Calculus for Economics, Commerce and Management
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Lecture – 13
Exponential function, exponential models, logarithmic function

So, welcome to this lecture. Let us just recall what we are done in the previous lecture. In the previous lecture we had started looking at various types of functions. We looked at linear functions, we looked at quadratic functions and then we looked at how to define or what motivates one to define what are called exponential functions. So, we gave a rough outline of how exponential function can be defined, hope you have revised that. Let me just summarize what is exponential function what are its properties. So, that we are able to go ahead.

So, exponential function is defined for every positive real number a of course, not equal to 1 and this function is a function from the real numbers taking values non negative values and it is denoted as f of x is equal to a x, it is something like a raise to power x with the following properties.

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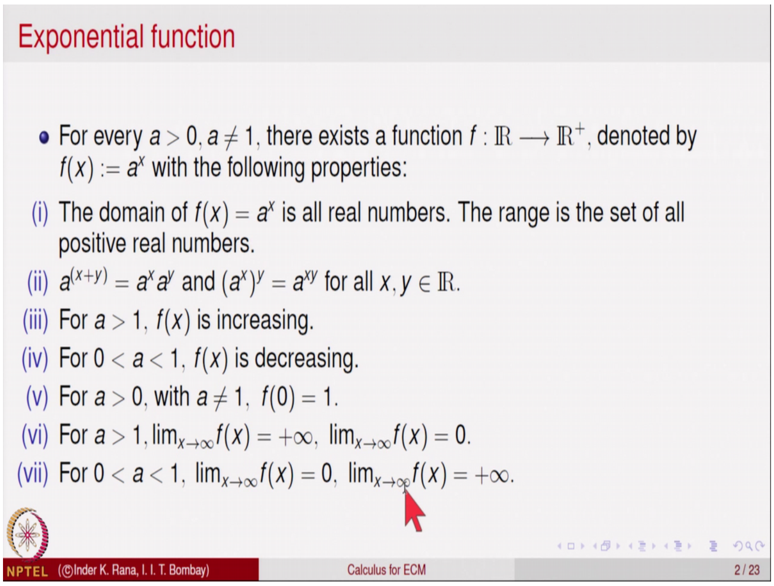

One, the domain of this function is all real numbers. So, it is defined for all real numbers. The range is a set of all positive real numbers; that means, for every positive real number there exist a real number such that x such that a raise to power x is equal to that real number y. So, range means for every positive real number y there exist a real number x such that a raise to power x is equal to y.

So, the domain is the set of all real numbers and the range is set of all positive real numbers. It behaves very much like the ordinary powers the log of exponents of work namely for this function a raise to power x plus y. So, what is this? This is a image of f of x plus y. So, a raise to power x plus y is same as a raise to power x into a raise to power y; that means, if you take two real numbers x and y and take the image of x plus y, then the image of x plus y is equal to the product of the images of x and y respectively. And similarly if you take a real number x take its image that will be a raise to power x that will be again a real number you can take its image again under the function f. So, this will give you a raise to power x raise to power y and we claim that this function has the property that this is same as a raise to power this powers multiply essentially x into y.

And for every a bigger than 1, so this number a is bigger than 0. So, I am assumed it is not equal to 1. So, it is either bigger than 1 or less than 1 if it is bigger than 1 then it is a increasing function and for less than 1 it is a decreasing function. So, increasing means as you go from left to right the graph will be going up wards and decreasing means as you go from left to right the graph will be bending downwards it will be dropping down.

For every positive real number a greater than 1 not equal to 1 of course, f of 0 is equal to 1; that means what? That means, there is for this 1 is a positive real number for this, this image pre image is nothing, but 0. So, 0 is mapped into 1 for so; that means, a raise to power 0 is equal to 1 for every a, and it keeps. So, for a bigger than 1 we said it is a increasing function and now it says that limit of x going to infinity f x is equal to plus infinity; that means, what? That means, as you keep on increasing f going as f as x keeps on increasing it goes to infinity and there is a type where should be minus infinity. So, limit x going to minus infinity is equal to 0.

So, as you keep on going to the left the function approaches the value 0 and on the right the function goes on increasing as large as you want. So, here is it should be minus infinity for a between 0 and 1 the other way around happens namely limit x going to infinity of f of x is 0 if a is less than 1 and it is limit of x going to this is minus infinity again is equal to plus infinity if x goes to infinity minus infinity for a less than 1. So, the behavior of a bigger than and 1, a less than 1 is almost opposite of each other.

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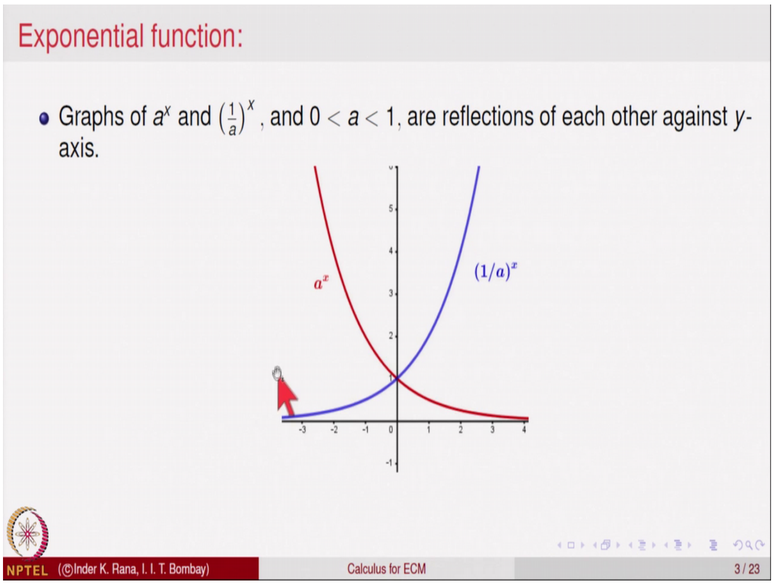

So, it quite clear that the graph of a raise to power x and 1 over a raise to power x are reflections of each other. So, if one tries to plot them this is what it look like for the number bigger than 1 this is the blue line when a is, here a is less than 1, so that is why it is written as 1 over a. So, for a real number bigger than 1 its power is this blue and it less than 1 non negative of course, then it is this. So, their reflections and 0 of the both values for both is equal to 1.

How this graph is I have plotted this graph using a software, but why does the graph look like this for the exponential function. One of course, we have said that as you go from left to right it is increasing and it goes on increasing and it on keeps on decreasing as you go to the left side the why should look like a nicely rounded kind of graph that requires more properties of functions calculus properties we will see it later on that this is actually the graph of both of them. So, this has to illustrate why graphs are important and what is the relation between the function raise to power x and 1 over a raise to power x geometrically how they look like.

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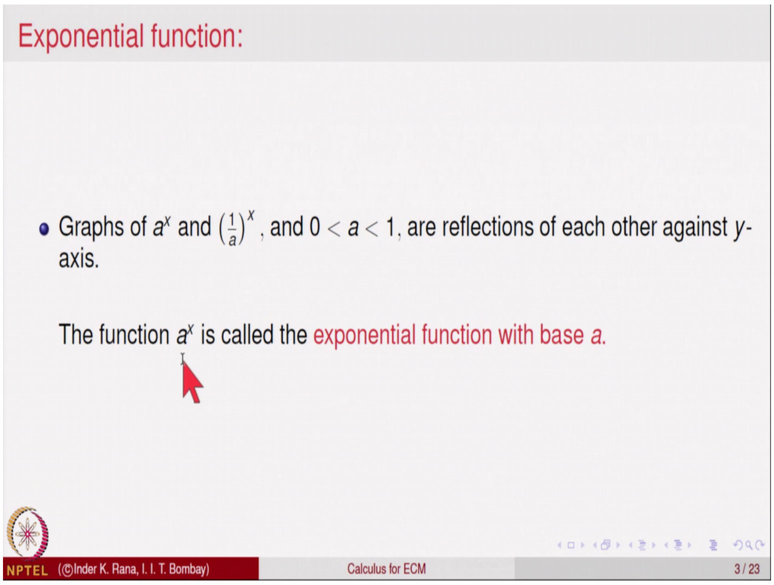

The function, both this functions or any for any a bigger than 0 this function x going to a raise to power x is call the exponential function and this number a is called the base of that function. So, it is exponential function with base a.

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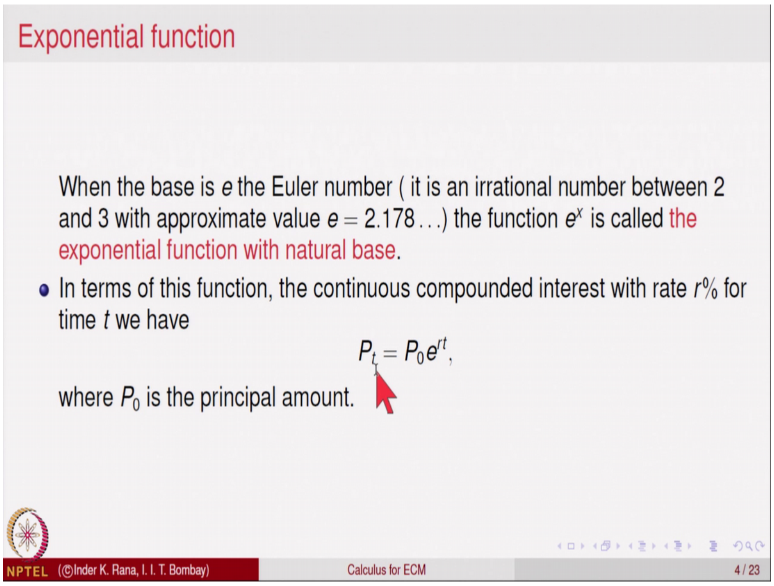

When this base e, base is a number e, e is a very special number and the real line which is bigger than 1 it is a number between 2 and 3 and its approximate value is 2.178, it is a irrational number. So, you cannot write exact value of this in the previous lecture we had seen how to define e using limits of sequences this is called an Euler number. So, when this particular base e is taken then we say that the exponential function has natural base.

So, when we say we are considering exponential function with natural base; that means, the value of a is taken as e which is the Euler number. So, for example, this is we saw in the previous lecture also in terms of this function the continuous compound did interest with rate of interest r percent in time t is given by, so P t is the principle that it will grow into in time t, P 0 is the initial investment or the principle amount, r is the rate of interest and t is the time. So, e raise power rt the exponential function appears in representing if you want to represent continuous compounded interest at the rate r for time t.

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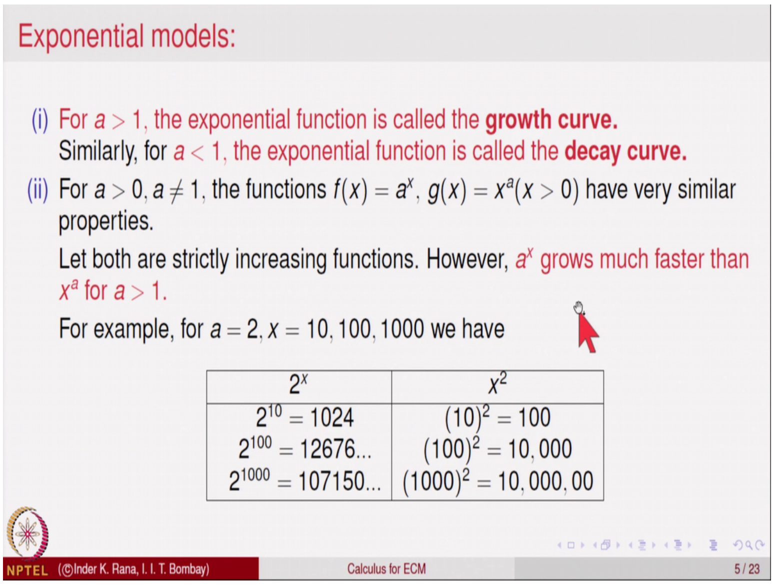

Let us look at some observations that for a bigger than 1 we said that a raise to power x is a increasing function, values increase. So, normally for a bigger than 1 the exponential function a raise to power x is called the growth curve and for a less than 1 because it is decreasing that is called the decay curve.

The functions f x is equal to a raise power x and x, to just look at the exponential function a raise power x and the function x to the power a, a is fix in both of them. So, it is here is the base being raise to power which is varying here the power is fixed and the raise you can think its varying. So, what is the difference between these two functions a raise to power x and x raise to power a? Both are very similar functions as we have seen a raise power x is increasing function x to the power a also is increasing function, but there is a difference between the two namely the a raise power x grows much faster than x to the power a just to give you a illustration let us look at a equal to 2 and x is equal to this number.

So, here is values of 2 to the power x and here is a power x square. So, 2 to the power 10 if you compute it is 1024 and 10 to the power 2 is just 100. So, you can already see the difference appearing for the value x is equal to 10 and we take x to be 100, then 2 to the power 100 we can compute it is 12676, whereas for 100 to the power 2 it is just 10,000. So, all more difference appears. When 2 to the power 1000 you can compute it is the value which is this and we are not written the other values you can see that this value is much bigger than this value. So, this says that the exponential function grows much faster than the power function.

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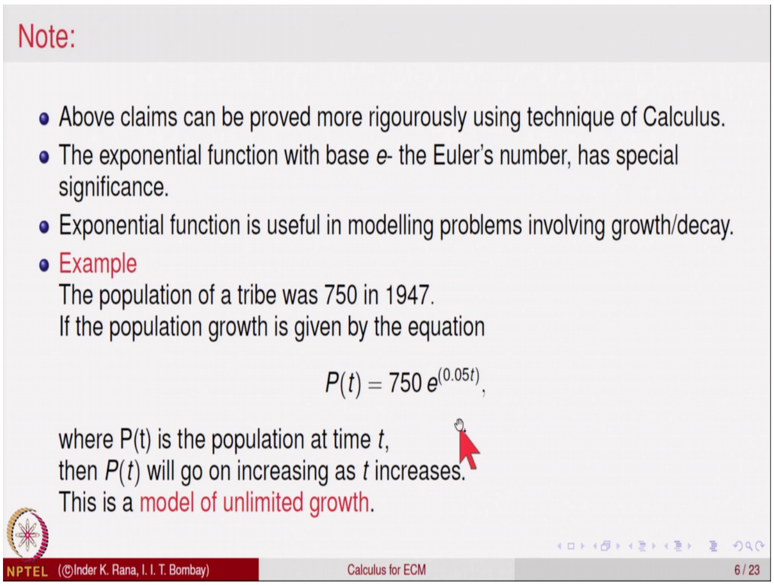

One can prove that this growth is faster using higher order calculus so we will not be doing that. The exponential function with base e the Euler number has the special significance it is useful in modeling problems involving growth or decay. We already seen for the compound interest the exponential function appears that is a growth function.

For example, let us look at the population of a tribe say it was 750 in 1947. So, here is an example how this exponential function is used and population is growing at certain rate. So, let us say this is the equation which gives the population at time t is 750 into e raise to power 0.05 t. It is very much similar to the compounded interest where the rate of interest you can think of is this. So, the growth condition, the growth curve whenever we want to model it is e raise to power some constant times t as time t varies that will give the growth. When t is equal to 0 this value is 1 that gives you the population the principle amount or the starting point. So, in this example if you want to know when you start observing what is a population. So, that is t equal to 0. So, that is 750 here. So, that is also given where as time passes the population is growing and we are saying that this model for the growth is given by this equation.So, the basic thing is exponential raise to power a t or alpha t or c t some constant time t where that constant is positive. So, that is the growth model.

So, this will keep on increasing at as time t increases. So, this unlimited growth it will grow as much as, if you do not put any checks it will grow very much it can grow as large as you want it as time grows.

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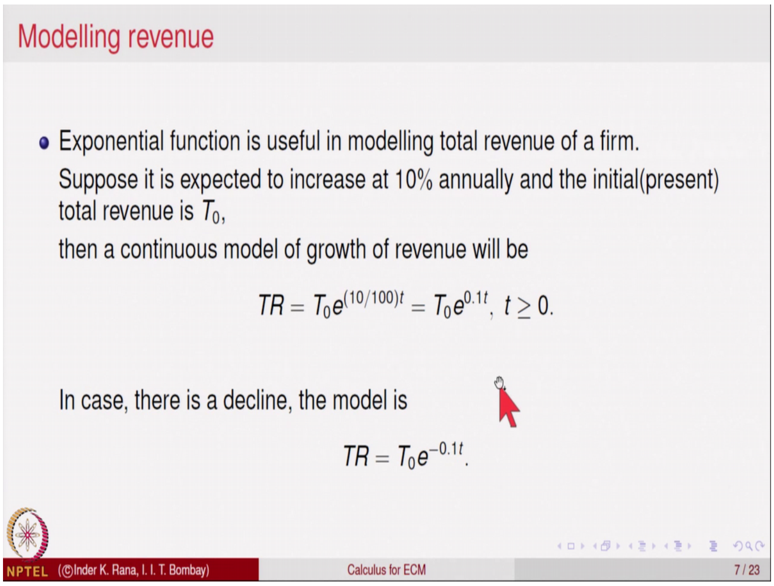

So, you can also use it for modeling total revenue of a firm. For example let us look at example suppose it is effected that the revenue of a firm increases at the rate of 10 percent annually. So, the revenue of a company or a or a firm is increasing at 10 percent annually and the initial that is say when you start observing the total revenue is t 0. So, then what is the growth model. So, total revenue will be t 0. So, e raise power 10 percent is 10 by 100 into t so that is the growth model for the revenue of a company is growing at the rate of 10 percent. So, in case you want to say there is a decline. So, how does that is represent, how is that represented by exponential function. The only difference comes is it is e raise power minus that rate at which it is decreasing times t.

So, if there is instead of growth increase the decrease in the revenue then the model will be total revenue is t 0 e raise power minus 0.1 t and if it is growth it is e raise to the power plus 0.1 times t. So, that is the difference the negative sign makes it either increase or decrease. So, in this earlier model the revenue will keep on increasing right and in this model the revenue will keep on decreasing.

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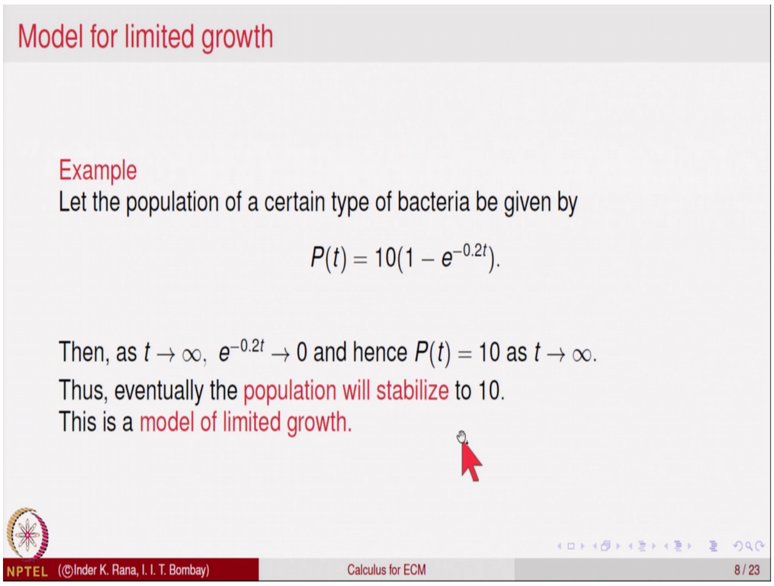

One can also have a in between thing whether the revenue or the growth is not as large as you want it, but it is gets limited to something. So, let us look at look example of that. So, let us say the population of certain bacteria is given by this function. So, P of t is 10 times 1 minus e raise power minus 0.2 t. So, in this scenario what is when t is equal to 0 this value is 1 e raise power 0 is. When, so that is 1, so that is 0. So, let us say that is 0 right.

So, now, what is happening? So, let us observe that as t increases what is happening to 0.2 t that is going to increase, but there is a negative sign. So, e raise power that is going to decrease. So, this is going to decrease, this quantity as t increases this quantity going to decrease because the e raise to power something that is same as 1 over e raise to power 0.2 t. So, 1 over 2 is exponential of 0.2 t as t increases the denominator will increase. So, 1 over that will decrease. So, this is going to decrease. And what it is going to decrease to? Essentially you can feel heuristically that this power becomes 0. So, this is going to become to 10. So, one says as t goes to infinity as t increases right e raise power minus 2.2 t goes to 0 and hence P of t the population stabilizes or eventually will reach to 10 as t goes to infinity right. So, here the population is there is a limited growth right. So, is stabilizes eventually to 10 right.

So, now, let us just point out here see as t goes to infinity we have taken this value as set of it goes to 0 is coming closer and closer to 0. So, this is the concept of a limit that we shall study soon. So, this is at present rate is only intuitively heuristically that has t goes to infinity right. So, as t goes to infinity this term the power is going to go to minus infinity; that means, this is going to 1 over infinity that is eventually sort of going to become smaller and smaller that is going to become 0. So, this is going to become 10.

So, this is institutive of way of saying that the exponential function also can be use for limited growth or limited decay. So, population here saying will stabilize to 10 right. So, these are the applications of, exponential function it can be used for unlimited growth unlimited decay or limited growth or limited decay. So, this is a model for the limited.

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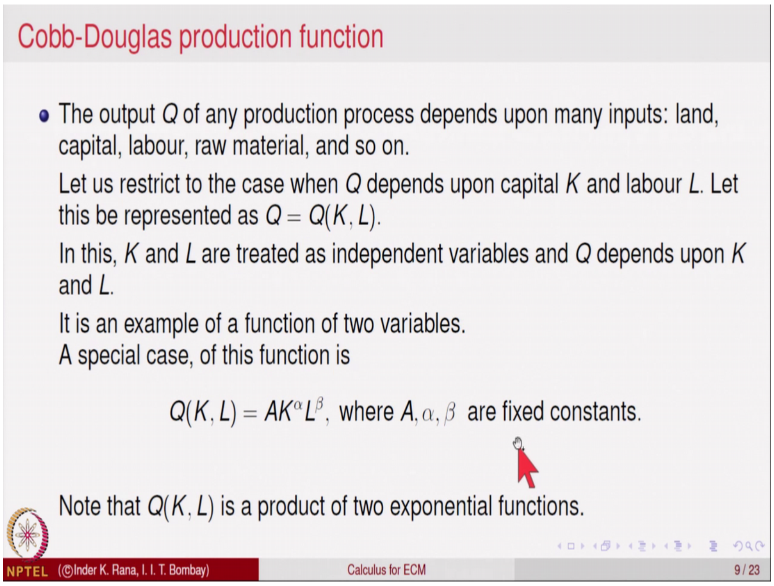

Let us look at one very particular special function which will not be dealing much at hopefully at in this course because that requires the concept of functions of two variables. Look at this function. The output of any product it depends upon many inputs right when something is being produced it depends on many things land, capital, labour, raw material and so on.

So, let us say for the time being for our consideration the output the production depends upon two things only K and L, where K is the capital that is amount invested right and L is the labour that is being implied. So, we want to analyze the production of certain product if visa we two variables namely K as the capital and L as the labour. So, this is a K how much money you invest and how much is the labour you imply that two are independent they are not related to each other. So, one says K and L are independent variables and Q is a function of two variables K and L.

As a particular special example of this relation Q K L is when this Q K L is equal to a into capital K right raise to power alpha and the labour L raise to power some beta. So, here are the two exponential functions coming into picture where A, alpha and beta are fixed constants. So, this is a model which is known as Cobb Douglas model for production of as certain quantity which depends on two variables namely capital and the labour, where this is a product of two exponential functions.

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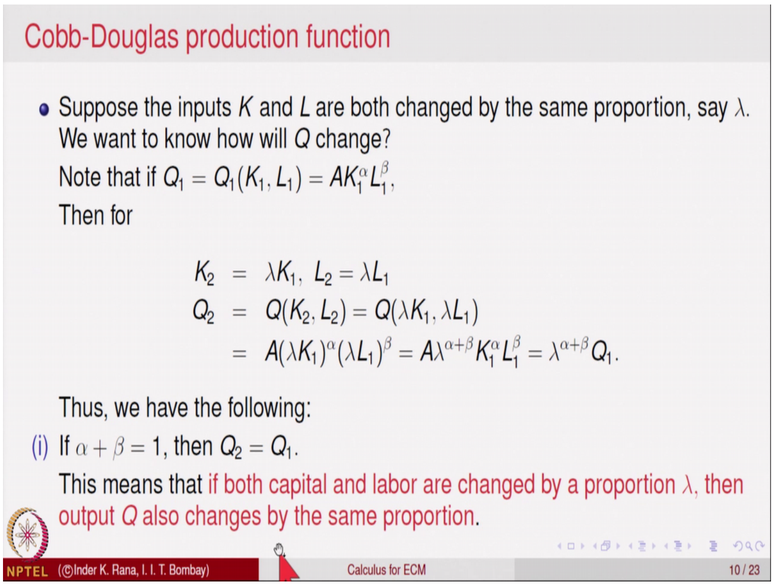

So, now, suppose the inputs K and L, the K is the capital L is the labour both change are changed by the same proportion right we want to know what will be the effects. So, they are changed by the same proportion say lambda right. So, that what does that mean? So, we change the capital by a scalar alpha that is a constant of proportionality is lambda and

L also is changed by the same constant of proportionality that is lambda. So, we want to know how was the Q change.

So, let us observe. So, the new Q 1 right will be Q 1, K 1, L 1 right if the new changed K is changing to K 1, L is changing to L 2. So, if there is Q 1 it is this then what will be K 2? K 2, we know that it is proportionately changing. So, K 2 is lambda K 1 and L 2 is lambda L 2 L 1. So, proportionately K 2 by K 1 is lambda and L 2 by L 1 is lambda. So, from some value K 1 L 2 it is change the capital is changed to K 2 L 2 and the proportionality whether same amount of same proportional amount same proportionality you change. So, what does that means? So, when you are changed by the same proportion that is reflected in this mathematics that is K 2 is lambda times K 1 and L 2 is lambda times L 1.

So, with this what is Q 2 this was Q 1, so Q 2 is you want to change it to Q 2 L 2 right. So, that is Q times lambda K 1 lambda L 1 because K 2 is lambda of K 1. So that means, you put the values here it is a times lambda K 1 to the power alpha lambda L 1 because what is Q? Q is a K raise to power alpha. So, we want as K 2. So, this is K 2 that is lambda K 1 to the power alpha lambda of L 1 to the power beta. So, when you simplify lambda of alpha lambda of beta both come out as and by the law of exponents you get lambda to the power alpha plus beta. K 1 to the power alpha and L 1 to the power beta. So, that is lambda if you write it outside. So, lambda times alpha plus beta inside is A L 1, A K 1 alpha and L 1 beta that is your Q 1. So, Q 2 is equal to lambda times alpha plus beta Q 1.

So, what does, how do you interpret this result. So, this says if K 1 and K, if K and L l are changed proportionately right thus we have the following if alpha plus beta is equal to 1 right, then Q 1 is equal to Q 2 right. In this Cobb Douglas model if it was K times alpha L times beta, if alpha plus beta is equal to 1 and you change proportionality Q and if you change proportionally K and L and if that alpha plus beta relation is 1 then Q 1 is equal to Q 2; that means, there will not be any change in the production if that condition is met. However, this means both capital and the labour are changed by a proportion lambda then output Q also changes by the same proportion right, so same.

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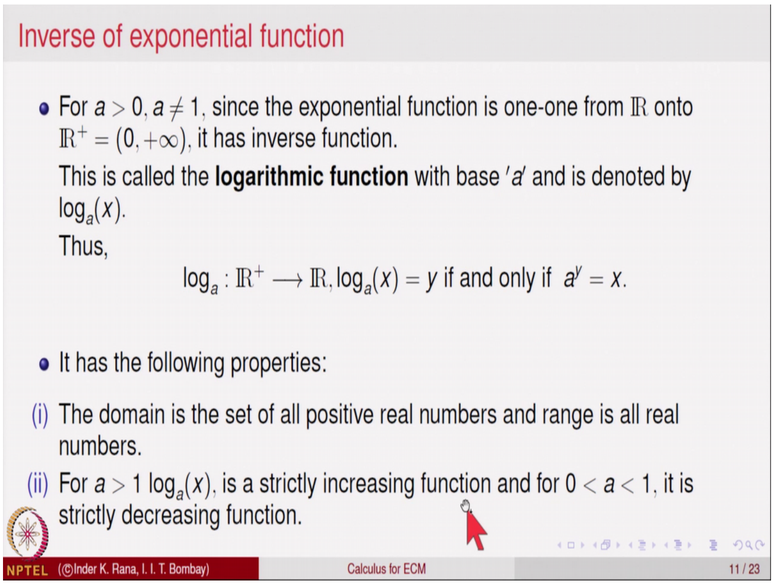

Now, let us come back to the exponential function. The exponential function which is defined for a bigger than 1, a not equal to 1 was a function with which was a 1 1 function from the domain was the real line and the range was the whole of r plus.

So, because it is a 1 1 onto function it has a inverse function and that inverse function is denoted by logarithmic function log function with the base a, if this is a then it is called the log function logarithmic function with base a. So, this is a function is a inverse of the exponential function exponential function for was from r to r plus. So, inverse function is from r plus to r and it is defined by the property that log of a right this is denoted by log to the base a of x is y if and only if a raise to power, a raise to power y is equal to x. So, that is a definition.

One can write down the properties of this log function essentially what we are doing is we are reflecting x against y. So, to do that, note that the domain is the set of all positive real numbers and the range is set of all real numbers. For a bigger than 1 it is a strictly increasing function, for a less than 1 it is a strictly decreasing function right.

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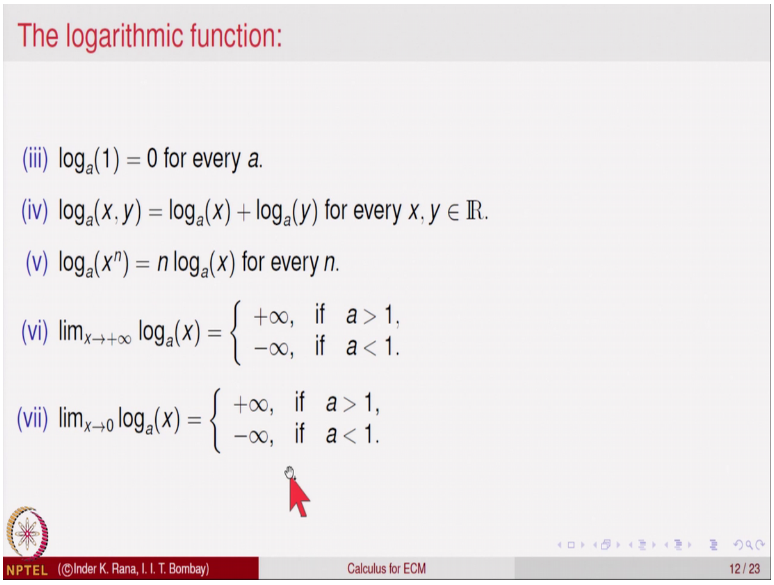

Exponential of 0 was equal to 1. So, log of 1 is equal to 0 we had exponential of x plus y is equal to exponential of x into exponential of y and so log function has the property the log of x y x this is not common this should log of the product x y is log x log y and similarly if you repeat it log x to the power n is n times log x and it goes to if a is bigger than 1 then as x goes to infinity this goes to infinity. So, it is a increasing function which goes to infinity as x goes to infinity and it or minus 1 it goes to minus infinity or as x goes to 0 as x goes to 0 log of 0 goes to for a bigger than 1 to plus 1 and less than 1 it goes to minus infinity. So, these are basically properties of the log function which can be deduce easily from the properties of the exponential function.

In higher calculus one looks at the properties of the inverse function visa viz of the given function and its graph is what it this look like.

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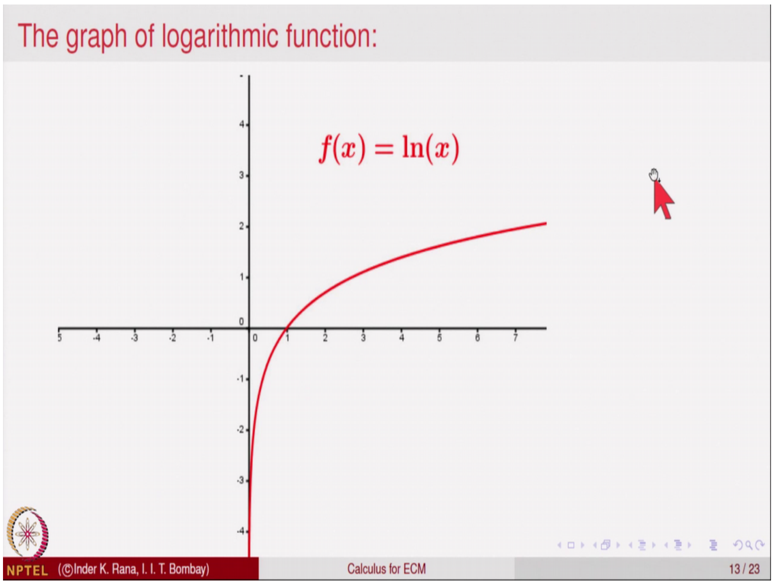

This is the graph of the function we have when the base is natural base; that means, a is equal to the Euler number then this is the graph of the function we are written. Basically it is the same for all functions it is a increasing function it keeps on increasing goes to infinity as x goes to infinity and it goes to 0, it goes to minus infinity as x goes to 0. So, it is basically if you draw a line y equal to x and deflect that is a graph of e raise to power x. So, it is a reflection against the line y equal to x.

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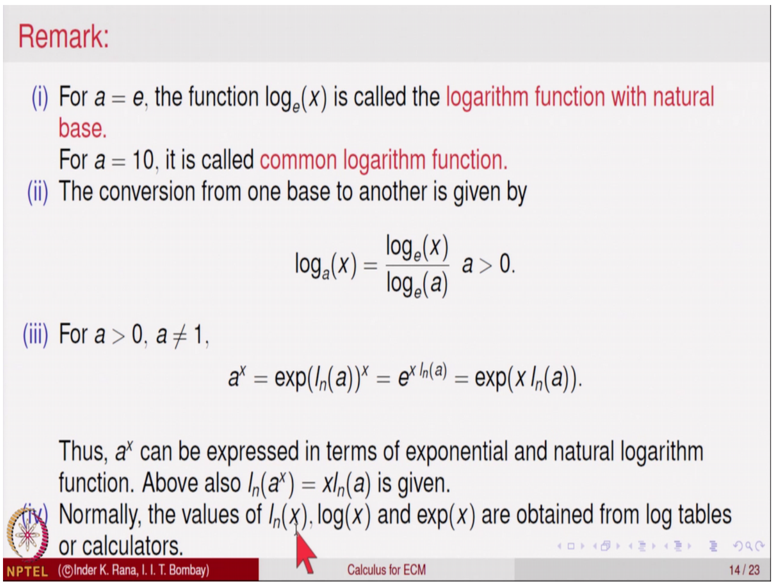

Just few remarks that for the base a equal to e it is called the log with the natural base otherwise it is called the log or the common logarithm when a is equal to 10 it is called the log function, just log function or the common logarithmic function. This is the formula which allows you to change from base a to the natural log. So, log a to the log x. So, the base a is log x to the natural base divided by log of that base to the power natural log.

So, these are some relations between the exponential function and the log a raise power x is same as exponential of l n of a to the power x and that is same as e raise power l n of a and that is equal to this, because they are inverse of each other so that is. So, one can express x and a raise power x in terms of natural log using this formulas. Normally the values of in computation problems the values of log x l n, x exponential x are used either from the log tables or from the calculators.

So, let me stop here for today's lecture by saying that we have looked at what is exponential function, we have looked at its properties today and we have seen how it is used in various modeling of economic and commerce problems. It can be used as a for a growth model with a raise power c t or alpha t, alpha positive gives you the growth model and alpha is negative it gives you the decay model. And then you can also change it a bit 1 minus e raise to power minus alpha t to saying that it stabilizes somewhere right some value. So, this is our applications of the exponential function. Not only in economics commerce and management it appears in many modeling problems in applied mathematics also.

Thank you.