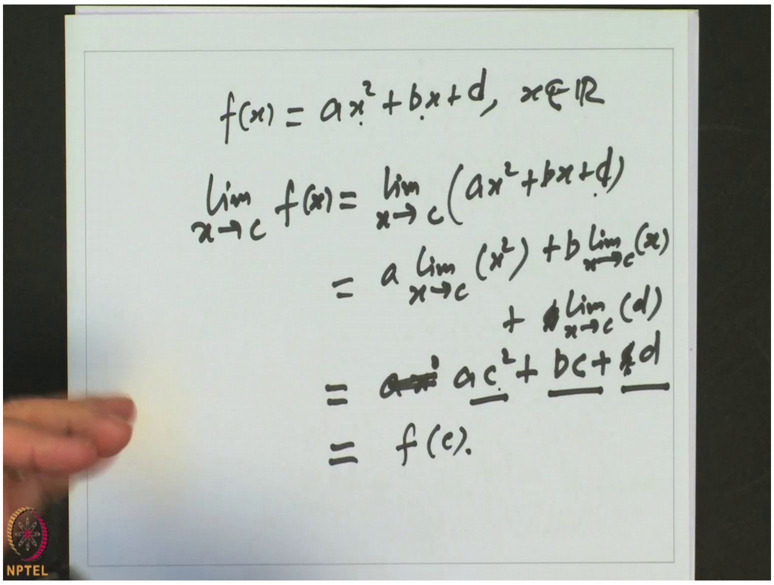

If I take it is equal to a x square plus b x plus c x belonging to r then limit x going to c of f of x will be equal to limit, x going to c of a x square plus b x plus c. And now we can apply algebra of limits that is a times limit x going to c of x square plus b times limit x going to c of x plus c times limit x going to c of there are this is a confusion. So, let me change this c, and to something else because the same c is a appearing same appearing. So, one should be careful in notations. So, let us call it as d. So, limit of this d. So, then this is limit of d. So, that will be equal to a x square x going to c.

So, that is a c square plus x going to c. So, that is b c plus c times d and that is same as right. So, there is there is no c here there is limit x going to c of d. So, that is equal to d only right. So, that is d. So, third time is limit of d. So, that is equal to f at c again a x square a c square plus b c plus d a c square plus b c plus d. So, that is true and that is also true for a general function. So, we can write that all polynomial functions are continuous. So, what is a polynomial function? A polynomial function is we looked at quadratic we can look at third degree a and so on.

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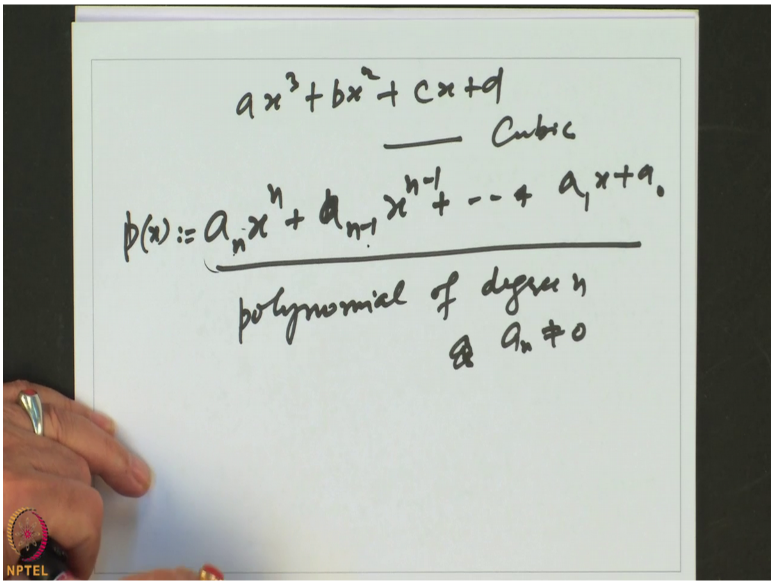

So, equation of the type a x cube plus b x square, plus c x plus d is normally called a cubic and similarly forth degree and so on. So, in general you will have say a n x to the power n, plus a n minus 1 x to the power n minus 1 plus so on, plus a 1 x plus a 0 you can call. So, such a thing is called a polynomial of degree n provided a n provided this coefficient a n is not equal to 0 right. So, an is called the coefficient of x to the power n, a n minus 1 is called the coefficient of x to the power n minus 1 and so on.

So, these are the coefficients. So, this is what is call a polynomial p x normally written as p x, and a ns are called a n a n they are called coefficients. So, a n should not be equal to 0, because this is 0 then its starts only with power n minus 1. So, to say it is a polynomial of degree n, the leading coefficient a n should not be equal to 0. So, all these are continuous functions because x to the power n is continuous by multiplying the function f x equal to x n times limit is same. So, by algebra of limits all polynomial functions are continuous functions.

So, we will not go very rigorously into this, but all continuous functions are next we would like to describe, when a function is not continuous at a point what does that mean. So, we say a point x is equal to c, where f is not continuous is called a point of discontinuity of the function a point of course, should be in the domain of the function, a point x equal to c in the domain of the function is set a point of discontinuity if f is not continuous at points. So, if a function is not continuous at point, then that point is called a point of discontinuity.

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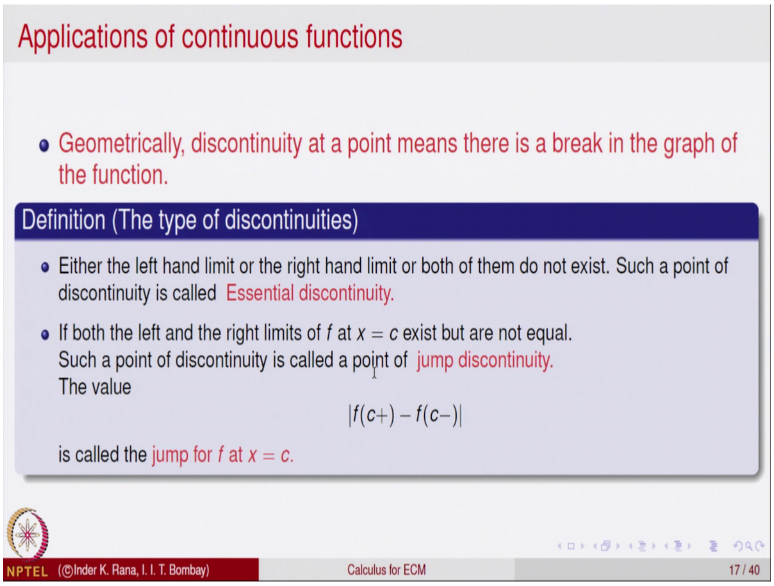

So, let us look at what does this continuity mean. Discontinuity mean essentially that there is a break in the graph of the function. We will try to make this statement more precise soon let us see what does it mean.

So, we will describe various types of discontinuities, when you say a function is not continuous at a point; that means what. So, let us look at what does means.

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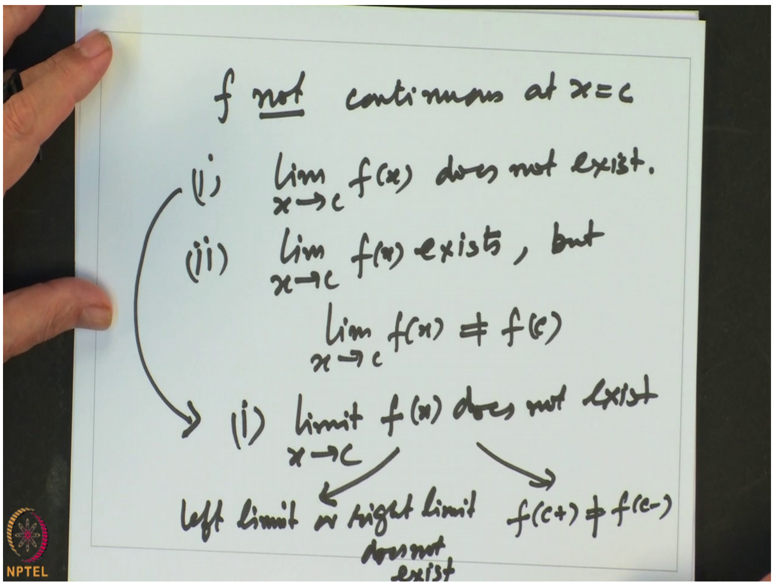

So, f not continuous at x is equal to c so; that means, what? One possibility is right that limit of x going to c, f of x does not exist, that is one possibility. Second is limit x going to c f x exist, but this limit x going to c f x is not equal to the value of the function at that point. So, that is a second possibility right because what was continuity? Continuity said that the limit should exist and should be equal to value of the function at that point. So, the way this cannot this may not happen this may breakdown is one possibility limit does not exist at all so.

Now, question of saying it is equal to the value of the function, but second possibilities limit exist, but not equal to the value of the function at that point. Now we will expand this a bit more. So, there are 2 possibilities for one the limit does not exist, limit f x does not exist limit x going to does not exist; there are 2 possibilities one at the left limit or right limit does not exist. So, either the left or the right or both probability do not exist. So, no question of existing the limit at least one of them should not exist the other is right a limit does not exist both limit exist. So, f of c plus exist, f of c minus exist, but they are not equal. So, that is another possibility the left limit exist the right limit exist, but they are not equal right. So, then also the limit will not exist right and of course, the second possibility was.

So, these are the various ways a function can be discontinuous at a point. So, let us give them names for each one of them. So, either the left hand limit or the right hand limit or both of them do not exist. Either the left hand limit does not exist or the right hand limit does not exist right or both do not exist right, such a point is called a point of essential discontinuity. So, this is called an essential discontinuity when at least one of either the left or the right limit does not exist. Other possibility is both the left and the right limit at the point x is equal to c exist, but are not equal both exist, but are not equal. So, that will also imply that the limit does not exist. So, the left limit is not equal to the right limit. So, one says this point this is called a point of jump discontinuity for the function.

So, this called jump discontinuity point for the function. So, this is the point, where the left limit exist the right also exist, but they are not equal and the difference between the 2 that is called the jump for the function at that point. So, left limit exist f of c minus, the right exist f of c plus look at the absolute value of that, that is called the jump of the function at that point right there is another possibility.

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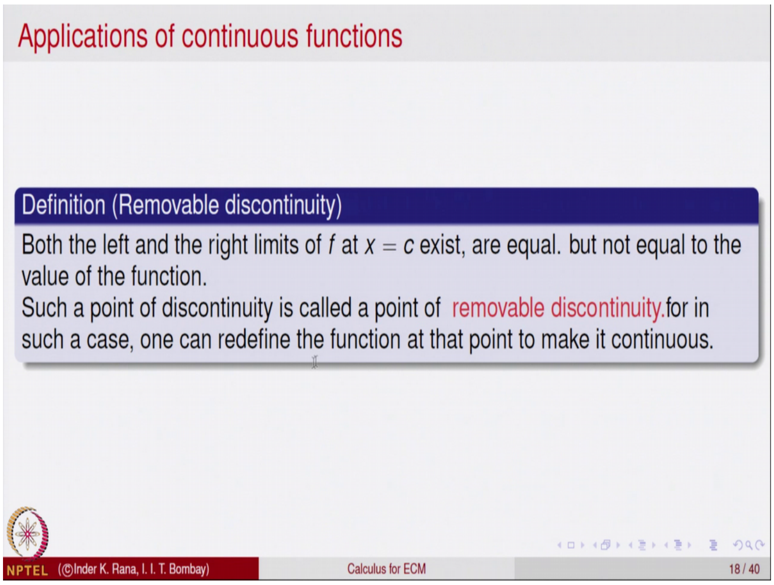

So, what is the possibility left? The left limit exist the right limit exist, but they are not they are equal, but not equal to the value of the function, that is another possibility right. So, that is what is called the removable discontinuity; that means what? One can redefine if need be one can such a point is called a removable discontinuity, because essentially means for in this case, one can redefine the function to be equal to the common value the left limit at the right limit and one can remove the discontinuity from the function.

So, this is called the removable discontinuity of the function. So, there are 3 types of discontinuities, one which is called essential; that means, it cannot be removed. So, because either the left limit does not exist or the right does not exist. So, both do not exist. The second is called the jump discontinuity where the both left and the right limit exist, but are not equal. So, that is called the jump discontinuity. And the third is called the removable discontinuity where the left limit is equal to the right limit, but not equal to the value of the function at that point. So, that is called the jump discontinuity right that is called the removable discontinuity. So, before we going to examples let me give you some examples of such functions where such kind of discontinuities can come.

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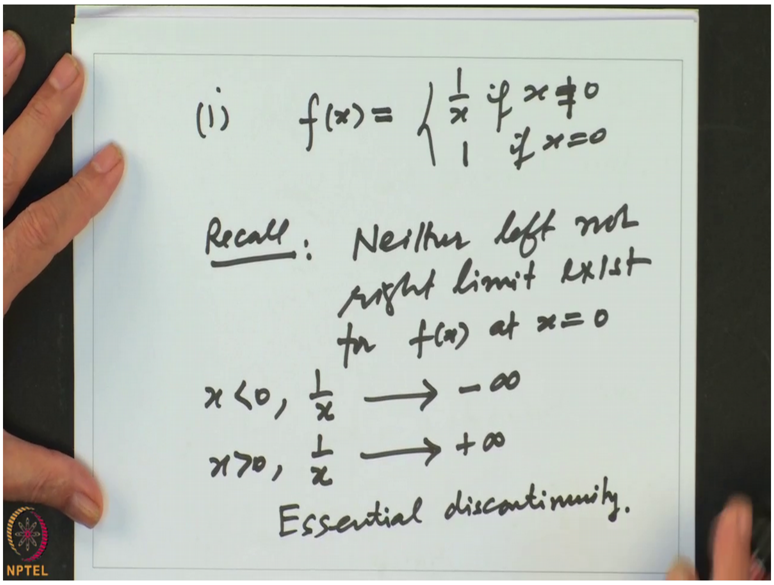

So, let us look at first one we have f of x is equal to say let me call it as 1 over x, f x is not equal to 0 and call it 1 if x is equal to 0. So, my function is defined as 1 over x, if x is not equal to 0 and define equal to 1 f x is equal to 0 right. Recall we prove that neither left nor right limit exist for f x at x is equal to 0, because y is that? Because if x is less than 0 then 1 over x you will be a quantity, when x is less than 0 it is a negative quantity and if x is becoming closer and closer to 0 this will go to minus infinity. And if x is bigger than 0 then 1 over x is positive and x becoming smaller and smaller that will go to plus infinity. So, neither the left nor the right limit for this function at this point exist, even the function is defined equal to 1 does not matter this is called essential.

So, this is example of a essential discontinuity right. So, let us look at another example of a function just now we had looked at that example.

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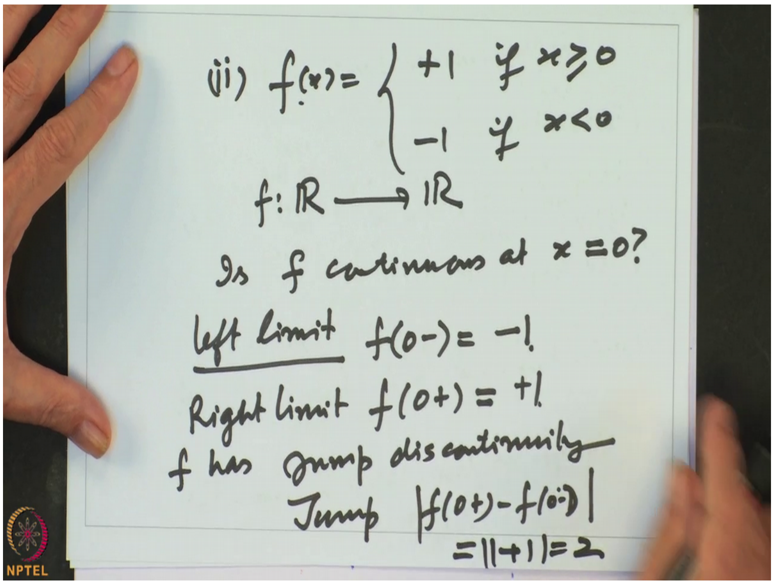

So, let us look at an example f of x, which was defined equal to plus 1 if x is bigger than 0 and minus 1 if x is less than 0. So, let us say bigger than or equal to 0 does not matter. Because now if I put bigger than or equal to 0 then f is a function, which is defined some whole real line to real line yeah. So, the question is f continuous at x is equal to 0. So, let us look at what is the left limit that is f of 0 minus. So, that is if x is less than 0 then the value is minus 1. So, whatever sequence I take less than 0, f of x n will be minus 1.

So, this is equal to minus 1 what is a right limit? F of 0 plus, when x is bigger than 0 or equal to 0 the value of the function is one. So, if I take a sequence x n bigger than or equal to 0 and converging to 0 it will converge to 1. So, that is equal to plus 1. So, both left and right limit exist right. So, f has jump discontinuity and what is the jump? Jump is equal to f 0 plus minus f 0 minus and that is what is that equal to. So, that is 1 plus 1 that is equal to 2. So, there is a jump of 2 at that point. So, that is jump discontinuity and let us finally, look at an example. So, let us look at f of x this is a good example we have not discussed this earlier.

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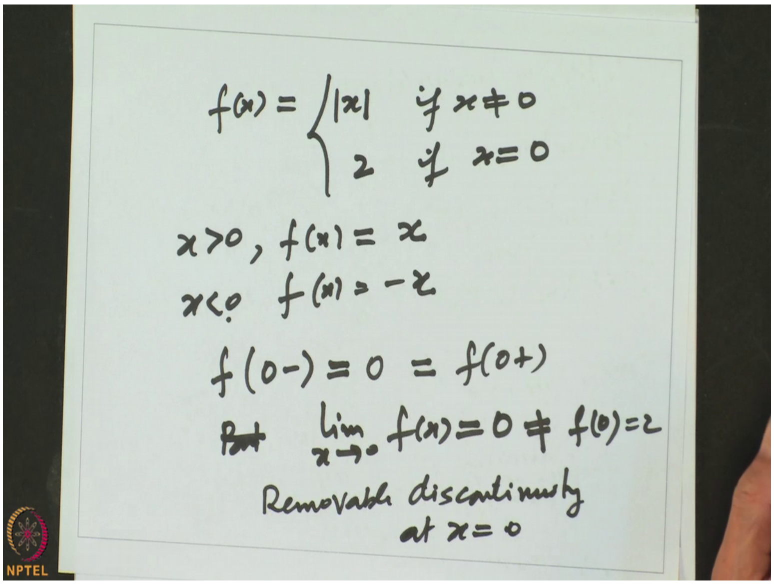

So, let us look at mod x if x is not equal to 0 and is equal to z 2 if x is equal to 0. Then for x bigger than 0 what is f of x that is equal to x, and x less than 0 f of x is equal to minus x. So, what is the left limit at 0? Left limit at 0 minus will be equal to when we approach a point 0 from the left. So, that will be equal to 0 right and that is also equal to f of 0 plus, whether you approach from the left or right the value will be 0. So, left limit exist the right limit exist, but so; that means, the limit x going to 0 of f x exist and is equal to 0, but not equal to f at 0, because that value is equal to 2. But I can forget that value. So, it is a removable discontinuity at x is equal to 0.

So, this function as a removable discontinuity at the x point 0. So, let me summarize what we have done in today's lecture, we have looked at the concept of left limit right limit and then we looked at the concept of continuity of a function. So, basically for the definition of a limit existence of a limit the function need not be defined at that point c, but for continuity of a function at a point x is equal to c, the function should be defined at that point 1 then second condition the limit of the function at that point must exist and be equal to the value of the function. And then we looked at how a function can be discontinuous. So, there are 3 different ways one the left limit or the right limit or either of them does not exist, that is called essential discontinuity. Second when the both left and right exist, but are not equal that will imply that the limit does not exist.

So, that is what is called the jump discontinuity, and the third the left and the right both exist and are equal, but not equal to the value of the function, that is called the removable discontinuity. So, we have looked at the concepts of limit, we have looked at the concepts of continuous functions, for continuity limit is required and we will see now in the [ca/next] next lecture some applications of this concepts of limit and continuity in our subject of economics.

Thank you.