Calculus for Economics, Commerce and Management
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Lecture – 15
Limit of a function at a point

So, welcome back to our discussion about limit of a function at a point, that was what we started looking at in the previous lecture. So, in this lecture we will continue our discussion about the same. So, let me just recall what we had defined in the previous lecture, and do some 1 or 2 examples to illustrate this idea a bit more. So, let us look at; what is called the limit of a function at a point.

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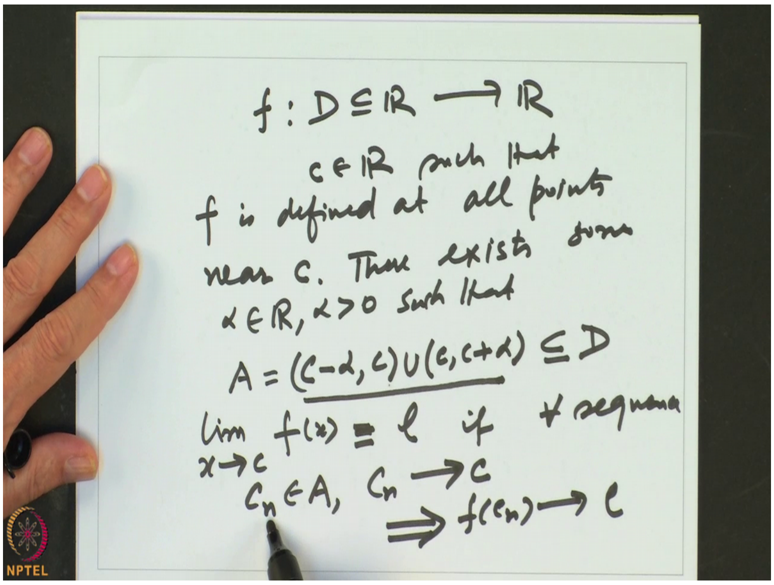

So, we said f is a function, say defined on a domain D in R to R. So, f is a function defined in a domain D and let us take a point c belonging to R such that f is defined at all points near c. See we are not saying c belongs to the domain of the function, but it should not be defined.

So, one way of saying is that there exist some say alpha belonging to R alpha bigger than 0, such that if I look at on the left side then I have got c minus alpha up to c, at least in this portion the function should be defined. So, at all points on the left and all points on the right. So, we should have from c to c plus alpha. So, may be at the point c; it may not be defined, but release this should be part of the domain. So, in that case we can define what is called the limit, we say that the limit of x going to c of f x is equal to is equal to l if for every sequence C n in this part.

So, let us call this as A, for every C n belonging to A, C n converging to the point c should imply that f of C n converges to l. So, for every sequence at all points, where it is defined. So, for every sequence C n in the domain, C n converging to c f of C n should converge to the value l, then we say that the limit of the function exist and is equal to l. So, that is a concept of limit.

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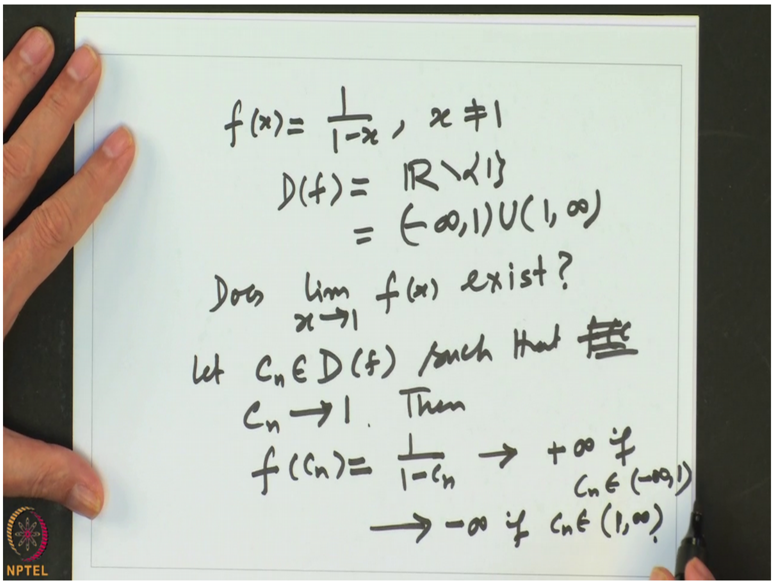

So, we looked at the previous example that we looked at was f of x is equal to 1 over 1 minus x, x not equal to 1. So, in this example what is a domain of the function? The domain of the function is equal to whole real line, minus the point 1 that is a domain of the function right at the point 1 it is not defined.

But. So, which is also same as minus infinity to 1 union 1 to infinity. So, that is a domain of the function right. So, let us take a sequence, we want to know does limit x going to 1 f x exist. So, that is the question you want to analyze. So, to answer this let us take a sequence c n, C n belong to domain of f, such that f of such that C n converges to 1. We want to analyze a limit at 1. So, pick up any sequence C n in the domain right. So, that C n converges to 1, then let us look at the sequence f of C n. So, what is f of C n? By the formula it is 1 over 1 minus C n now as C n goes to 1, 1 minus C n that goes to 0. So, this denominator becoming smaller and it is going to 0 by the limit theorem for sequences.

So, this will go to a value which is equal to if this quantity is positive; that means, if all the C ns are less than 1 right then this will go to plus infinity. If our c ns belong to minus infinity to 1 right and this will converge to minus infinity if C ns belong to 1 to plus infinity. So, in either case we see that the limit does not exist. So, whether we go take a sequence C n converging to 1, then f of C n does not converge to a suitable value so; that means, we can write it as that the limit.

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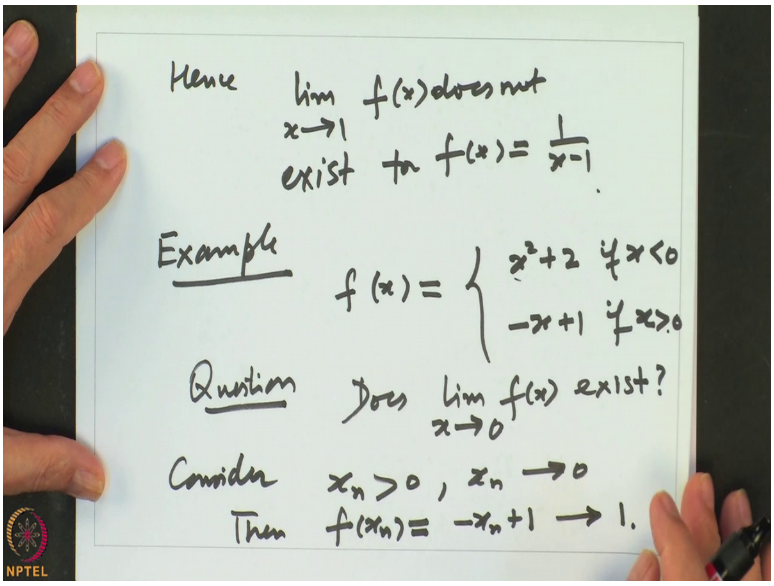

So, hence limit x going to 1 of f of x which is does not exist for f of x equal to 1 over x minus 1 right let us look at another example.

So, let us look at another example, let us say f of x is defined as x square plus 2, if x is less than 0 and defined as minus x plus 1 if x is bigger than 0 right. So, we want to know. So, the question does limit x going to 0 of f x exist. Well now here we have to start looking at the function and try to make a guess. For x less than 0 it is defined as differently and it is defined differently at for values x bigger than 0. So, it may be a good idea to look for sequences which are less than 0 and bigger than 0 and test whether at least for those sequences where are the image sequences converge somewhere or not. So, let us look at. So, consider x n bigger than 0 and x n converging to 0.

So, when x n is bigger than 0 and x n converges to 0, then what is f of x n? For x bigger than 0, f is defined this way if the formula is minus x 2 plus 1. So, that gives you that this value is minus of x n plus 1, for every n f of x n is this. Now x n is going to 0. So, by the limit theorem on sequence is minus x n will go to 0, minus x n plus 1 will go to 1. So, this value is equal to 1 right let us look at a sequence where x n is less than 0.

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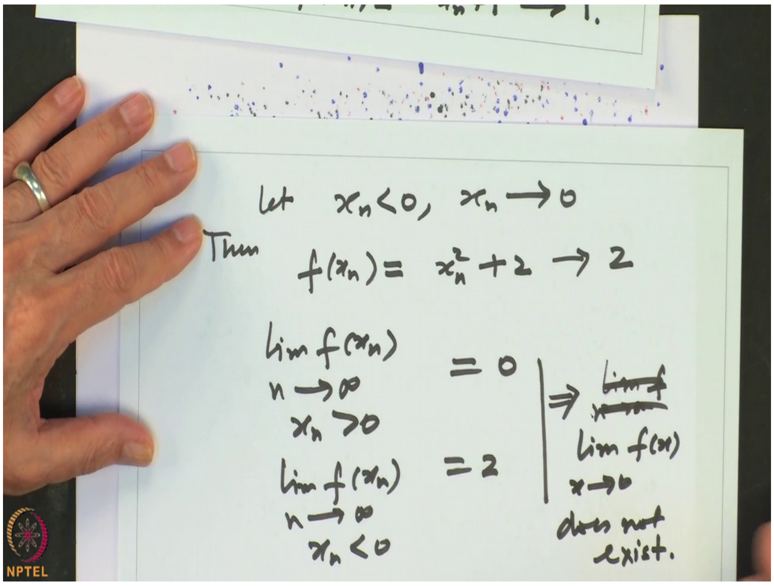

So, let x n be less than 0 and x n going to 0, then what is f of x n? F of x n is equal to by the formula. So, just recall the formula is for x n less than 0, the formula is x square plus 2. So, it is equal to x square x n square plus 2 now x n goes to 0.

So, by the limit theorems if x n goes to a and y n goes to b then x n y n goes to a into b. So, using that we get that this converges to x n goes to 0. So, this converges to 2 right. So, this converges to 2 because x n goes to 0, x n square goes to 0 this goes to two. So, so we have got if I look at a sequence. So, if I look at. So, we can write it as limit of f of x n, n going to infinity x n bigger than 0 is equal to 0, where as limit f of x n, n going to infinity for x n less than 0 is 2. So, this 2 indicate. So, imply that for we have got different sequences both all going to 0, but the values of the image sequences are different.

So, that is says that the limit n going to infinity of limit sorry limit of the function f of x as x goes to 0 does not exist that does not exist. So, what we have been trying to do is, trying to in the concept of limit when we are trying to approach the point for where we are trying to find out the limit x going to c. So, we are approaching the point c via sequences. There is another way of looking at this concept of limit via the notion of neighborhoods, which is not very difficult, but in fact, it is similar to what we are doing. So, let me introduce that idea also. So, that we are able to use whichever is required.

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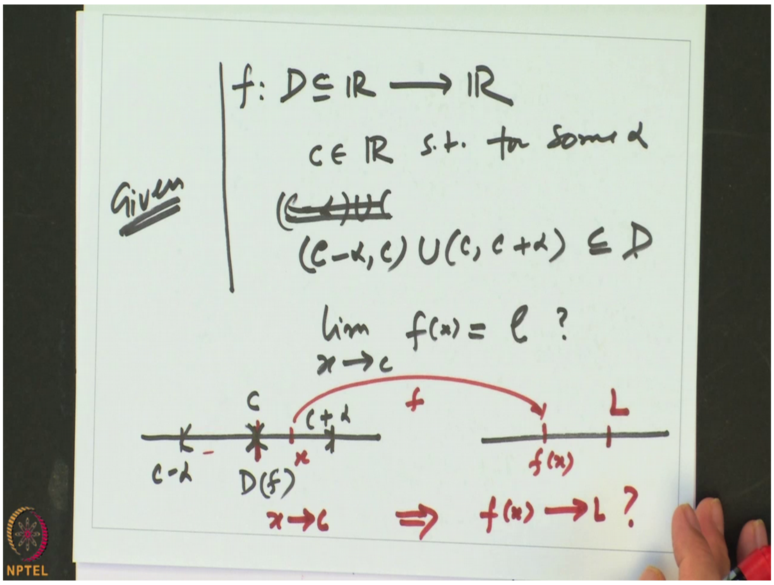

So, let us once again f is a function defined in a domain D subset of R to R, c is a point in R such that for some alpha c minus alpha union sorry.

C minus alpha to c this interval union, we want to say that it is defined at all points near c. So, for some interval around c except at the point c. So, that is c to c plus alpha is inside the domain. So, that condition is given. So, this is what is given to us we want to say whether the limit when can we say that the limit x going to c of f of x is equal to l now let me draw a picture. So, this is the domain part, and this is the range part of f. So, this is the point c. So, for every point c near. So, this is c 1 alpha and this is c plus alpha. So, here is some c plus alpha so; that means, for all points in this part the function is defined they are the part of the domain.

So, if I take a point x way does the value go. So, for a point x, it goes to some value f of x right. So, that is domain function now we want to say that as you approach the point c right as you approach, your f of x will approach some value l or not. So, x approaching c does it imply n f of x approaches L. So, we want to give this a precise meaning, one we have already seen via sequences if I can say for every sequences x n going to c f x n converges to l then we say that is the limit I want to define it slightly differently here.

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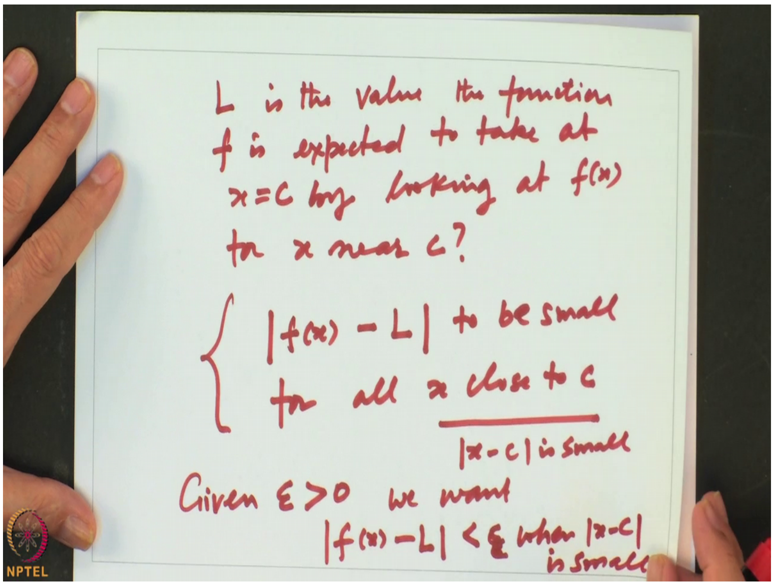

So, what is that L; right? So, L is the value. So, L is the value, the function f is expected to take at x is equal to c by looking at f of x for x near c. So, that is what we want to do.

So, at a point x near c, f of x is the actual value the function takes, and l is the value that we are expecting the function to take. So, how much error or how much is the error. So, this is the distance between the 2. So, this is the error 1 is making this is the value probably we expect the function to take, this is the actual value being taken. So, this is the error the function is making, and we want this error to be smaller this to be smaller for what x for all x close to c. So, this is. So, this is what we want to do. So, we can also visualize it x close to c means what. So, let us make that also precise x close to c is same as saying x minus c is small right that is a distance. So, distance between f x and l is small for all x close to this.

To make this slightly more precise, now, we want this value to be small right we want to say as x comes closer to c f of x is coming closer to c right. So, how close that we will say beforehand I want it this much close. So, 1 writes given a number epsilon bigger than 0, I want this error you can think this as a error being made this is the actual value, this is the expected value this is error. So, we want error to be small given epsilon we want f x minus l to be less than epsilon we want the error to be small. By the way this Greek letter epsilon is something equivalent to the English alphabet E. So, error we write as a epsilon. So, this is epsilon. So, we want this to be small when x minus c is small, but how small; right; what are the smallness for x minus c.

So, we say it more precisely as that once you are given a margin for error, I should be able to say that for all points sufficiently close this is true. So, that sufficiently close is specified as follows.

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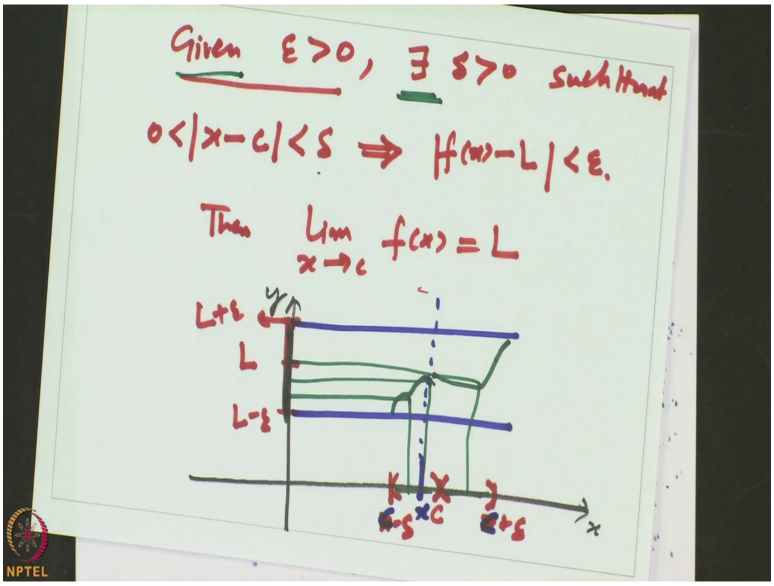

So, given epsilon bigger than 0 that is a margin for error, there exist some positive real number delta bigger than 0 such that if x is close to c by delta by the distance delta of course, not equal to c. So, we put bigger than 0, then this should imply that f of x is close to L by that please specify it margin for error. So, this is precisely saying if this happens then we say a limit x going to c f x is equal to L.

So, in a sense here we are saying all points close to c f of x is close to this, earlier we add the closeness in terms of sequences. For every sequence when a sequence x n is converging to c it is going to come close as close as you want right and f of x n is converging to l. So, that is going to come close as close as you want. So, the notion of limit one is via sequences namely this limit exist if for every sequence x n converging to c f of x n converges to the same limit L or we can also say it by this way that for every margin for error for every given epsilon, there exist a delta bigger than 0 such that whenever x is close to c by delta f x is close to this by epsilon. I can also exhibit to you this as a picture. So, let us look at in the picture this is x and this is y right.

So, what we want to do is, there is a point c where we want to analyze the limit and we want to say that the limit is equal to some value L right. So, when you say given epsilon bigger than 0 right x minus c there is a delta such that this happen. So, given epsilon you want to say distance between f x and L is less than epsilon that is same as saying if this is L minus epsilon and this is L plus epsilon. So, you want your f of x to be inside this value right for what x whenever x, x minus delta and x plus delta when there is a delta. So, that x minus c the distance. So, if I take this portion and then I take any point if I take any point here say this is c minus and this is c plus, if I take any point x here in between and I look at where does the graph hits.

So, the graph of the function should lie between should be visible in this; that means, the graph of the function for these values should look like something this right, that is how the graph should look like because for every point if I look at the value right it will be inside here. If I look at a point the value will be inside here, if I look at a point the value will. So, this is the error for margin and this is the neighborhood or this is the interval around c which we should be able to find. So, this is given and there exist error for margin is pre specified, and the interval around delta you have to find out if you want to say this is equal to this. So, this is what is called the epsilon delta definition of the concept of limit of a function at a point.

So, either one can be used depending upon the convenience. So, let us look at one example in this way also.

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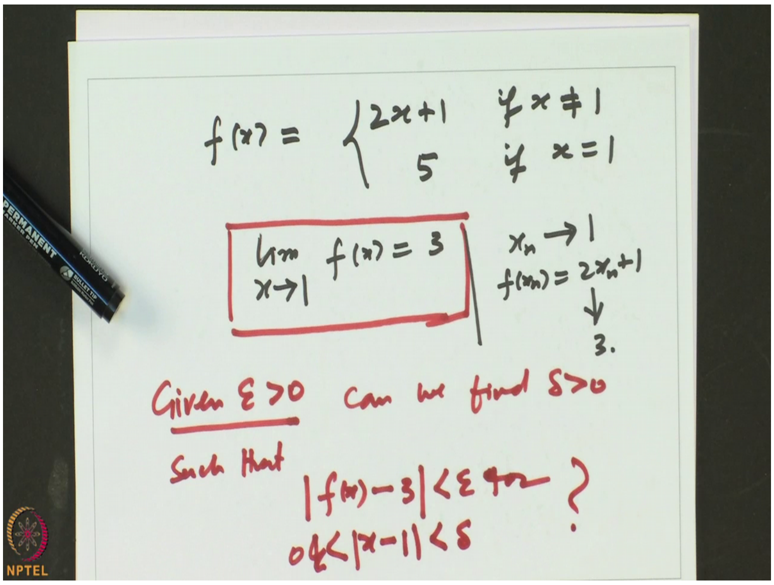

So, let us look at that example that we had f of x is equal to 2 x plus 1, if x is not equal to 1 and we add this equal to 5, if x is equal to 1 right. By the sequences we said limit of x going to 1 f of x is equal to 3 right that limit is equal to 3 because if I take a sequence. So, the reason was if x n converges to 1 then f of x n is equal to 2 x n plus 1, and that converges to 3. So, the limit should be equal to 2 plus 1. So, that is what we established by the definition of sequences let us try to do a analysis.

So, let us say given epsilon bigger than 0, can we find delta bigger than 0 such that mod of f x minus here is the value expected is 3, less than epsilon for 0 less than mod of x minus 1 less than delta. So, this is what we want to do. So, what we want is we want to make f x minus 3 small. So, let us start analyzing that quantity what is f of x minus three. So, let us analyze that quantity a bit.

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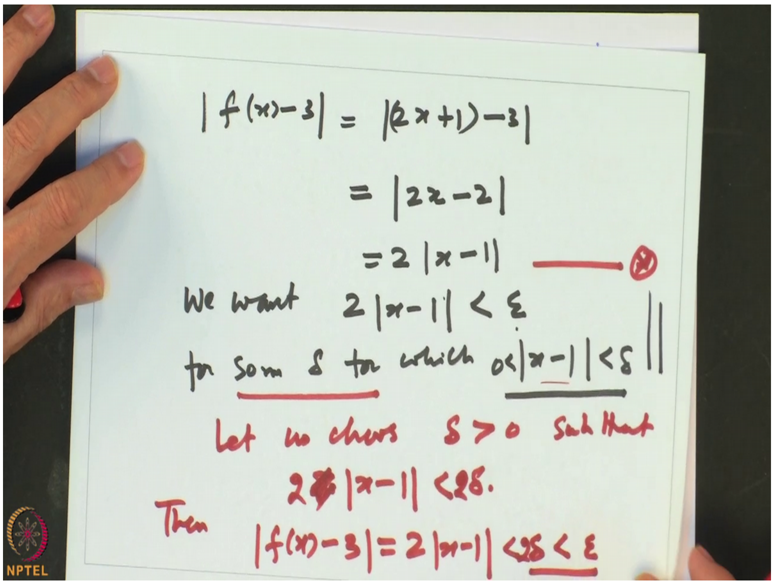

So, the quantity which we want to make smaller than epsilon is f of x minus 3 right. So, let us put the value of f, f is equal to when it is not equal to 1 the value is 2 x plus 1. So, this is equal to 2 x plus 1 minus 3 right.

So, let us simplify this. So, this is 2 x minus 2, which is equal to 2 times x minus 1 right. So, we want 2 times x minus 1 less than epsilon for some delta for which mod x minus 1 will be less than delta. So, we want whenever this is true this should be true. So, now, let us just compare I want 2 x minus 1 to be less than epsilon, but if I find a delta that is going to give me x minus 1 is less than delta. So, both this things motivate 1 to say let us chose delta bigger than 0.

So, we have to find some. So, let us chose delta bigger than 0 such that 2 times x minus 1 is less than delta right we can always do that right. So, that; that means, x is between. So, in a neighborhood around 1 minus delta 2 1 plus delta. So, we can do that then what is f of x. So, look at this equation. So, we had f of x minus 3 was equal to 2 times x minus 1. So, if x is delta is such that 2 times this is less than 1, then x minus 1 is less than right the x minus 1 is less than is such that either way we can do it, but let us choose delta say that this is less than delta, then this is going to be less than delta. So, if delta is less than epsilon that will do the job right.

So, we can chose delta less than epsilon provided delta is this or another way of that would be let us chose a delta such that let us cancel out this 2. So, anyway we are going to chose this is less than delta right. So, or if is 2 times this and it will be 2 delta. So, I will be 2 delta and that should be less than epsilon. So, this is how you analyze if you want f x minus l less than epsilon. So, first analyze f x minus l you will get this value, and somehow you should write to bring in x minus c. So, that is x minus c is 1 here I have to try to bring in x minus c. So, that is this quantity. So, we because x minus 1 is going to be less than delta, but delta is going to be such that this is true. So, this quantity is going to be less than 2 delta. So, 2 delta should be less than epsilon.

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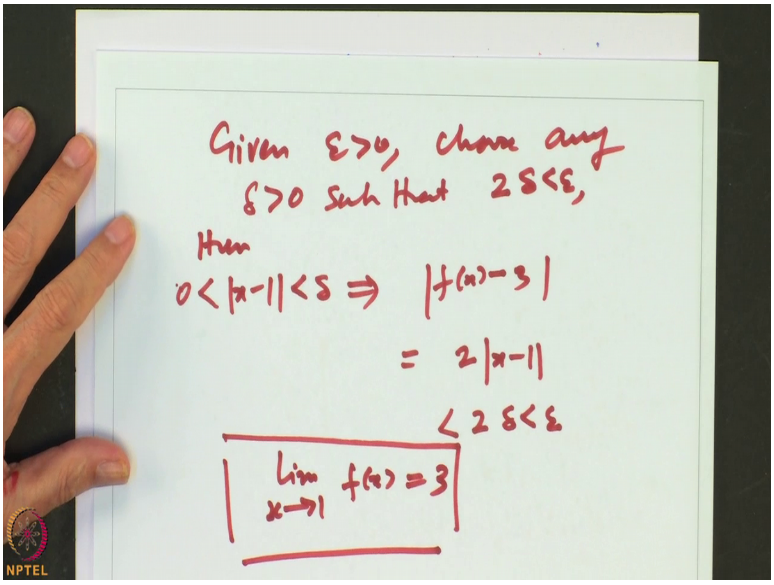

So, claim is given epsilon bigger than 0 chose any delta such that 2 delta is less than epsilon n then 0 less than x minus 1 n less than delta will imply f x minus 3 which is equal to 2 times x minus 1 which is less than 2 delta will be less than epsilon. So, that will prove that the limit by this method x going to 1 f x is also equal to 3. So, that will also prove that way. So, this is called the epsilon delta method of checking ha whether something the limit exist or not will not will not specify too much for the epsilon delta method of finding limits, most of the time will give you some more rules to analyze limits whether epsilon delta or sequence method, and using those rules 1 will be able to analyze the limits of function at a point whether it exist or not.

So, let me just summarize we have try to give you a feeling for what is meant by saying that the limit of a function at a point exist, that says that you want to say that limit of f of x, x going to c exist. So, first of all the point c need not be in the domain of the function that is 1, but we need that the function should be defined at points close to the function close to the value c. So, at all points close to this value c the function should be defined that should be in the domain. So, we; that means, what? Given a point c there are points there are sequences in the domain which we can find which converge to the value c. And the limit exists says that for any sequence at C n which converges to c f of C n should converge to the value l same value l then we will say that the limit exist.

And we also give a alternate definition, which is the neighborhood definition it says given epsilon bigger than 0 right. You look at given an epsilon bigger than 0 you the error f x minus l you want to makes less than epsilon. So, for that you should be able to find a delta bigger than 0, say that for all points in a neighborhood of delta of length 2 delta at the point c of course, not equal to c the f of x should be close to l by at the most it should not go away from l by distance l. So, given epsilon bigger than 0 there is a delta such that x minus c strictly bigger than 0 less than delta implies absolute value of f x minus c is less than epsilon. So, we will continue this discussion of limit and its applications in our next lecture.

Thank you.