Calculus for Economics, Commerce & Management

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Lecture – 14
Limit of a function at a point, continuous functions

Welcome to today's lecture till now what we have been doing is basically looking at setting up the base for the calculus tools to be developed. We looked at set theory, we looked at the concept of sequences, we looked at the concept of how functions arise in trying to represent various physical scenarios, we looked at various types of functions and so on.

Today's lecture is going to be the key lecture for the whole course, we are going to look at the concept of what is called the limit of a function. So, limit of a function is the core of calculus; whether it is differential calculus or integral calculus the concept of limit is the basic idea, it is the fundamental idea in calculus. So, we will go it a bit slow and try to understand this concept very clearly, what the concept of limit of a function means. If you recall we have already looked at the concept of limit of a sequence.

So, what was the limit of the sequence essentially the idea was because sequence is a ordered collection a 1 a 2 a 3 and so on a evolving process. So, we wanted to know what happens when in the evolving process, the n the stage that looking at becomes very very large. So, essentially there we wanted to look at the behavior of the function for n very large. Eventually what happens as n tends to infinity one would say one becomes very large. So, n becomes comes closer and closer to infinity, the infinity is not a number. So, you can think of when n is becoming larger and larger is a n coming closer and closer to some value. So, that was the behavior of sequence limit of a sequence we looked at something similar is going to happen for a function.

So, we are going to understand what does it mean that a function approaches a particular value as the points the independent various approaches some value given value. So, to understand this concept of limit what does this approaching means, and how do we analyze it mathematically. I am going to look at a example in a software called Geogebra, I do not have to use that software at all I can do it on a piece of paper, but it becomes very more dynamic and interesting to look at in the software called Geogebra.

So, I have developed a small applet in Geogebra and I am going to interact with that, and explain to you what is the concept of limit of a function.

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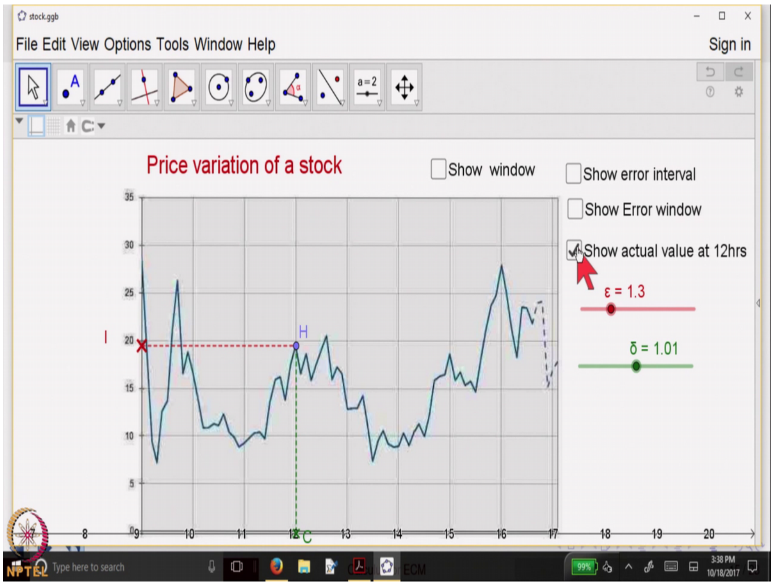

So, let us open the applet. So, here is the applet that I am going to use. So, this is an applet in a software called Geogebra.

So, in this what we are going to look at is, the price variation of a stock. So, I am given on a particular day, we looked at the price of a particular stock of a company from morning 9 ‘o’ clock till evening 5 ‘o’ clock. So, here at the bottom is the time frame. So, 9 ‘o’ clock, 10 ‘o’ clock, 11 ‘o’ clock, 12 ‘o’ clock, 1 ‘o’ clock, 2 ‘o’ clock, 3 or 4 and 5. So, this is a time frame and on this side on the y axis you can think of is the price of the stock. So, at 9 ‘o’ clock the price was somewhere here. So, that was equal to somewhere near 31 or 32 or something this price whatever that quantity is. So, the price dropped. So, I am just interpreting the graph. So, when you are going down; that means, the price is dropping down. So, as the time progressed share market started the price of that stock debt, and then suddenly the price started growing going up increasing it increased.

So, there was a peak here, and then something happened and then again it started dropping, then it recovered a bit drop and so on. So, there were many fluctuations in the price of that stock on the day, and eventually it close somewhere here. So, this is the price time graph of that particular stock on a particular day fine. Now what I want to do? So, let us the question is at 12 ‘o’ clock what was the actual value of the stock in the market. So, to find out the value, will draw a line here it cuts the graph at this point say H and then we will go vertically and find out the value. So, let us say the actual value. So, this is the value. So, at time 12 ‘o’ clock the value was somewhere here somewhere around the value was just below 20 rupees or something like that that was its value right.

I can see that value actually this software tells me how to see that value, I can click here and find out what is the value the property of that. So, let us not go into that right, see it is between 15 and 20. So, somewhere probably 19 rupees or so, that is the actual price. But let us ignore that price let us close our eyes and something somewhere somehow in that graph something happened to the graph and the whole of the graph is not visible.

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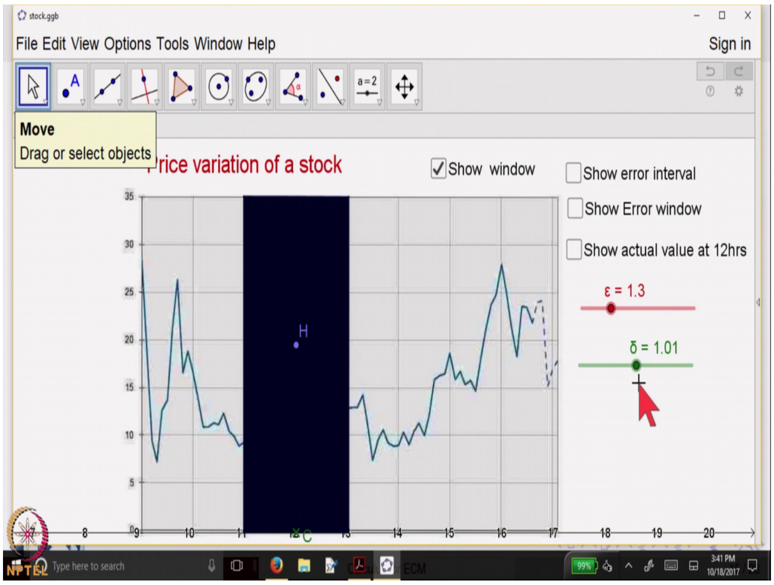

So, let us say only this part of the graph is visible. So, what I have done is, I have blackened out a part of the graph right now what is visible to me is, what happened to the stock price up to this time point that is 11 ‘o’ clock, I do not know what happen to the stock price in between right the data is missing because of some reason, and that at 1 ‘o’ clock again the data started appearing.

So, probably the machine which was recording the data how is the price fluctuating went bad was out of order and data was not recorded, and then I have the data for the full day. So, now, look at this scenario I have the data for this price of the stock before 11 and after 1, and I want to predict what could have been the price at 12 ‘o’ clock. So, this is the problem we want to analyze. So, the problem to be analyze is, I have the data available for the price of the stock from 9 ‘o’ clock to 11 ‘o’ clock, and then from 1 ‘o’ clock to 5 ‘o’ clock. The problem is can I predict some suitable value for the price of the stock at 12 ‘o’ clock, can I predict what could have been the value at this time 12 ‘o’ clock well. There are many ways of sort of guessing the possible value, say at this point probably the value was somewhere around 9 rupees and at this point at 1 ‘o’ clock the price was somewhere say equal to 13 rupees.

So, here it was 9 rupees here it was 13 rupees probably we can say let us take the average of the 2, and say that was average price. So, average price probably 13 plus 9 22 may be 11. So, may be this was the price at that time. So, that is possible case, but somehow somebody else comes out on the scene and says I have a data available for all points for all time points close to 12, only at 12 I do not know what is the price meaning what somebody else was recording the price of the stock during the time right in that persons machine everything got recorded except the value at the point 12. So, how do I show it so; that means, more data was available. So, let us change the window and see what happens? So, in the second person’s machine the following was visible see. So, more and more data became visible.

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So, at 11 and before 1. So, more and more data is visible, more and more data is visible, more and more data is visible. Supposing we just stop here now the data is available up to the time something around 1130 and 1230 onwards. So, between 11 30 and 12 30 data is not available. Could I use my earlier scenario of saying that the average value should have been the one at the point 12 probably? Here if I look at 1130 the price was around just about 15 rupees maybe it was 16 and the price at 12 30 was just around maybe 17.

So, probably the price would have been 16 if I take the average as the consideration. So, that is one way of predicting a value for this price of the stock at 12 ‘o’ clock I could probably take the average of the 2 values or somebody might say let us join these 2 by a line and then draw the vertical line and see what could have been the value, that could have been another way of predicting a value for the stock price at this point. Well since the more data is available. Let us see what more is available to us let us let us shrink this window which is blackened out slightly more, slightly more and slightly more slightly more.

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So, at all time points near 12 ‘o’ clock the data is visible for example, if I do this now this much only data is only blackened between this time period, between this time period was probably about 10 minutes, data was not available. Then if I try to take the average it is from here to here the average is probably something different. So, depending on how much data is available the method of saying that the value at this point 12 can be taken as the average or by joining this by a line and taking does not really give me a very satisfactory answer right.

So, what could be a way of saying that whatever I predict right will give me a nice value? So, let us assume that I am able to find out all data is available to me everywhere except the point except the value at that point right very small.

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So, at 12 ‘o’ clock exactly may be 11 59 the data is available, and then 12 one minute past 12 the machine starts working only at 12 ‘o’ clock it is not working. So, what could have been the value at 12 how do I predict it. So, that is a nice value. So, well one could be that I take various values right I take points, and move on the graph from the left and try to see what could be the value at a point nearby is it come closer and closer to some value and similarly from the right I go and come and closer and closer to the value right.

So; that means, I try to approach 12 ‘o’ clock from the left as well as from the right and see whether the values of the stock stabilize somewhere, they go closer and closer to something or not. And if this closeness if they do approach some particular value then we can say that should have been the value, that should have been given to the function to that stock market price.

So, to make this idea more precise right, what we are saying is as we approach. So, at 12 ‘o’ clock, we do not know; what is the value of the stock. So, what we can do is instead of taking average or doing anything, we try to approach this point time point 12 from the time points before it, and see what is happening to the value of a stock, probably that is coming closer and closer to a particular value and probably when I move from the right that is also coming closer and closer to a value.

So, if I can predict that value that must be the suitable value that function should take at that point. So, that is precise is the idea of the limit. So, let us try to make it more precise by looking at a function ok.

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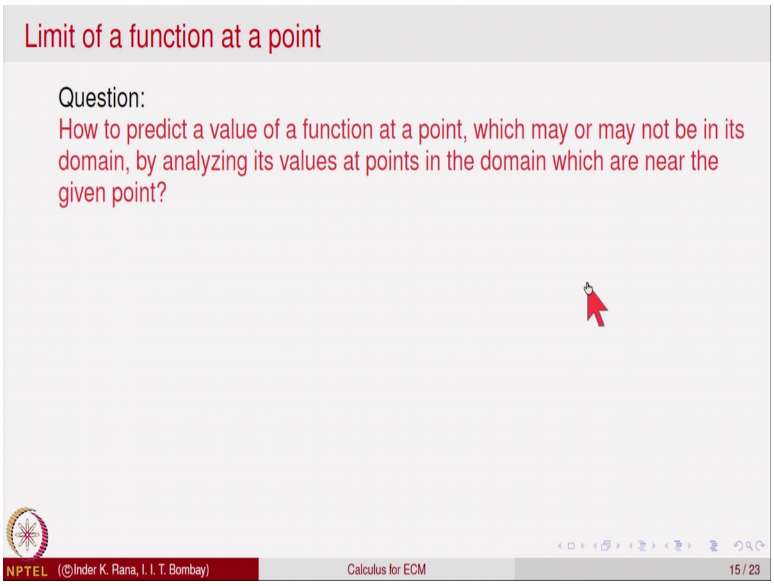

So, the question we want to analyze mathematically is, how to predict a value of a function at a point, which may or may not be in its domain the function may actually may not be defined at all what we want to do is, but it is defined at all points nearby. So, a function is defined at all points near by a point right and we want to know what could be a suitable value for the function at that point, by looking at the values of the function at points nearby. So, that is the problem we want to mathematically tackle and give a suitable answer to it.

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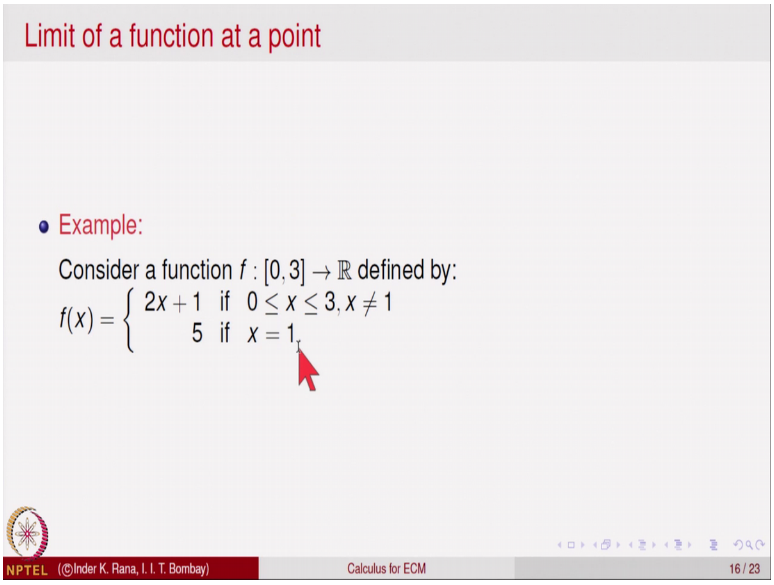

So, I am going to look another example now, we looked at an example of the price of a stock fluctuation in a day and said how we should be able to predict namely approach that point where the data is not available, and see what is a values of the function at those points approaching to.

So, let us look at. So, here is a problem, we are given a function f with domain as 0 to 3 taking real values. So, this is a function taking real values. So, how is the function defined? It is defined as 2 x plus 1, if x is between 0 and 3. So, for all points between 0 and 3 the value defined is 2 x plus 1 except at the point x is equal to 1. At x equal to 1 the function is given the value 5. So, because it is essentially a linear function except at the point x is equal to 1.

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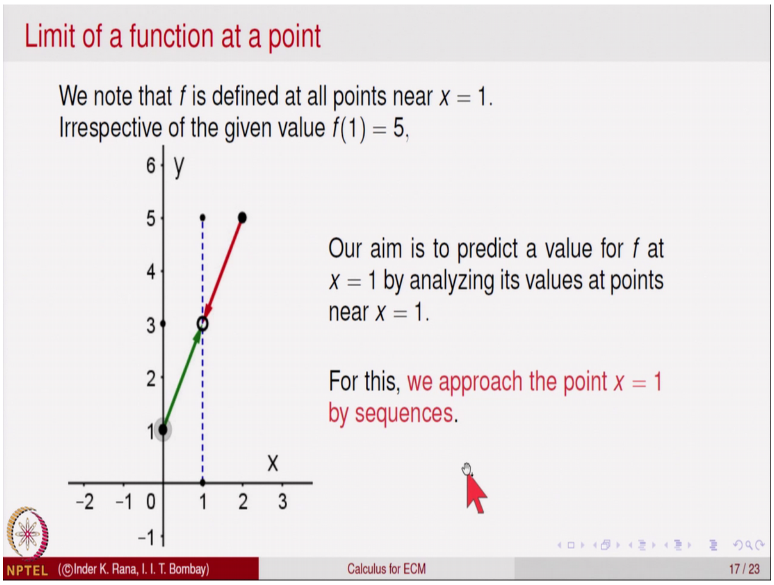

We can sketch a graph of this function. So, this is what the graph of the function looks like right. So, it is a linear function everywhere, except at the point x is equal to 1 and the value is 5. So, that is a value.

So, what we want to do? What we want is at the point 1 there is a value given by the function that is given to us, but let us ignore that value right let us not bother whether the function is defined at one or not, what we are going to do is we are going to look at the values of the function at all points near 1 right and try to say what could be a suitable value of the function at this point x is equal to 1. So, our aim is to predict a value for the function f at x is equal to 1 by analyzing its values at 12 points near x is equal to 1.

So, as in the previous example when we looked at, what we said was, let us try to approach the point 1 by one is in the domain, but let us ignore that, let us take other points and approach one by other points nearby points. So, how do can one approach a point on the line? One can approach a point by sequences. So, we can take a sequence of points which are coming closer and closer to 1; that means, we can take a sequence in the domain which is converging to 1.

So, irrespective whether it is on the left or on the right, we have to approach the point 1. So, we can take a sequence x n or in the domain right x n not equal to 1 because there; 1 we do not know what is happening. So, look at a sequence approaching one and look at the values of the sequence because this points are in the domain, what is f of x n that gives a sequence in the range of the function. So, it will be giving a sequence on this red or the green graph and say whether these values approach that sequence converges to something or not, if it converges then probably that is a suitable value.

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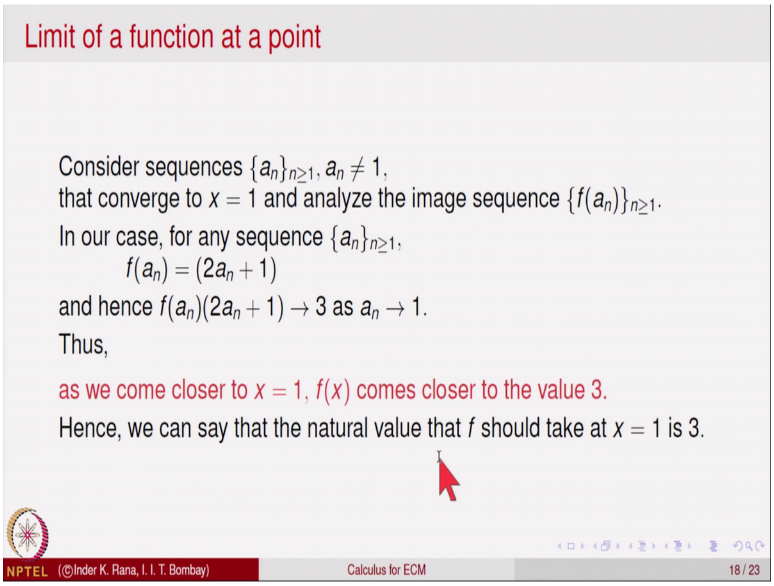

So, let us look at that. So, consider a sequence a n, a n not equal to 1 that converges to the value equal to 1, and analyze what is happening to the image sequence. So, what is the image sequence f of a n. If n is a sequence in the domain then for the function f, f of a n is the image sequence right.

So, what we do in our case because a n is the image domain sequence, in the domain f of x is equal to 2 x plus 1 whenever x is not equal to 1. So, f of a n is equal to 2 of an plus 1. So, we have a sequence in the domain converging to 1, look at the image of that domain sequence that is 2 a n plus 1. So, does this sequence converge somewhere? Well now we have to apply our limit theorems for sequences. So, let us analyze that, we know a n converges to 1 right the sequence a n right we have got a sequence a n which is converging to one if a n is converging to 1, 2 a n will converge to 2 by the limit theorems for sequences, and 2 a n plus one will converge to 2 plus 1. So, it will converge to 3. So, it says that f of a n which is 2 a n plus 1 will converge to 3. So, this fact is using limit theorems for sequences.

So, f of 2 n here should be equality sign, f of 2 n equal to 2 a n plus 1 that will converge to 3 as n goes to 1. So, if I take a sequence in the domain a n a sequence a n in the domain, a n not equal to 1 and if a n converges to 1 so; that means, if I am approaching the one point 1 in the domain, then in the range f of a n approaches the value 3 so; that means, probably we can say that if as x comes closer to 1, f of x come closer to the value 3.

So, a suitable value for the function at the point 1 should be equal to 3, by looking at the behavior of the function at all points nearby points. So, that value should be equal to 3. So, we can say that the natural value of the function at the point x is equal to 1 should be 3, if we have to give consideration to value of the function at nearby points right. So, that is the way we try to predict.

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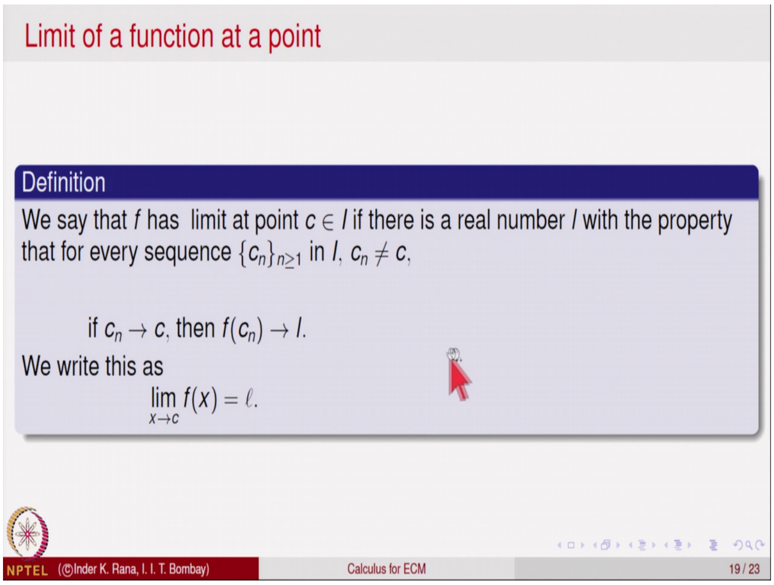

So, let us make it as a definition. So, we say.

So, here we are making this more precise. So, let us say we have got a function f, we say f has a limit at a point c. So, here f is a function defined in some domain taking values in the real line, we are not written that precisely. So, f is a function defined say in a interval around the point c. So, I is the interval. So, we say f has a limit at a point c in the interval I. So, domain of the function is the interval I, it taking values in the real line. So, probably one can write more precisely given a function f from an interval I with real values right. The limit of a function at a point c is a real number l. So, this is l with a property that for every sequence c n in the domain I, in the interval I if c n of course, not equal to c, if c n converges to c then the image sequence should converge to the same value well.

So, just let us analyze essentially the content of this definition is, pick up any sequence in the domain converging to c. C may or may not be in the domain. So, pick up any sequence in the domain converging to c, because c n is in domain look at the corresponding image sequence f of c n, f of c n is defined because c n is in the domain this is a sequence, one can analyze whether it converges or not. If it converges and if for every sequence c n converging to c, f of c n converges to the same value l then we say l is the limit and we write f of x, x going to c is equal to l this l does not seem same l. So, one should write differently. This is a number think this as l. So, this is same should be same as this and should be same as this. So, otherwise it may be confuse with cs c belonging to I domain of the of function is a interval I, and the range and the limit is equal to l.

So, this is l the some other symbol, we should use like here l provided. So, the limit f of x x going to c is equal to l, this only signifies the following. For every sequence x n, c n converging to the point c in the domain f of x n should converge to the value l in the range, then we say the limit exist and is equal to l. We will give some example more examples to illustrate this point.

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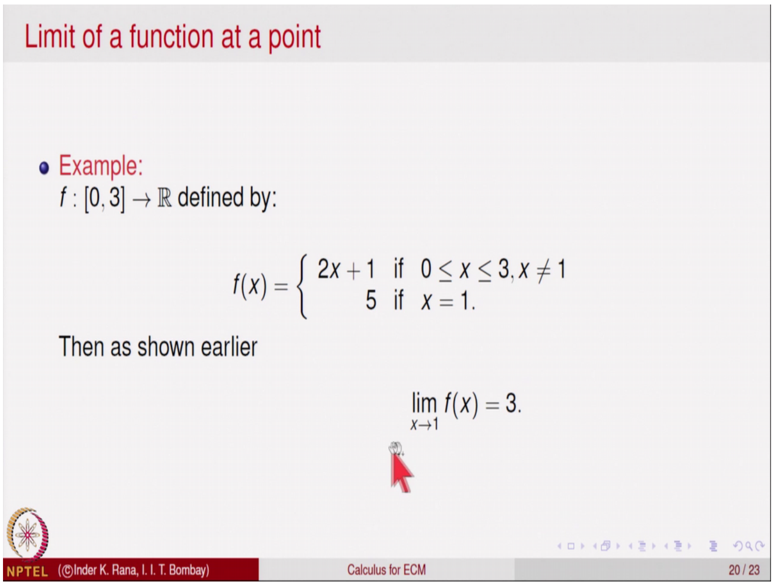

So, let us look at this example the previous example that we saw f of x is equal to this. So, in the formal definition, we saw that though the function is defined as equal to one at one that is not really the suitable value. We said the suitable value should be 3 because when we look took a sequence converging to one in the domain, f of c n converge to 3 for every sequence. So, one can write for this function limit f x, x going to one is equal to 3 right according to the definition let us look at some more.

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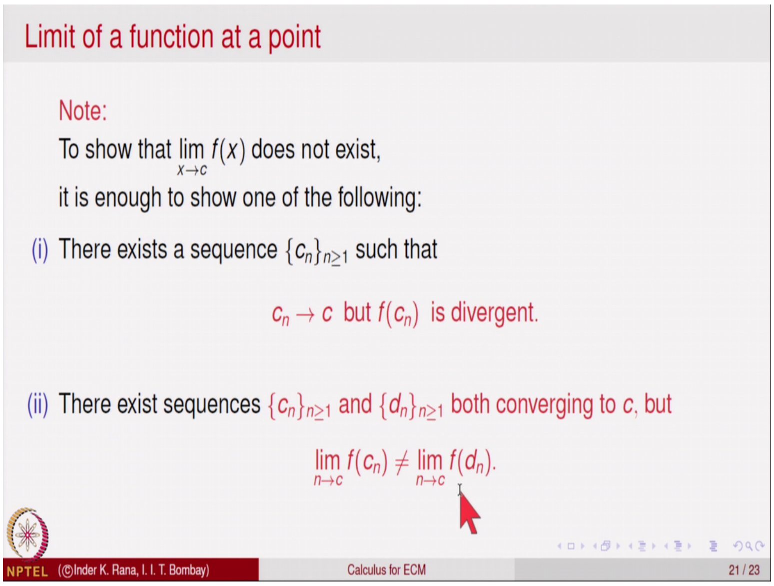

Now, before going further examples to show that the limit does not exist, what one is to show; see the definition of saying that the limit exist was for every sequence x converging c n converging to c, for every sequence c n converging to c, f of x n should converge to the same value l. So, saying at that it does not exist will essentially mean either of the following one right. The possibility that there is a sequence c n in the domain that c n converges to c, but f of c n is divergent it is not convergent right. So, showing that the limit does not exist, one way is produce a sequence c n which converges to c in the domain right c n is a sequence in the domain converging to the point c, but when the sequence and look at the image sequence that does not converge that is divergent.

So, in that case this limit of f x x going to c will not exist or at least I should be able to produce 2 different sequences c n and d n right both converging to c, but the image sequences they converge, but they converge to different values. So, limit of the image sequence f of c n for this sequence, the c n is converging to c. F of c n also converges, but the value to which it converges is not same as where f of d n converges. So, they are different; that means, if I take this path of c n approaching c, then I should be predicting this value where as if I take this path of d n converging to c, then I should be predicting this value both are different (Refer Time: 26:51). So, I am in a dilemma which is a right value I should predict. So, saying that the limit does not exist this is another possibility one is able to produce 2 different sequences c n and d n.

Such that both converge to the same value c right c n and d n are sequences in the domain both converges to the same value c, but the image sequences f of c n and f of d n, they do not converge to the same value right they converge to different value. So, we are not able to produce predict a suitable value so; that means, that for showing limit does not exist one has to either produce a sequence c n which converges to c, but f of c n is not convergent or produce 2 different sequences converging to the same value c, but the image sequences do not converge so that, will say that we have got limit does not exist, but to prove limit exist what we have to show?

We have to show that for every sequence x n converging to c f of x n should converge to the same value right. So, let us look at one more example. So, that the idea settles us down.

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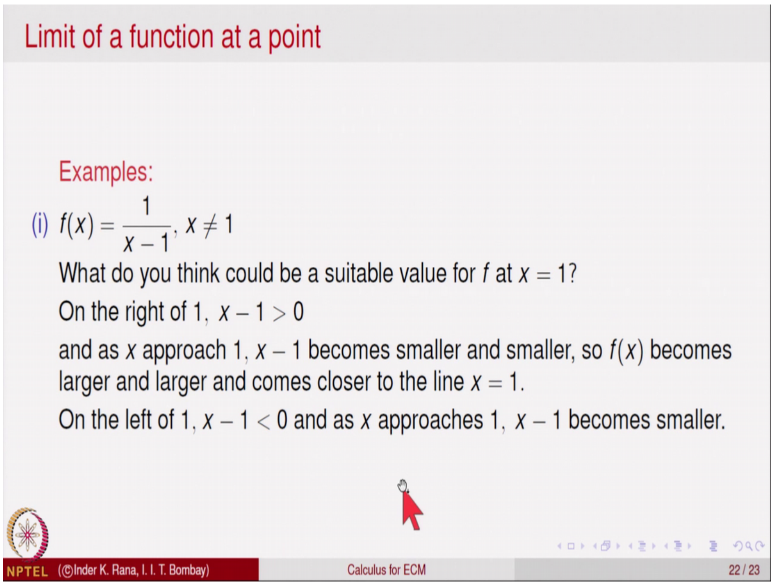

So, let us look at this example f of x is equal to 1 over x minus 1, if x is not equal to 1. So, what is the domain of this function? Domain of the function is all real numbers not equal to 1 right because at x is equal to 1 this value will be 0. So, this is not defined. So, the function is not defined at the point x is equal to 1, that point is not in the domain of the function. But that does not matter function is defined at all points near the value x equal to 1. So, we can look at the limit of the function for f x as x goes to 1 whether that exist or not. So, let us look at that do you think we can produce a suitable value well let us look at the particular cases. So, on the right of 1 if I look at when x is bigger than one that is on the right side this will be the positive quantity right. So, whenever I take a sequence on the right of 1, I approach the one from the right side these all will be positive.

So, the limit will be going to be a positive number right if at all it converges right; however, if x approaches one from the right this value is going to increase, because this value is going to become smaller and smaller, one over that is going to become larger and larger. So, as x goes to 1 from the right side 1 over of x minus 1 goes to plus infinity while if I look on the left side x minus 1 becomes smaller and smaller, but it is negative right so; that means, what?

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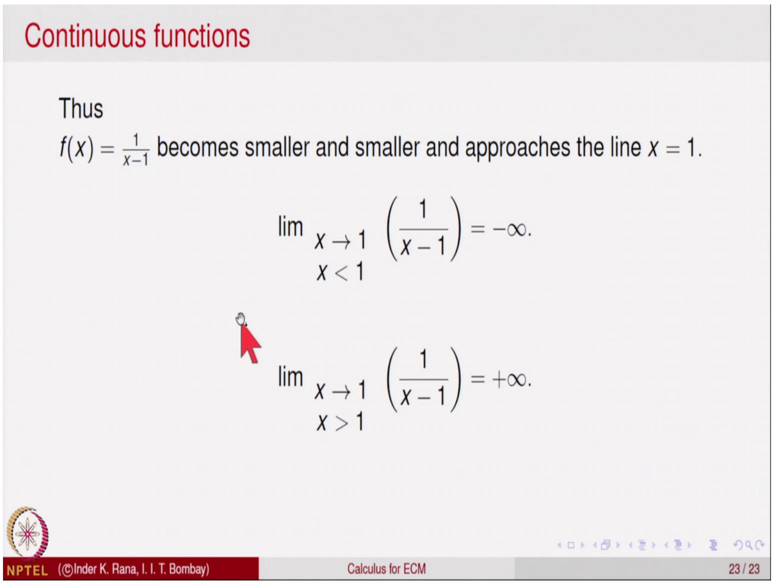

If I approach from the left right is smaller and smaller and approaches a line x is equal to 1, and if I approaches the line x is equal to 0 and if I look at. So, for x less than one this is negative it goes to minus infinity.

So; that means, on the positive side it goes to plus infinity; that means what? That means, as I approach the point 1 the value of the function for the sequence is either approaching plus infinity or minus infinity; that means, I cannot predict a suitable value for the function right. So, as x approaches 1; that means, for the line x is equal to 1, we are coming from the right side the value becomes plus infinity it keeps on increasing on the left side keeps on decreasing. So, we cannot predict a suitable value for the function right.

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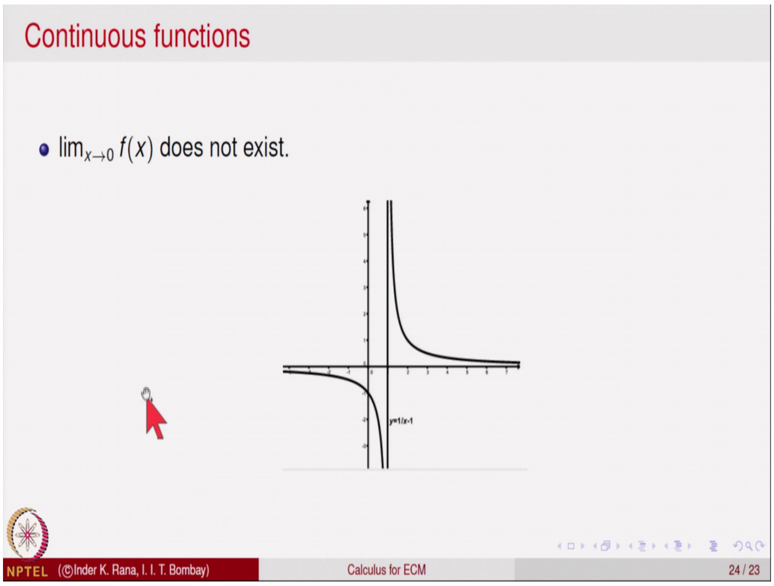

So, no suitable value can be predicted, and this is looks like the graph of the function if you want to sorry this is x is equal to 1 not x is equal to 0. So, that is a line x is equal to 1.

So, as I approach from the left the value keeps on increasing as I approach this line from the right from the left it value keeps on decreasing. So, that does not exist. So, we will continue with the study of these limits in the next lecture.

Thank you.