Calculus for Economics, Commerce and Management
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Lecture – 16
Limit of a function at a point, left and right limits

Welcome to today's lecture. If you recall we had started looking at an important concept namely limit of a function at a point. We had said that finding the limit of a function at a point is motivated by the idea that, how to predict a suitable value for a function at a point which may or may not be in it is domain.

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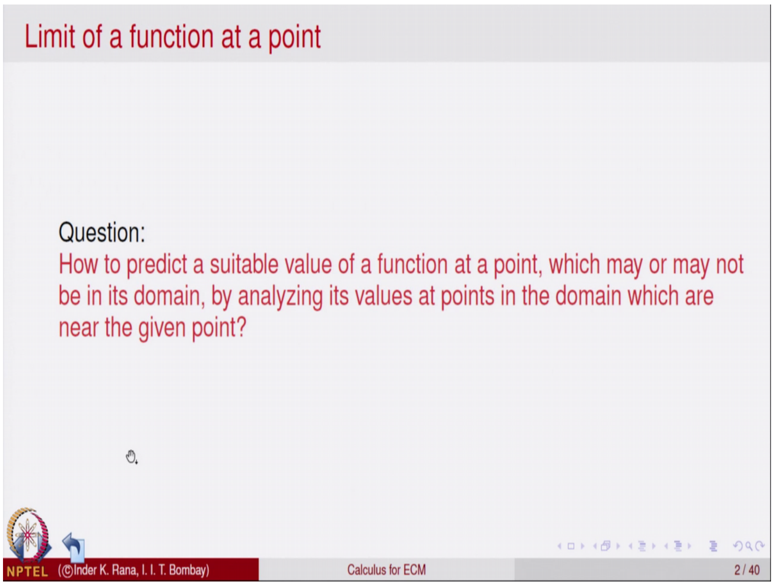

And we want to predict this value by looking at the value of the function at points nearby the given point. We gave 2 definitions of this concept of limit, one was via sequences.

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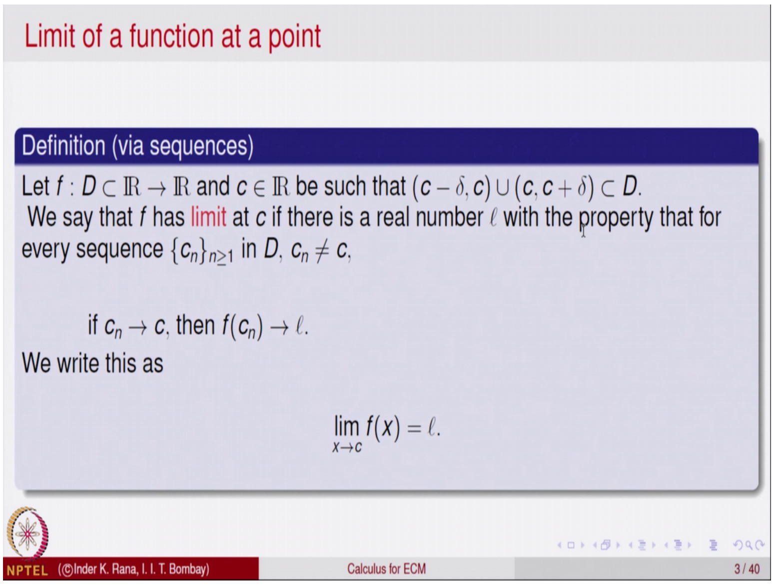

So, we said let us take a function f with the domain D in the real line taking values in the real numbers, and let us take a point c in R such that a interval c minus delta to c and c to c plus delta the union of these 2 intervals is inside D.

So, this is to ensure that the function is defined at all points near c, may not be at the point c c are may or may not be in the domain. But at least points on the left and interval points in the interval on the right are inside D. So, there are sufficient number of points around the point c, where the function is defined. So, given such a function we say that it has a limit at the point c, if the following happens there exist a real number l with the property that every sequence c n in D, in the domain such that c n is of course, not equal to c.

So, take any sequence in the domain and which is converging to c, then the image of the sequence c n f of c n must converge to l. Same number l that mean then we say that the limit of the function x going to c exist and is equal to l and we write this equal to this. So, the important thing is for every sequence c n in the domain converging to the point c, f of c n must converge to the same limit l, then l is called the limit of the function at the point c and is denoted by this to.

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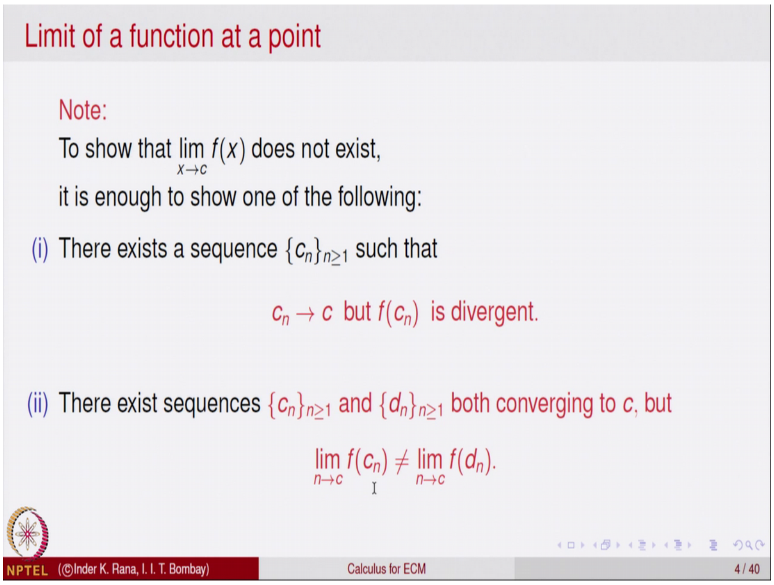

So, to why the definition of sequence limit of a function at a point, the limit will not exist if one can show that there is a sequence c n in the domain, c n converging to c, but f of c n does not convergent that is divergent. So, this is one way one condition which may be satisfied to show that the limit does not exist or one is able to find 2 sequences at least to distinct sequences c n and d n such that both are converging to c. So, both are in the domain and both are converging to c, but the limit of the image sequences f of c n is also exist limit of the image of d n also exist, but the 2 limits are different. In that case also the limit of the function will not exist.

So, if either this or this is one is able to find, then one can claim that the limit of the function at that point does not exist.

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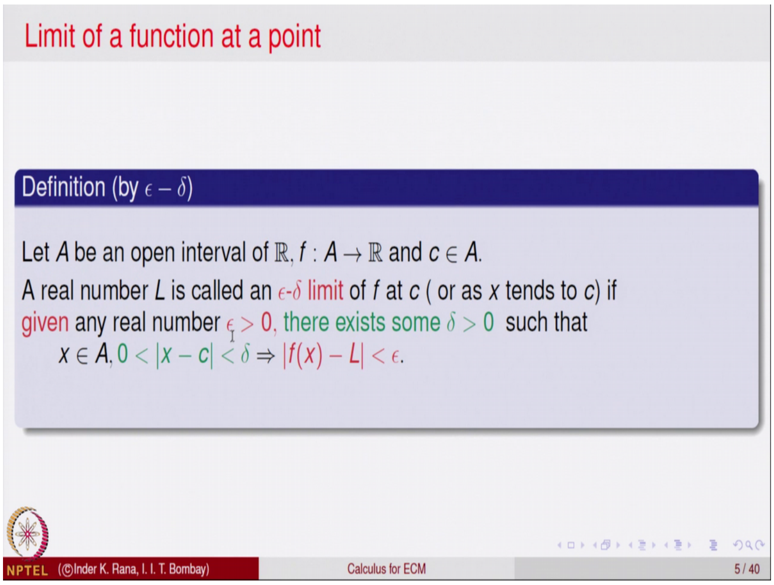

We also gave another definition of the limit of a sequence, limit of a function at a point as follows let us a f is defined an open interval around a point c in the domain, like we said c minus delta and c plus delta, it may or may not be defined again at that point c. So, saying that the limit exist or to be more sort of specific we say epsilon delta limit of f at c exist. If the following happens given any real number epsilon bigger than 0 there exist some delta bigger than 0 such that x minus c bigger than 0 and less than delta implies f x minus L is less then epsilon. This essentially saying that L is the value you want to predict, f x is the actual value you have gotten at a point x.

So, f x is minus L is the error you are making, and this error you want to make it small how small you want to make it? We will specify beforehand that will give you the margin for error epsilon bigger than 0, you should be able to find a interval around the point c, such that whenever a point x is close to c by a distance delta then f x is close to L by epsilon. This should happen for every x not equal to c, but distance between x minus c less than delta. So, this is you can call it as epsilon delta criteria for existing the limit, once again we write the limit x going to c f of x is equal to L. Let us we gave some examples last time how to find the limit of a function if it exist using epsilon delta definition, let me give one more example. So, that you feel comfortable.

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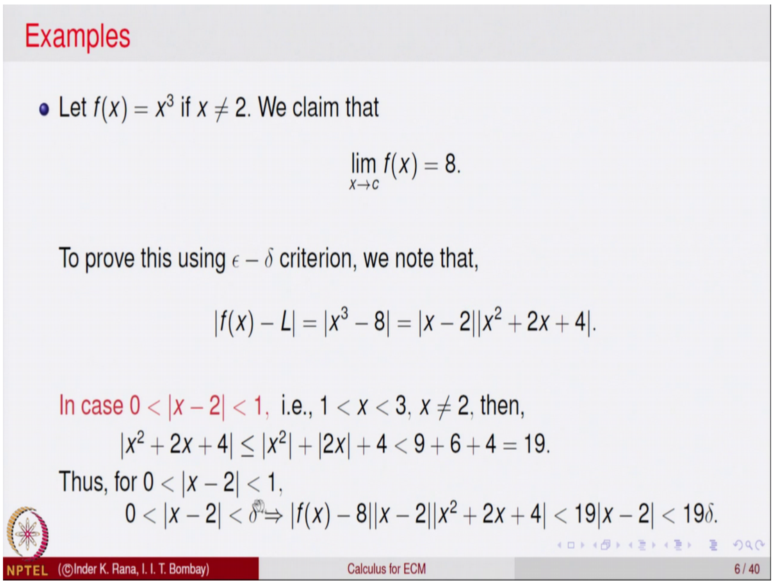

Let us if the function f x is equal to x cube if x is not equal to 2.

We want to check whether the limit of the function exist or not, we claim that the limit of this function is equal to 8, how did I guess that value? I am not just putting the value x is equal to 2 here, because that in general may not be possible and may not be true also, but what we are saying is if you take a sequence X n which is converging to 2, then f of X n will be equal to X n to the power cube right and that will converge to 2 to the power 3 and that is 8. So, by the concept limit via sequences we know that the limit should be equal to 8. So, that is what we have guessed that the limit is equal to 8. We want to prove it by the epsilon delta method.

So, what we want to do is, given epsilon we want to find a delta such that whenever x minus c here c is 2, we had looking at the limit at the point x is equal to. So, this point c is 2 actually here c is the point 2. So, limit x going to 2 of f of x is equal to 8, we want to check. So, given an epsilon bigger than 0, we should be able to find an l we should be able to find a delta such that whenever x minus 2 is less than delta then f x minus 8 is less than epsilon. So, f x minus L is the thing we want to make it small. So, the beginning of such analysis should start with saying, we look at the value f x minus L, because this is the; what we want to make it smaller than epsilon right. So, in our case f x is equal to x cube. So, we put that value L is equal to expected value of L f at 2 is 8. So, we will put this equal to 8. So, we want to make this quantity small whenever x minus mod of x minus 1 is going to be small.

Now, x minus 2 because limit at the point c equal to 2 we are going to calculate. So, the idea is somehow in this expression try to bring in x minus 2. So, here it is quite easy in a way that a cube minus b cube 2, 8 is equal to 2 cube. So, by using the algebraic identity of a cube minus b cube is equal to a minus b, a square plus a b plus b square. So, x minus 2 into x square plus 2 x right plus x b square and that is 4. So, using the algebraic identity of a cube minus b cube is equal to a minus b into a square plus a b into plus b square, we get this as factors and then use the property of the absolute value, at the absolute value of a b is equal to absolute value of a into absolute value of b.

So, using algebraic identity and property of absolute value, we get this expression. Now this x minus 2 we know we are going to make it less than some quantity delta right. The point is how do I make this quantity small. So, to make this quantity small there are many ways of doing that. Let us see that we are going to look at points near 2 right. So, let us make an additional hypothesis, that x minus 2 is going to be less than 1. We are going to look at only those points, which are at a distance of 1 from the point 2; that means, between 1 and 3 right. So, mod of x minus 2 less than 1 is 1 less than x less than 3 and this bigger than 0 means x not equal to 2. So, if you put this condition on the point x then we can estimate this quantity. So, to estimate this quantity we note that x square plus 2 x plus 4, the simplest way of estimating that would be by using triangle in equality.

This is less than or equal to mod x square plus mod 2 x plus 4 right. So, using the triangle inequality property of absolute value, we get x square plus 2 x plus 4 is less than mod x square plus 2 x plus 4. Now since x is going to be less than 3 here from here. So, x mod x square is going to be less than or equal to less than 9. So, this is less than 9 x, 2 x is going to be less than 2 into 3 that is 6 plus 4. So, that is less than 19. So, this quantity is going to be less than 19 so; that means, if x minus 2 is less than 1, then f x minus L is going to be less than 19 times x minus 2. So, let us keep that also in mind. So, in case we want what we wanted was x minus 2 is less than delta. So, if x minus 2 is less than 1 also x minus 2 is less than delta then f x minus 8 which was equal to this.

So, there is a equality sign missing here equal to and then it is x minus 8 that is equal to x minus 2 into x square plus 2 x plus 4. So, I am using this identity. So, this is less than 19 times x minus 2 right. So, this is less than 19 this is less than 2. So, putting that value combining this to equation with this and this value, we get that f x minus 8 will be less than 19 delta.

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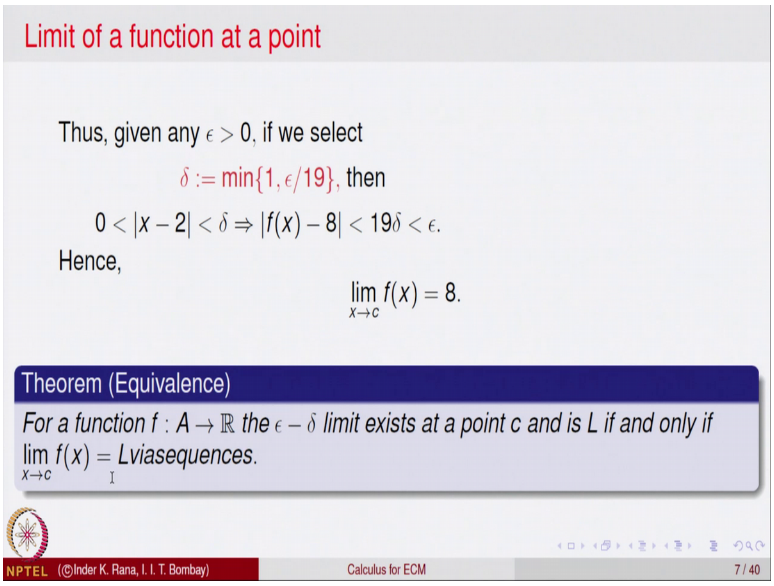

So, if we choose our given an epsilon now epsilon is going to be given. So, given epsilon we can select a delta, such that delta is minimum of 1 and epsilon by 19, how did I do that? Because in the previous slide if you look our x minus 2 is going to be less than 1, right and f x minus 8 is going to be less than 19 delta. So, if 19 delta is going to be less than epsilon then delta must be less than epsilon by 19.

So, that motivates one to find a delta. So, given an epsilon choose a delta such that both the conditions are met namely, delta is less than 1 and mod of remaining quantity. So, delta is also less than epsilon by 19. So, then what will happen? For such a choice of delta as we analyze earlier mod of x minus 2 less than delta will imply f x minus 8 which was less than 19 delta is less than epsilon. So, this will show that given an epsilon bigger than 0 any epsilon bigger than 0, we can find a suitable delta for the given function such that x minus 2 less than delta bigger than 0 implies f x minus 8 is less than epsilon. So, this is the kind of analysis one has to do using epsilon delta method of finding the limits, and we already seen a method of finding the limits using sequences. Both ways are in fact, equivalent.

So, that is a theorem namely for a function f the epsilon delta limit exist at a point c, and is L if and only if the limit via sequences also exist and is equal to L. So, either method is good enough to find the limit of a function at a point, to show that it. Many a times in computational problem sequences are easy to use to guess and prove that the limit exist or to prove that the limit does not exist at times proving some results about limits epsilon delta definition is useful. So, one can use whichever is appropriate at any level to find the concept of limit, but both are equivalent and we are given enough examples how to compute limits of a function at a point using either of it.

Let us now once you are familiar now with the idea of how to say that the limit a function exist or not, let us try to do some examples more examples quickly so that this ideas are ok.

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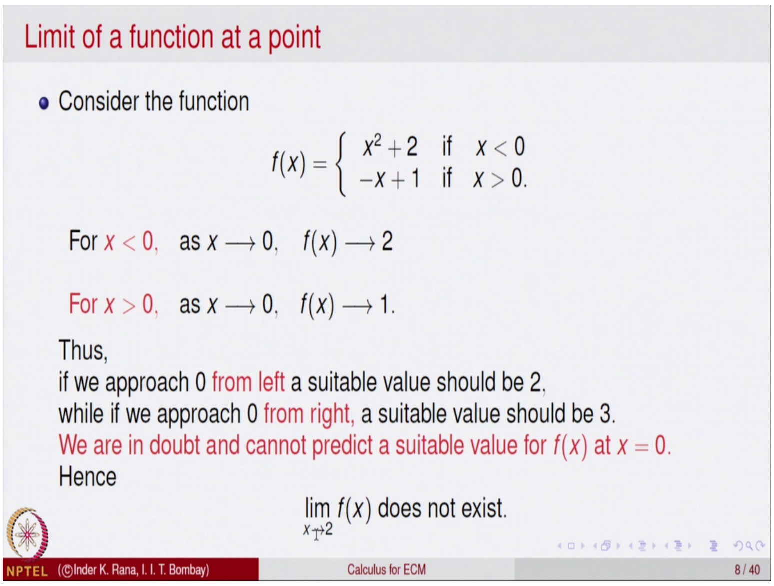

So, let us take a function f of x which is defined as x square plus 2, if x is less than 0 and it is defined as minus x plus 1 if x is bigger than 0. So, it is defined as this formula for x less than 0 and minus x plus 1 if x is bigger than 0. It is defined differently for x less than 0 and defined by another for different formula by for x bigger than 0. We want to at 0 it is not defined at all. So, one would like to find out what is the does the limit exist at the point x is equal to 0. So, that is a question we one would like to analyze. So, let us look at because it is defined differently. So, is good to analyze the concept the existence of limit when x is less than 0 and x is bigger than 0. So, if x is less than 0 then the formula is x square plus 2.

So, if we look at sequences converging to 0 purely from the left side, then if X n converges to 0 and only from the left side of 0 that is X n is less than 0. Then f of X n will be equal to X n square plus 2. So, that will mean that X n is converging to 0. So, X n square also will converge to 0. So, f of X n will converge to the value 2 right so; that means, if you are in the left side and x approaches 0, then f of x will approach 2. One is tempted to put this value here that x is equal to 0 here gives you 2, that works in many a situations, but not always. So, be careful always try to analyze at least using sequences. So, we are saying is if X n is a sequence less than 0, and X n converges to 0 then X n square also converges to 0.

So, f of X n converges to 0 plus 2 that is 2. So, for x less than 0 the sequence converges to f x as x approaches 0, f x approaches a value 2. And if x is bigger than 0 then this is a formula. So, if I take a sequence X n bigger than 0 and X n converging to 0, then minus X n will also converge 0. Since minus X n converges to 0, f of X n will converge to 1. So, if X n are bigger than 0 and converging to 0 then f of X n will converge to 1. So, one can say that for x bigger than 0 as x approaches 0, f of x will approach the value 1. So, if I approach the point 0 from the left side, f of x approaches to if I approach the point 0 from the right side f of x approaches 1.

So, I cannot say that there is a suitable value for the function at the point 0. Because if I look at from the left side the value should be equal to 2 if I look at the right side the suitable value for the function should be equal to 1 which is not true. So, this from the right this this is a type here that should be 1 so; that means, the limit of the function does not exist as for this function the limit does not exist as x goes to 0, x goes to again there is a typo I am sorry for limit f x, x going to 0 does not exist, right.

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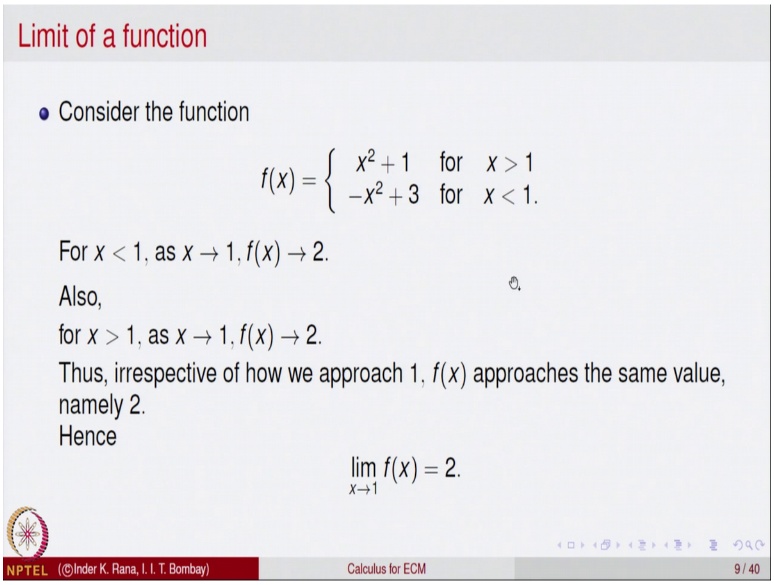

So let us look at some more examples for examples, let us look at this function f of x is equal to x square plus 1 or x bigger than 1, and minus x square plus 3 for x less than 1. We want to know whether the function has a limit at the point x is equal to 1.

So, once again we look at the left, we look at the right and then decide. So, when will if you take a sequence X n which is bigger than 1, all the elements x ns are bigger than 1 and x ns converge to one then X n square will also converge to one because X n converges to 1. So, f of X n which is equal to X n square plus 1 will converge to 1 right. And if I look at the values from the left side if X n converge is less than 1, and X n converges to 1 then f of X n will be equal to minus X n square plus 3 X n is converging to 1. So, X n square will converge to minus 1 so; that means, this value this will converge to the value 2 right because X n square will converge to the value minus 1 plus 3 to 2. So, both from the left as well as the right the value coming out to be 2, if x less than 1 x x converging to one f x converges to 2.

Converging means approaches this arrow means approaches. As x approaches 1, f of x approaches 2 from the left, and also from the right if x is bigger than 1, x approaches one then f o x also approaches the value 2. So, in this example there is no doubt about saying what could be a suitable value for the function at the point x is equal to 1. So, we can say that irrespective of how we approach 1 f of x approaches the same value namely 2. So, the limit exist and is equal to 2 right. So, my idea of giving these examples was to illustrate at times it is possible that as you approach a point c from the left of c or the right of c the function may approach a value and the values may be different.

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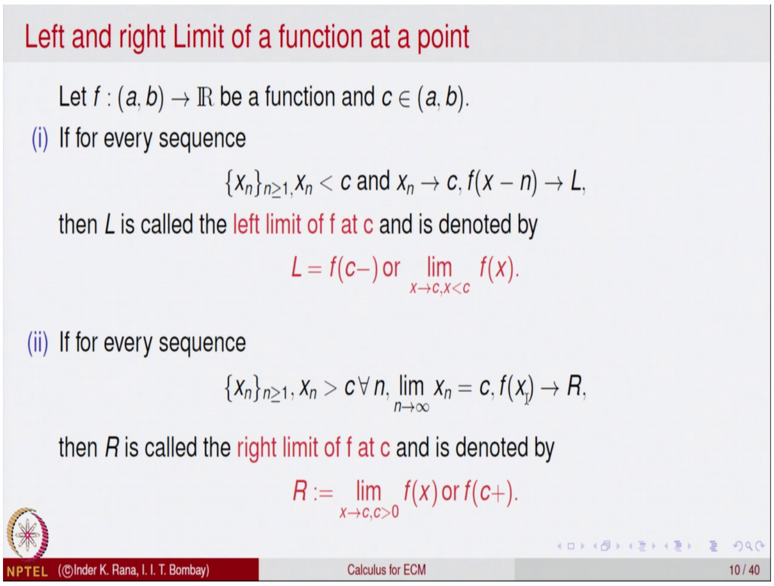

So, can we relate these 2 a general concept. So, here is the idea, what is called the left and the right limit of a function at a point. Let f be a function defined say let us say very for a simple domain say open interval a b, and c is inside the open interval a b. See in the idea we can when c is in the open interval a b then it is a defined at all points in a b. So, all points on the left as well as on the right it are defined. So, one does not have to put that extra condition that it is defined in some interval on the left and on the right. So, if for every sequence x n, X n less than c, X n converging to c. So, look at this what we are saying is if I looking at a sequence X n less than c, and X n converging to c, f of there is a typo here. So, that should be f of x lower n should converge.

So, this is not x minus n this is x evaluated at n. So, f of X n converges to one if that happens then we say that the function has a left limit at the point and it is denoted by L which is. So, so f of if X n less than c, X n converges to c implies f of X n right. So, as I mentioned this is f of X n not x minus n, X n converges to L, then we right that the f has left limit. So, it is denoted by either f of c minus, minus indicating you are coming from the left side or you write x going to c x less than c equal to x. So, either way this indicates that the function has a limit as you approach the point c from the left. And similarly from the right if X n is a sequence X n bigger than c and X n converges to c implies f of again they should be a f of X n converges to R then you say that the right limit exist.

So, basically we saying that as you approach the point from the left or from the right the limit exist, then one is called the left limit the other is called the right limit. So, once again let me apologize for the typo that this should be f of x lower n. So, the right limit is denoted by say R which is also written as a f of c plus; that means, you are looking at the values on the right of you are approaching the point c on the right of f. So, that is called the right limit. So, you can look at the left limit and the right limit and obvious observation is that the limit of a function at a point c exist, if and only if both the left and the right limit exist and are equal, and then that case the limit of the function is also equal to that common value.

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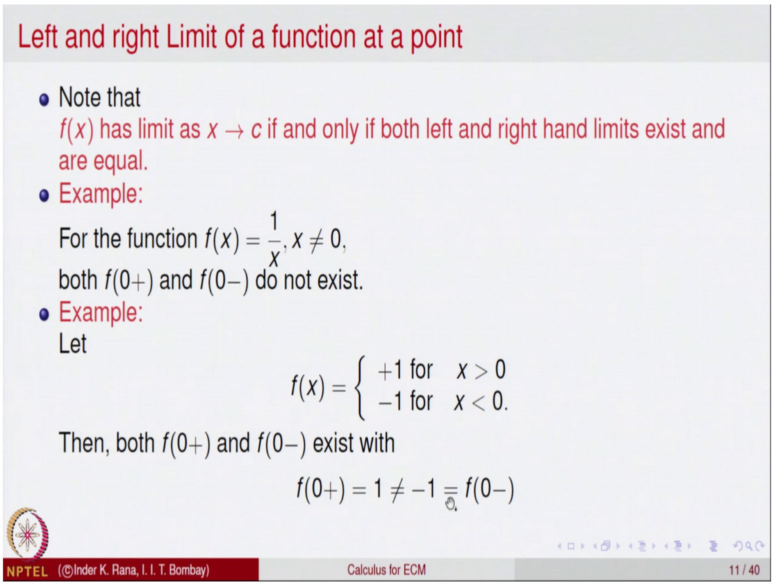

So, if either the left or the right limit does not exist, then the limit of the function will not exist.

And if both left limit and the right limit exist, but are different in that case also the limit of the function will not exist. So, the limit of a function f x as x goes to c exist, and is equal to say l if and only if both left and right limit exist and are also equal to that common value l. Then say that is effect obvious fact which can assume or you can just prove it from the definition of the existence.

So, examples let us look at some more examples of left limit and right limit for example, if you take the function f of x is equal to 1 over x. So, let us look at x not equal to 0 it is not defined. So, if we look at the value of the function at the point 0 right it is not defined. So, one can ask whether the limit exist at 0 or not. So, as you approach the value 0 from the left side. So, the points on the left will all will be negative values.

So, 1 over x will be a negative number and when x is close to 0 this will be quite a large value, but negative right so; that means, what as x approaches 0 from the left f of x will approach the value minus infinity, and similarly when x is bigger than 0 and you approach 0 then this value will be positive, but as x becomes smaller 1 over x will become larger and larger. So, f of x will converge to plus infinity. So, from the left the left limit and the right limit both does not exist because either you go from the left or from the right if you go from the left this value actually diverges to plus infinity, 1 over x diverges right. If you take a sequence X n converging to 0 on the left and 1 over X n will diverge to minus infinity if X n is positive, and converging to 0 1 over X n will diverge to plus infinity so, both do not exist.

Another simple example let us take the function f x is equal to plus 1 for x bigger than 0 and minus 1 for less than 0. And there this case it is very simple because the function is a constant function plus 1 on the right minus 1 on the left. So, whatever sequence bigger than 0 I take and look at f of X n that is going to be a constant sequence plus 1. So, the right limit exist and is equal to plus 1. And similarly if X n is less than 0, and look at f of X n then it is negative right X n converging to 0. So, constant sequence f of X n is minus 1. So, it will converge to minus 1. So, the left limit exist the right limit exist, but they are not equal, the right limit is plus 1 the left limit is minus one. So, this is a concept of left limits and the right limits, let us look at probably one more example before we close.

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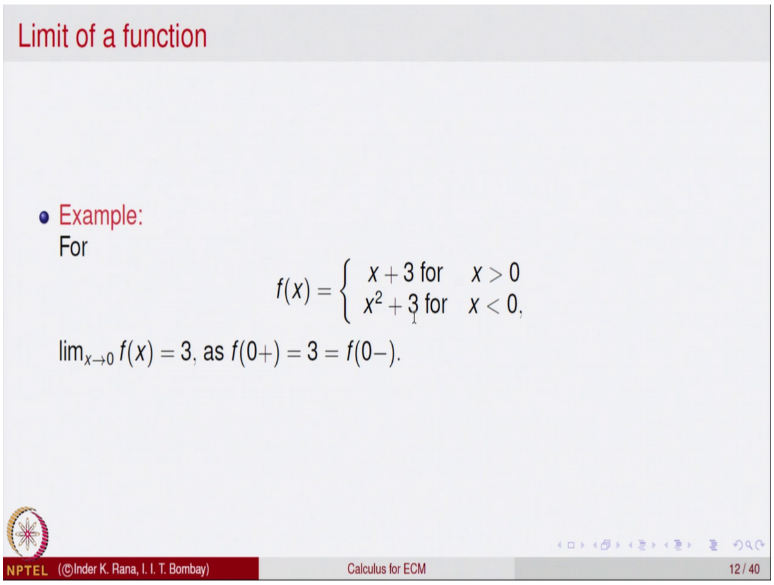

So, let us look at x plus 3 and x square plus 3 for x bigger than 0 and less than 0. So, if I take a sequence X n bigger than 0, then X n plus 3 will converge to the value 3 right. So, limit of the function from the right is equal to 3. If it is less than 0 then and X n converges to 0 then X n square will also converge to 0. So, the f of X n will converge to value 3. So, in this case the left limit is equal to the right limit. So, hence the limit will also exist and is equal to 3. So, we will stop here by looking at the examples of limits of a sequence by various ways. By now you should be confident about looking at simple examples and finding out whether the limit exist or not. We will stop here today in the next lecture we will look at some techniques of computing limits.

Thank you.