Calculus for Economics, Commerce & Management
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Lecture – 12
Quadratic functions, quadratic models, power function, exponential function

So, welcome to the study of quadratic functions. So, if we recall in the previous lecture we had started looking at finding the roots of a quadratic equation and we said that if we look at the quadratic equation a x square plus b x plus c equal to 0, then it has 2 roots x is equal to minus b plus minus b square minus 4 a c by 2 a provided b square minus 4 a c the quantity has a square root; that means, it is bigger than or equal to 0.

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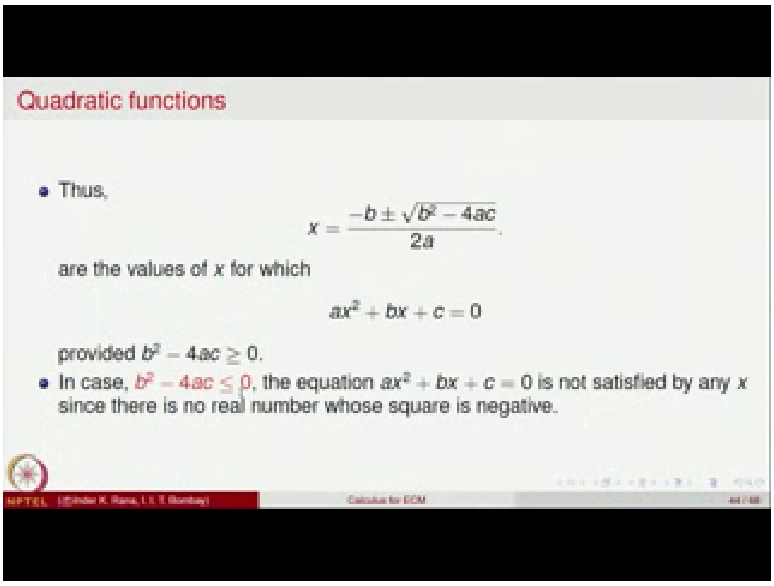

So, let us look at case when it is negative when b square equal to minus 4 a c is less than or equal to 0 this is a real number which is negative. It will not have any square root. So, no real root exist for this quadratic. So, actually roots exist as a complex number, but for our considerations we are looking at only the real roots.

So, for a quadratic a x square plus b x plus c equal to 0 if b square minus 4 a c is less than or equal to 0, then no real root say this this should be strictly less than 0 because only if it is 0 then the root will exist. So, there is a small type over here b square minus 4 a c strictly less than 0 no root will exist because then it is negative quantity.

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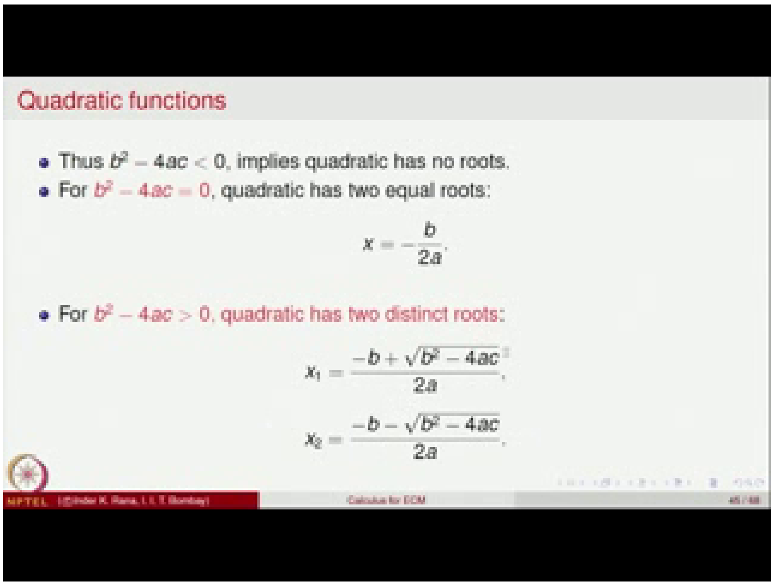

Right. So, no real root exist this is here that there is no real root when it is equal to 0; that means, 2 roots exist, but both are equal. So, this is called identical roots. So, the real roots exist both are identical. So, the roots are equal to minus b divided by 2 a. So, these are the roots of a quadratic. So, for b square minus 4 a c bigger than 0 as we have seen 2 distinct roots exist and they are given by these 2 values minus b plus square root minus b minus square root divided by that.

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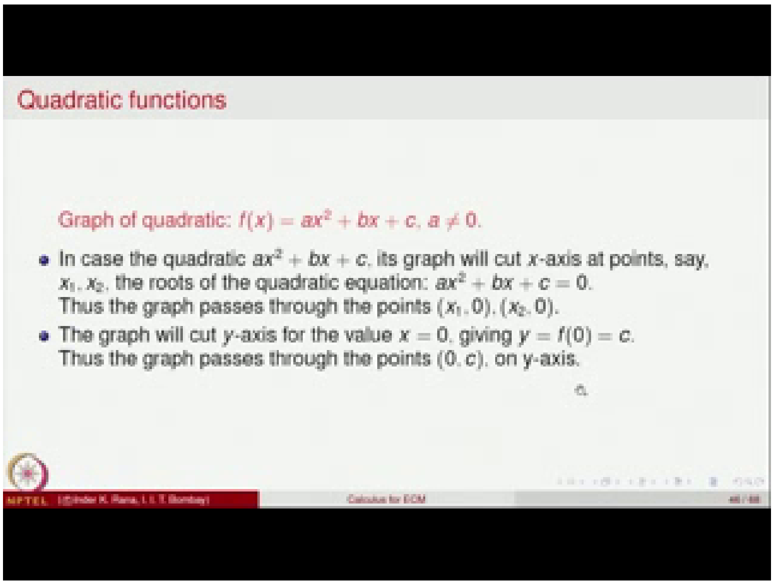

So, these are 2 distinct roots next what we want to do is we want to look at this function f x is equal to a x square plus b x plus c a not equal to 0; how does one represent it graphically what is the graph of this function; that means, we want to plot all the values x comma y such that y is equal to a x square plus b x plus c a not equal to 0 how does one do that let us try to analyze this a slightly. So, first of all this is a quadratic. So, let us assume that b square minus 4 a c at the most it can have 2 roots right. So, x one x 2 are possibly the roots it may not have any real root it may have a 2 distinct real roots or it may have only one real root both the roots may go inside.

So, all this data can be put as part of our information namely when is if x is equal to 0 y is equal to c. So, the graph also passes through the point 0 comma c it cuts the y axis in at the point 0 comma c. So, it cuts the x axis at 2 points possibly, it cuts the y axis at the point 0 comma c. So, these are points of intersections with the axis 2 axis.

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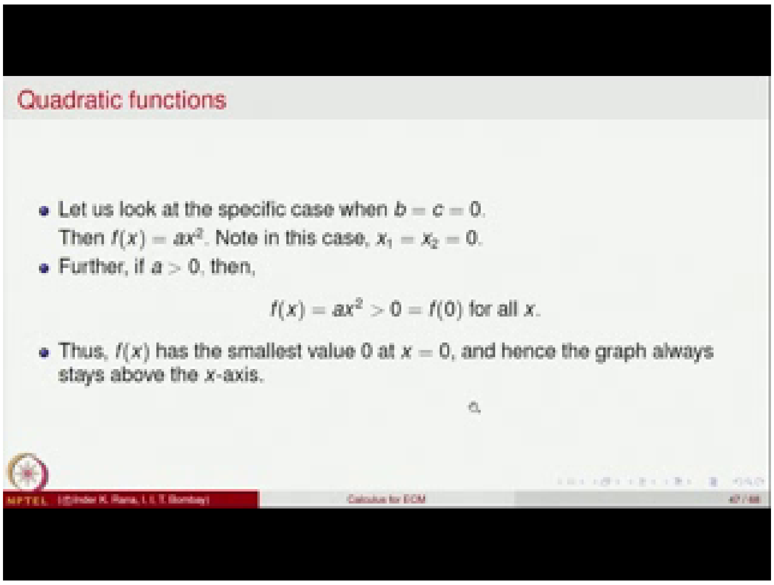

Next let us look at a slightly simpler situation when in the quadratic the coefficient b and c both are equal to 0. So, in that case the quadratic equation becomes f x is equal to a x square. So, a simple observation that for the quadratic f x is equal to a x square; what are the roots a x square equal to 0. So, x is one equal to 0 x 2 is equal to 0. So, both the roots go inside and it is 0.

So, the curve passes through the origin 0 0 and right it actually is touching at the point 0 0 because both the roots exist it cannot go below 0 0. So, for a bigger than 0 when a is bigger than 0 f x is a x square right x square is always a non negative quantity. So, a x square is bigger than 0. So, for every other value at 0 the value is 0 for every other value f of a is bigger than f at 0 for all x. So, what does that indicate that indicates that that f x has a smallest value at 0 and that value is 0 hence the graph of the function will always be above the x axis every other point which is on the graph of the quadratic y equal to a x square will be above the x axis. So, in the graph we will be touching at 0 0 it is always above the x axis it cuts the y axis right c is 0.

So, that is also gone ok.

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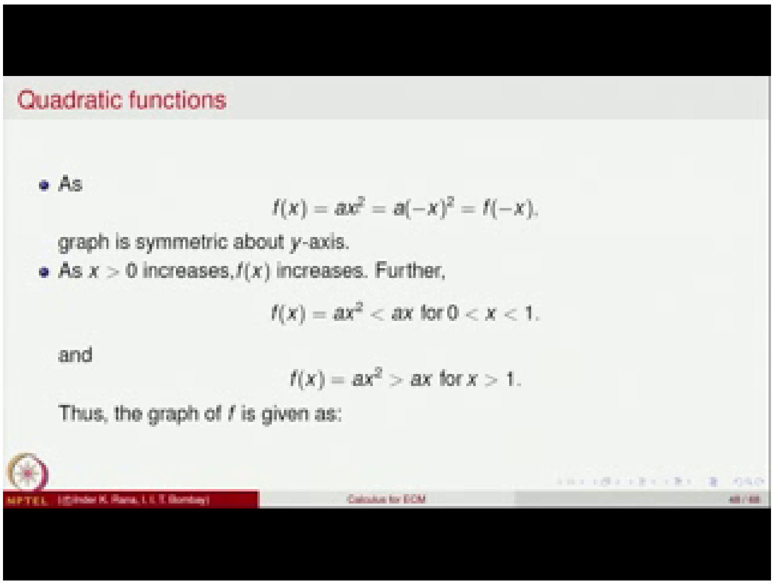

And let us observe one thing that f x is a x square is a times minus x square is the same as f of minus x so; that means, f of x is same as equal to f of minus x. So, this implies that the graph is symmetric around y axis if you shift change x 2 minus x the value of y remains the same. So, x comma y if it is on the graph then minus x comma y is also on the graph of the function.

So that means, it is symmetric right, we are trying to get a picture of the function visualization if x is equal to; bigger than 0 right, then f x is increasing of course, because f x is equal to a x square. So, if x is bigger than 0, then it is increasing right further note that a x square is always less than a x between 0 and 1 because x square will be much smaller between 0 and 1. So, the graph between 0 and 1 will lie below the line y equal to a x and it will grow faster it will be above a x square is bigger than a x.

So, it will take over the graph of a x for x bigger than 1. So, when if a is equal to 1 that is equal to y equal to x square.

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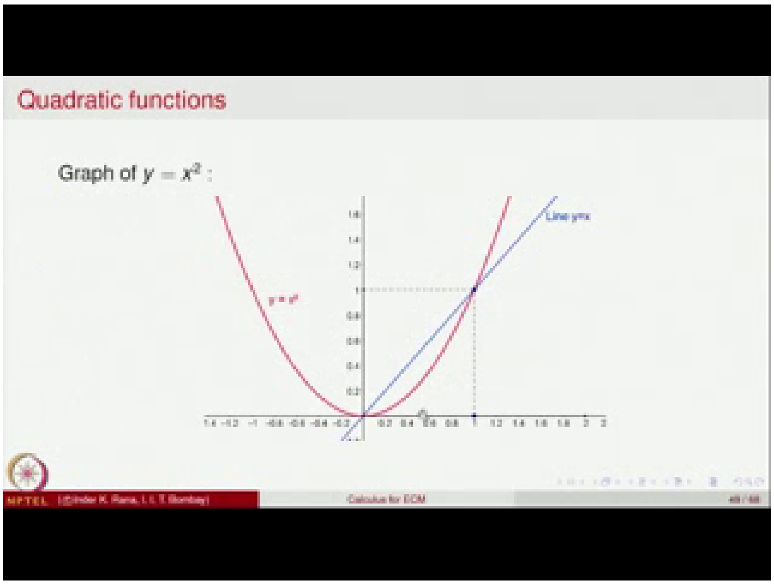

Then we can easily plot the graph. So, this is a graph of the function y equal to x square where you can taken a equal to 1 just to simplify and understand what is the graph right. So, this is the graph of the function y equal red one is the graph of the function this is the line y equal to x because a equal to 1. So, up to though the value one the graph is below the line and then it overtakes and grows faster and it is symmetric around the line y axis here one thing would like to mention that we have very nicely drawn this kind of a smooth kind of a curve for the graph how do we know this graph actually looks like this right or it does not go kind of wavy kind of it does not go like this, right. So, all these things will see later when we look at the notion of derivative of a function.

So, at present we are just intuitively we can take some values of x between 0 and one and join them as we do in school and see that probably this will look like a graph of the function this graph is drawn with a help of a software it is called geogebra; g e o g e b r a, geogebra you can search the internet and this is a open source software for plotting functions and doing many other things in geometric. So, along with in this course, I would say yes that you download that software and start getting to know how to import a function and get its graph that will help you to understand the graph the functions and many other things better because visualization of something always helps ok.

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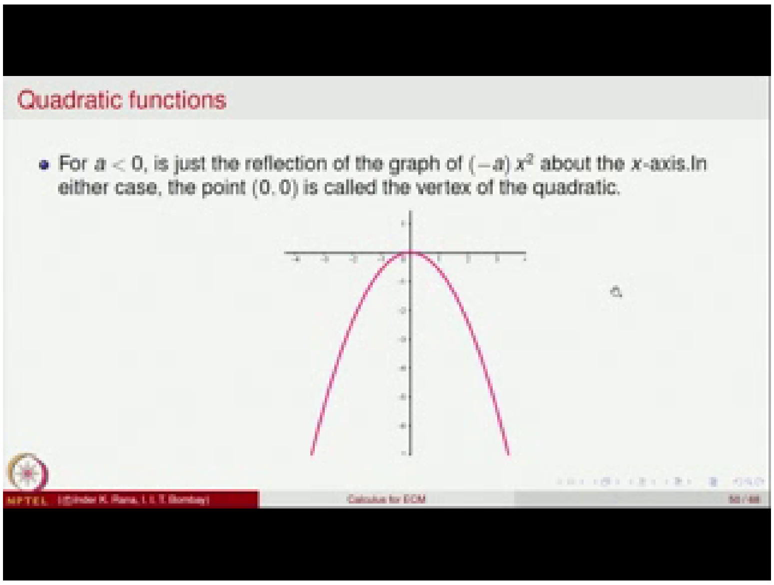

So, this was when a is positive when a is negative right for example, y equal to minus x square all the values are going to be negative.

So, it is become a graph below the x axis. In fact, the graph of y equal to x minus a x square is just the reflection of the graph of y equal to a x square when a is positive.

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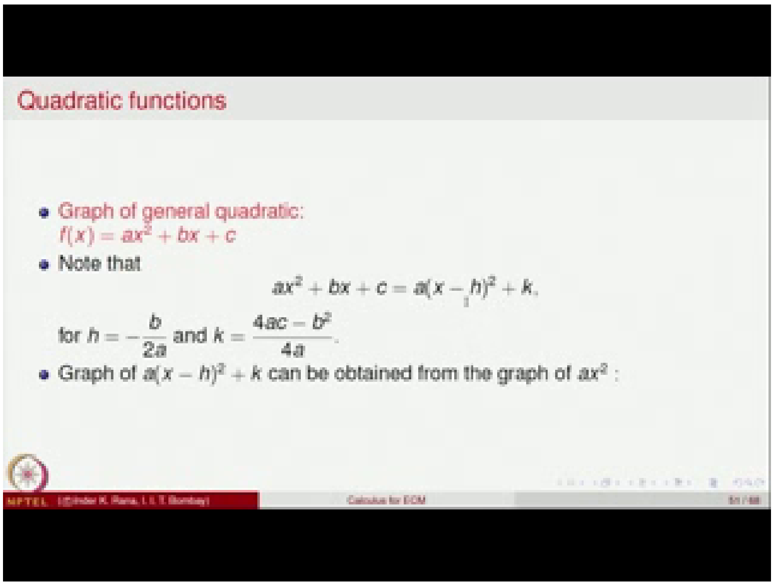

So, graph of the quadratic for a general quadratic. So, we look at the graph of the quadratic when a was one say b is 0 c 0 what happens if a b and c are not 0, how do we plot the graph of the function well what we are going to do is we are going to bring this equation a bit similar to our equation of y equal to x square and then sort of say as the graph of the function. So, for that let us observe that a x square plus b x plus c, we want to write it in the form of a multiplied by x minus h whole square plus k.

So, we want to transform this equation a x square plus b x plus c as a times is so; that means, we want to a is already given to us b is given to us c is given to us we want to find constants h.

And k such that for every x these 2 equations are satisfied and for that to be true for example, if x is 0 that says c is if x is 0 from here we can calculate what is c? C should be equal to b by 2 a and k should be equal to this quantity. So, from here, we can calculate or we can compute the coefficients of coefficients of the liked a powers and compute. So, once you have done that the graph of. So, using these relations, let me just repeat that saying this is equal to a into x minus h square plus k; that means, coefficient of like powers must be equal. So, k is equal to c, if you want do what you can do is k is equal to c x minus h whole square you can open it out and then compare.

So, because if you want to write this then the right hand side will be a x square plus x minus h whole square will be x square minus 2 a h plus h square i mean you open out you will get a times x square minus 2 a x h plus a h square plus k equal to this. So, when you compare constant term here will be equal to constant term on this side coefficient of x is b on this side. So, we will get a coefficient of b on that side and that will be minus 2 a minus 2 a and coefficient of x is a x pair. So, coefficient of x square is equal to a; that is equal to a. So, from those 3 equations you will get in session only 2 equation you will get the value of h and k. So, if this equation is to be transformed into this equation then h is equal to this value and k is equal to this value.

So, i strongly says that you work out this and see that the values are correct. So, once you have done that the equation will look like a x minus h square plus k. So, that how does it compare with the equation y is equal to a x square. So, that says that the y coordinate in the earlier one is shifted by k units and x coordinate is shifted by h units; that means, this is only a shift of origin the earlier origin 0; 0 is shifted to the origin h k in the new equation and; that means, it is something similar to a x square plus p only we have to translate our origin to the value h k.

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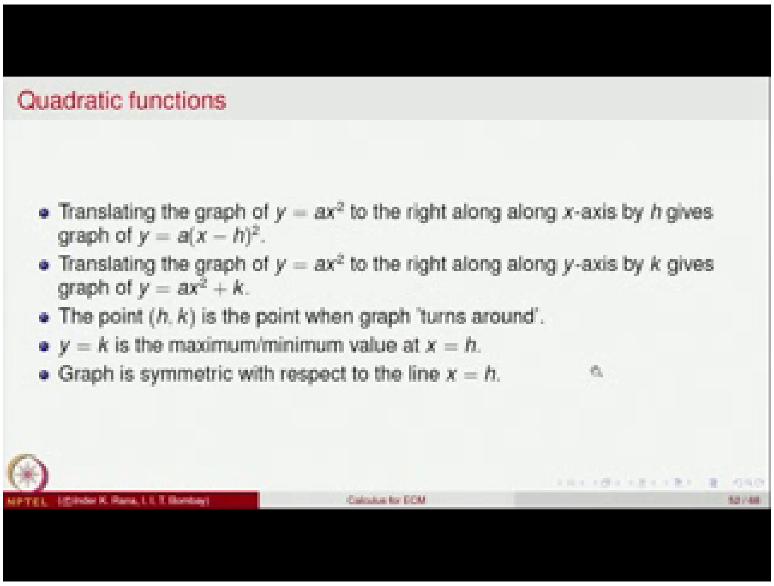

So, in intuitively. So, translating along x axis by h units in k by k units to along y axis first along x axis is h units. So, it will transform into this and k units along y axis we will transform into that equation. So, that gives you essentially one says that the origin has been transformed origin has been shifted origin has been translated to the point h k.

So, the curve will turn around the graph will turn around at the point h k earlier it turning at 0 0 depending on a and positive or negative k will be maximum or the value minimum the maximum value will be k for the minimum value will be will be k at the point x is equal to h. So, and the graph will be symmetric around the line x is equal to h. So, these are just observations translated from the equation y equal to x square.

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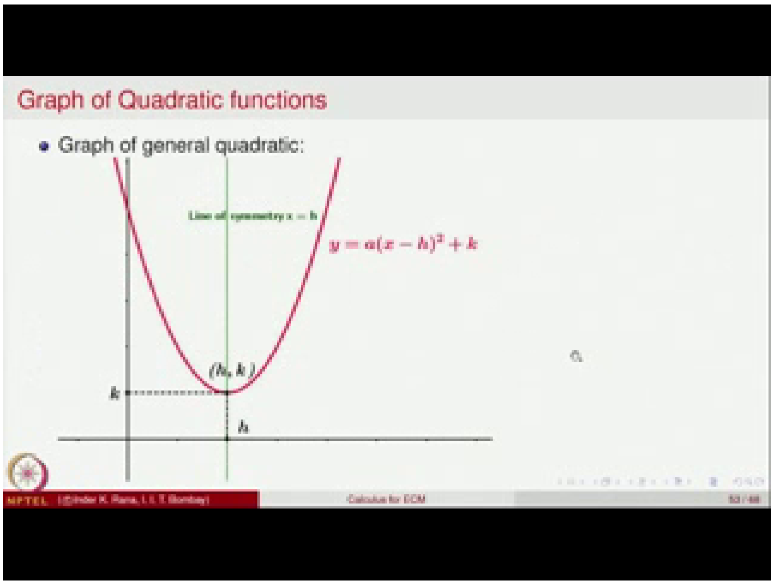

So, the graph of the function y is equal to the quadratic a x square plus b x plus c when transformed into a times x minus h square plus k where h and k are given by these values as we mentioned earlier, the graph will look like this normally this point is called the vertex of the parabola and this is called the axis of the parabola and this is the point where it will cut. So, that is the point 0 comma c.

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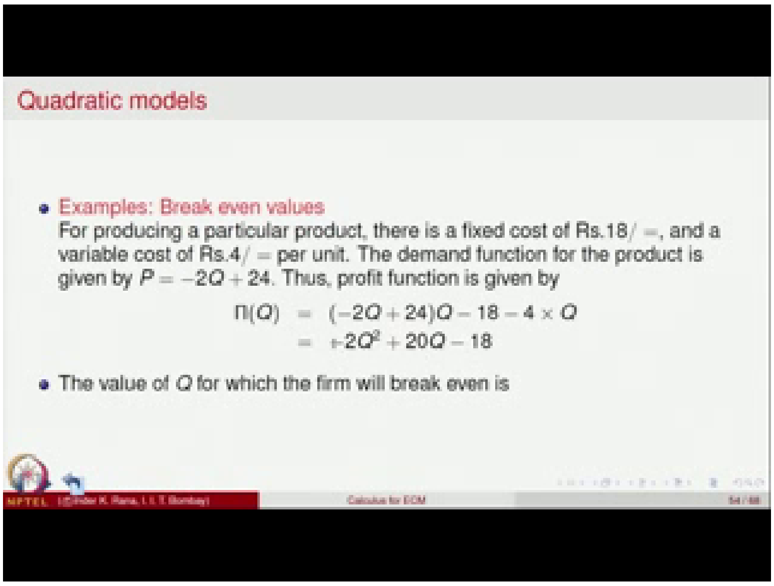

So, this is the equation. So, let us see how this quadratic equations appear and what do they indicate in our study of economic models; models in economics. So, let us look at an example the production of particular product there is a fixed cost namely 18 rupees in producing that will be the rental or electricity or such things. So, in producing a particular product there is a fixed cost whether something is produced or not that much amount is spent and is the variable cost rupees 4 per unit right and the demand function price and demand function of the product is P equal to minus 2 Q plus 24. So, this is a data given we would like to know what is a profit function for this. So, how do you calculate the profit; profit is the amount spend minus the amount earned. So, let us compute that P Q pi Q the profit is the amount earned.

So, this the price at which something is being sold right and if Q units are sold. So, product P into Q that gives you the, that gives you the; that gives you the income. So, this is the income. So, this is the price you are demanding this is the Q unit sold that that is the income and what is the amounts spent 18 rupees are spent anyway that is a face cost plus the variable cost 4 rupees per unit. So, 4 Q is the variable cost. So, this is the cost n making a one unit and this is the amount you earn by selling one Q units. So, the difference is the profit. So, minus 2 Q square plus 20 Q minus 18, if producing a particular product 18 is the cost fixed cost and there is a variable cost of 4 rupees per unit and this is a price verses demand equation.

Then this is the profit function which you immediately recognize as a quadratic. So, let us find out when will the company breakeven; that means, pi Q is equal to 0. So, we have to find roots of this quadratic equation.

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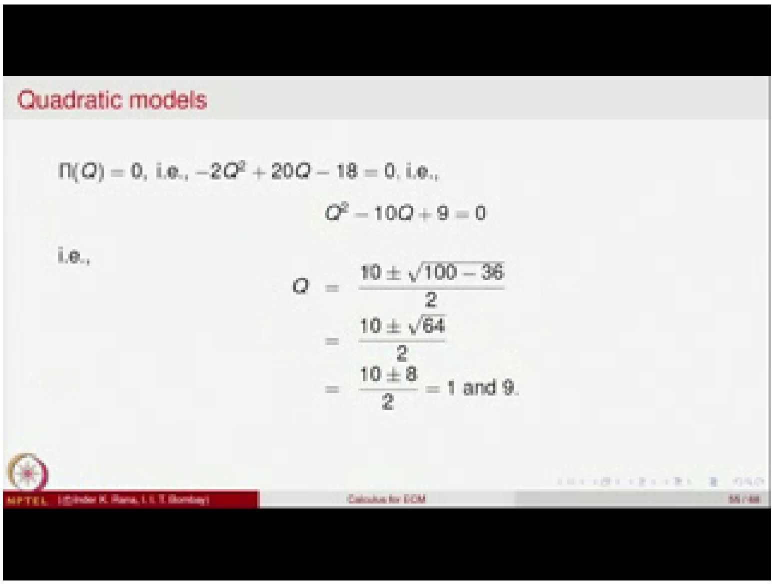

So, to find out that we put Q square minus 10 Q; this equation equal to 0 you simplify Q square minus 10 Q plus 9 equal to 0 when you solve b square. So, Q is equal to minus b; b is minus 10. So, 10 plus minus b square 10 square is 100 minus 4 a c. So, 9 into 4 into a is 1. So, 36 so that gives you the value equal to 1 or 9. So, there are 2 possibilities that Q is equal to 1 or Q is equal to nine; that means what the company will break even when it is producing only one unit right or it will break even when it is equal to 9.

So, that essentially means what let us try to interpret that what more we can reduce. So, is a quadratic a less than minus right? So, it is minus 9. So, a is minus 9.

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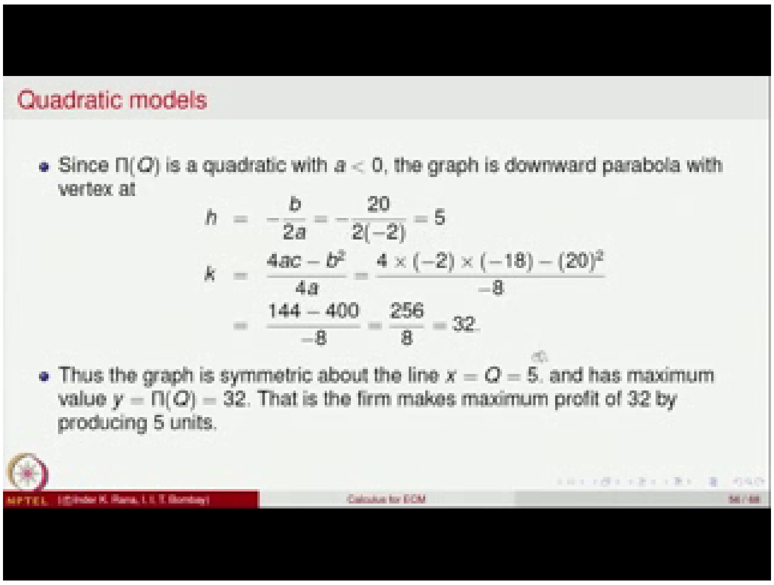

So, a is less than 0. So, graph is pointing downwards is a downward parabola with a vertex at the point we find h k that gives you the vertex minus b by 2 a. So, that gives you 5 and k is 4 is c minus b square by 4 a when you multiply. So, the simplification that comes out 32. So, for h is equal to 5 k is equal to 32, right that is the vertex of the parabola and what does the vertex indicate that indicates that for the value h equal to 5 k will be equal to 32. So, 32. So, 32 is the maximum value that Q can take right. So, for x is equal to h is equal to 5 q is equal to 32.

So, when 5 units are being produced the maximum profit will be earned and that is equal to 32.

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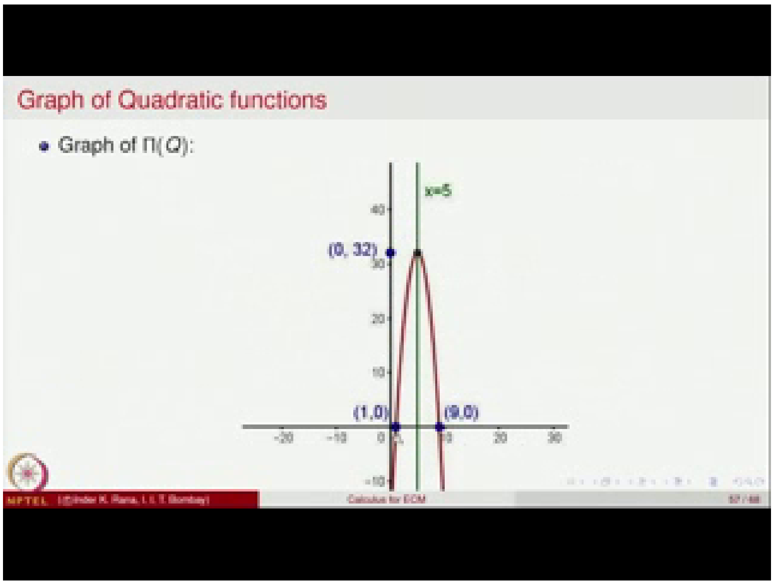

So, the graph looks like this. So, it is graph below the x axis means a profit companies in the negative, then it breaks even at the 0.10; that means, when one unit is being produced, it breaks even and then goes up and when produces 5 units the profit is 32 that is the maximum it can achieve and then again it if it produces more it will go the price; profit will go on decreasing and it will becomes 0, when it is producing 9 units and then it will go in the loss. So, this is how you interpret your profit function right. So, we have looked at the linear functions as a linear models we looked at the quadratic models and now, there are some other functions which appear in the study of economics commerce and management.

So, they are some special non-linear functions force quadratic function was non-linear, but there are some more which we will like to introduce now so that we can make a use of them.

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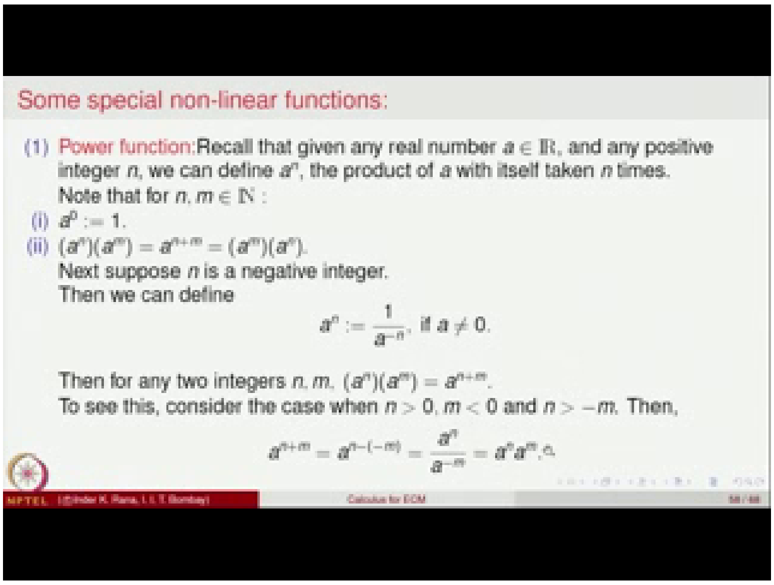

So, to motivate and introduce those let us look at what is called the power function, if you recall; what is a power of a number it is a number multiplied again and again with itself. So, given any positive real number n a raise to power n can be defined even for a integer a raise power n can be defined. So, for a positive you define a multiplied with itself a 0 is 0 and define by next one as multiplied.

So, appetitive multiplications we can do and for a negative number you define a raise to power n to be one over a raise to power minus one if a is negative a raise to power 0 is already defined. So, this is what is called powers when it is a integer we want to go a step further, but keep in mind the formula that a raise power n into a raise power m is equal to a raise to power n plus m, even for integers this make sense. So, you can give a elementary proof of this or you can just write rewrite and then do it.

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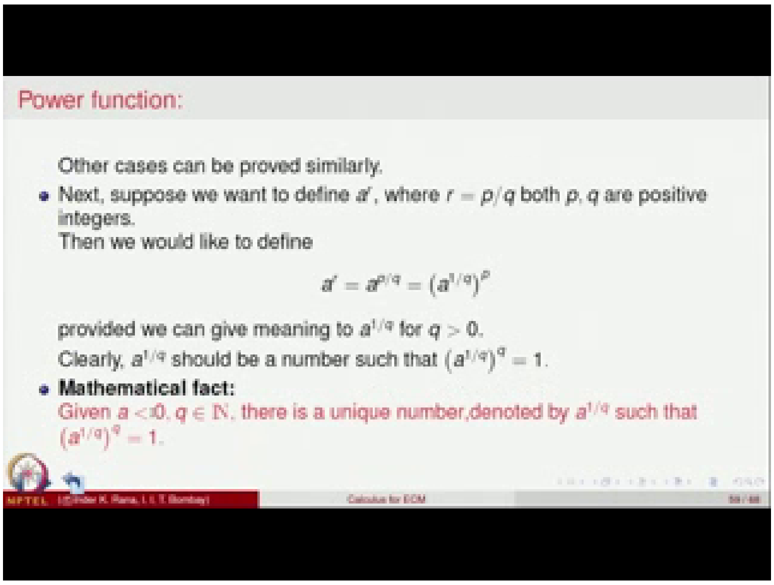

So, other cases can be proved similarly for integers now let us next look at a rational number r and we want to define a raise to power r where r is say p by q where p and q are both positive. So, we are looking at positive rationales. So, one can define it as a raise to power r to be equal to this is what we want to define.

So, what we do is a raise to power one over q? Q is a positive integer. So, one over q need to be defined and then to the power, this can be defined if we can give a meaning to a raise to power one over q where q is a positive integer will not be doing in this course what it; it is, but using advance p analysis; one can show given any positive number a and given a positive number q you can find a number called a raise to power one over q such that a raise to power 1 over q raise o power q is equal to 1. So, this is a mathematical fact; sorry this this should be a bigger than 0 this is a type over here for a bigger than 0 a belonging to n this is defined.

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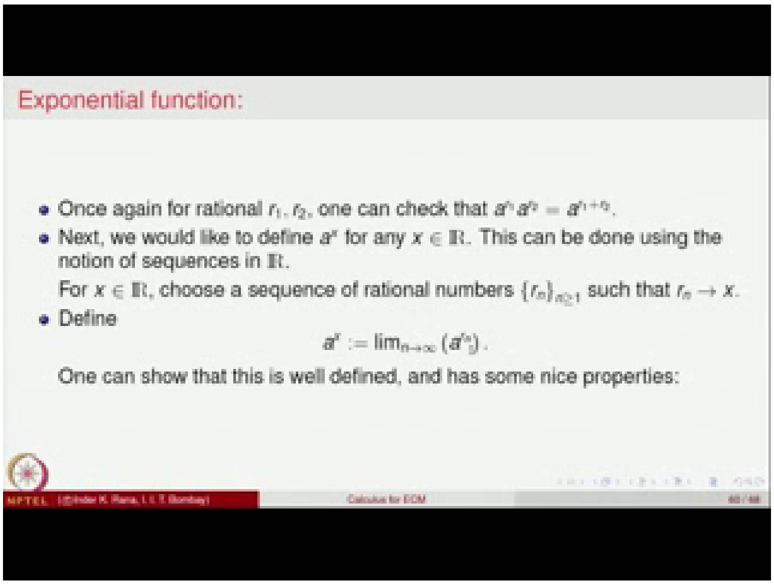

So, for rational numbers one can define a raise to power a rational number and then using definition you can prove or let us assume all this facts.

But I am just giving you a root how it is not that is easy to define the power function. So, a raise power r 1 r 2 is equal to a raise power r 1 plus r 2 once a raise power a rational number has been defined for a real number x if you want to define a raise to power x this can be done using the notion of sequences given a real number x we take any sequence of rational numbers which converging to convergence to it such sequences exist that is we called the denseness property of rational numbers and then a raise power r n is defined. So, we take the limit of that again it needs to be shown mathematically that this limit is always exist and it will be unique and so on.

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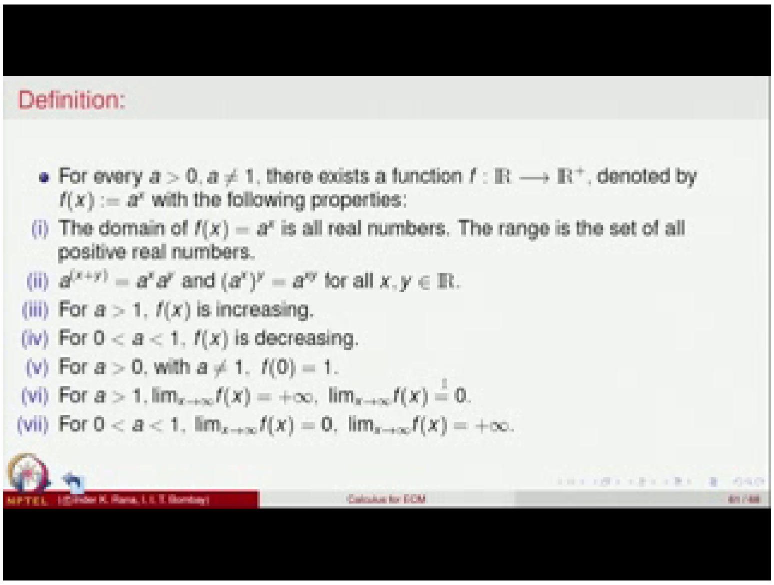

So, a raise power x can be defined and one can prove it as nice properties similar to that of the earlier one rationales.

So, for a bigger than 1; of course, a not equal to 1 a equal to 1 it will be constant function a raise to power x can be defined with a following properties. So, i just indicate it a root how this can be defined if you like you can just take it as a definition that we have a function f x equal to a raise to power x with the following properties that for every x a raise to power x is defined its domain is all real numbers range is set of all positive real numbers it obeys the law of exponents a raise power x plus y is a; a raise power x into a raise power y and a raise power x whole thing raise to power y is equal to a raise to power x y if a is bigger than one then this function is monotonically increasing or increasing and if a is between 0 and 1.

When we recall a is bigger than one then it is a decreasing function monotonically decreasing function f of 0 a is 0 for a bigger than one and a not equal to 1 f of 0 is equal to 1 one can put that as the value it keeps on increasing for bigger than one and when you are coming closer it becomes 0. So, between a and 1 this becomes. So, this gives us the graph of the 2 functions.

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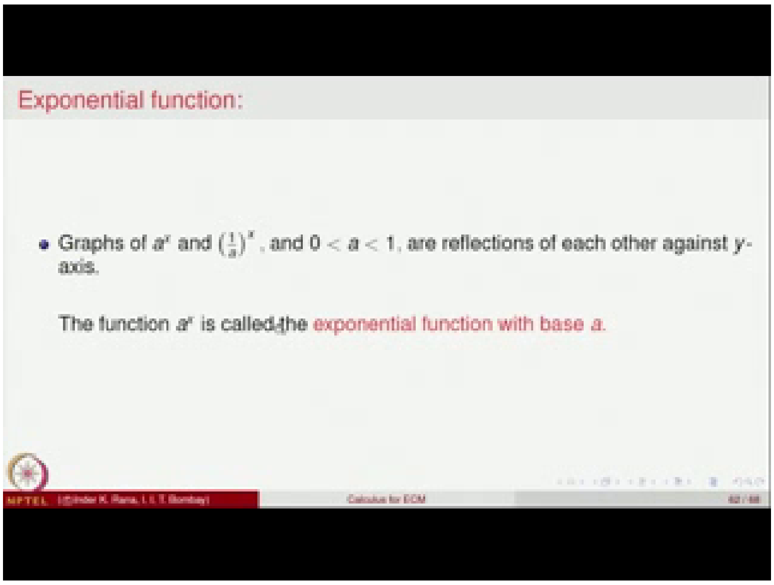

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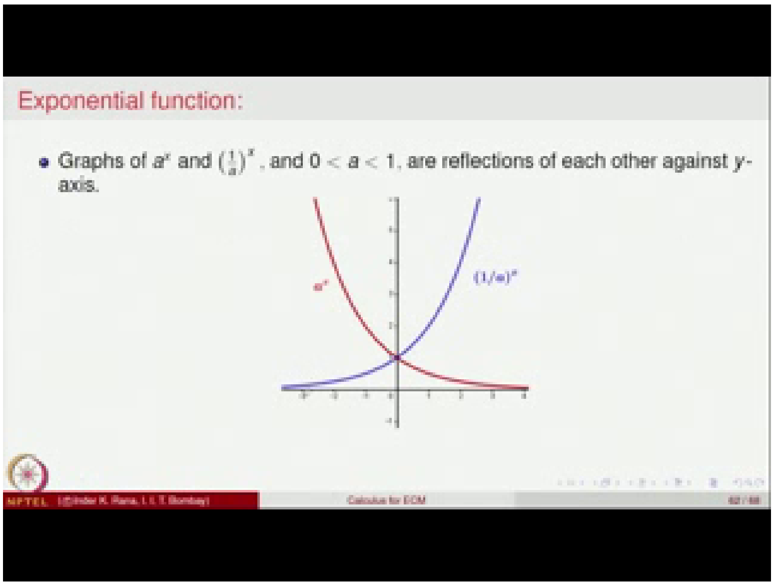

So, these are the graphs of the 2 functions. So, what we are saying is when one over a is bigger than 1; this is the graph of the function and a is less than one the red is is the graph of the function when a. So, this is defined for all values of x a raise power x is always bigger than 0 right and you can see one is the reflection of the other one can easily show that also.

So, these are the graph of the functions we will be using this properties of this function and its graph later on in our analysis this function is normally called exponential function with base a. So, a raise power x when a is bigger than 0 is called exponential function. So, a raise power x is called exponential function.

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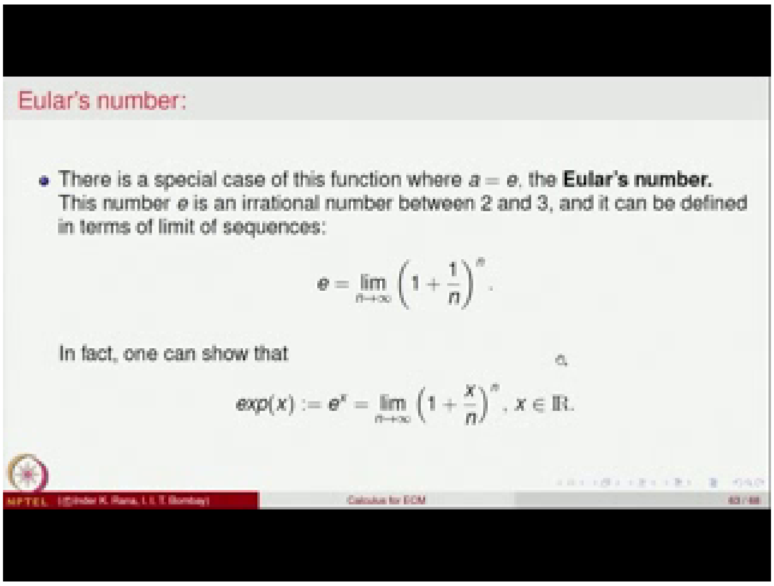

There is a special value of a when a is equal to e there is a special number called the Euler number when a is equal to that number the function is e raise to power x this number e is not easy to describe it is a irrational number and one way of describing this is to show that the sequence one plus one over n divided by whole thing is raise to power n converges and that number is called e that is one way of defining the number e.

So, for this particular value the function e raise to power x is called the exponential function. So, one way of directly defining the exponential function is this, but then one will have to prove those properties which is not easy. So, this is exponential function let us see what is y one has one needs exponential function in our analysis.

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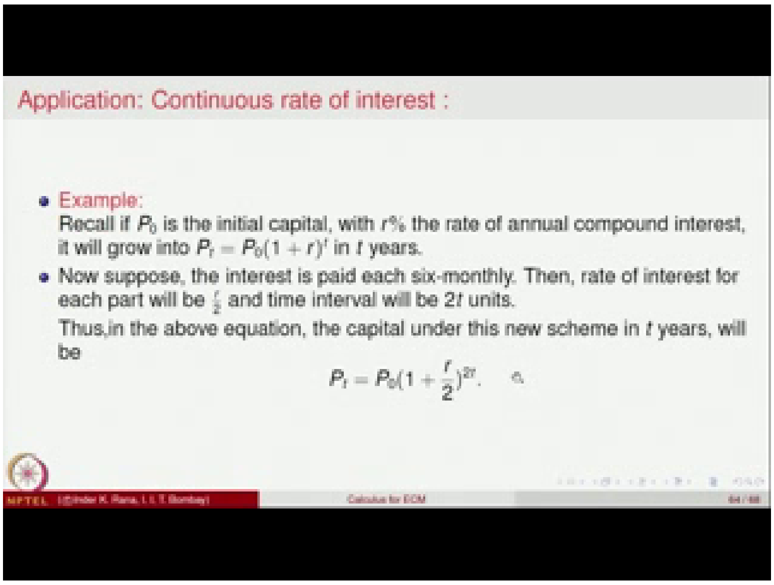

Let us recall if P 0 is initial amount with r percent rate of interest and the interest is compounded is compound interest then after t years t is the number of years for which it is compounded the amount is given by P t equal to P 0 1 plus r raise to power t we had seen this earlier and we got this the geometric expression.

So, this was a geometric progression P t is a geometric progression t is the number of years supposing the bank or wherever the interest is giving the interest rate is applied every six months if the interest rate is applied every six months; that means, we have to break the year into 2 parts. So, r by 2. So, time interval will be 2 two units. So, rate of interest will be half.

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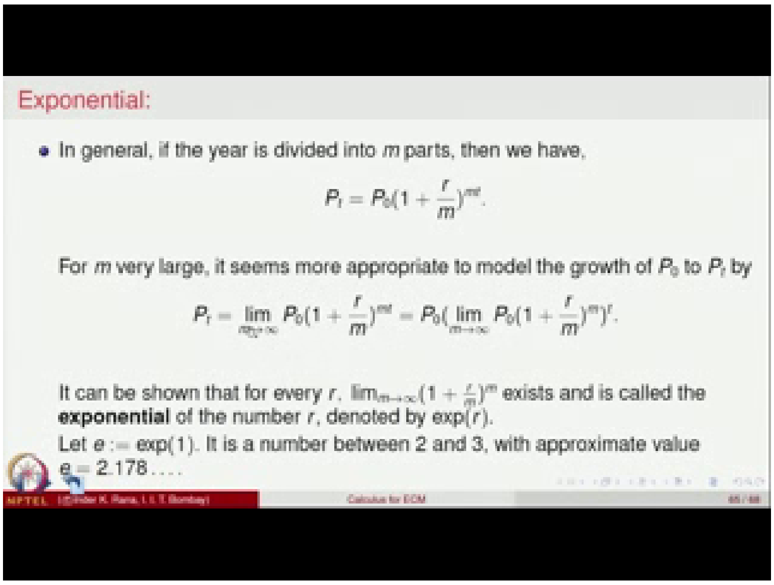

So, P t will become equal to this right that is how we will come if the rate of interest is half yearly it is applied every 6 months what about if it is implied more m times a year. So, inductively one can say that the amount will grow into P 0 into one plus r by m raise to power m t and suppose that the number of parts becomes larger and larger. So, and let us say that m becomes large.

So, larger and larger one says that m goes to infinity in that case we need to look at the limit of this quantity as m goes to infinity. So, that is equal to this and by our earlier formula one plus one by m raise to power m limit if this is justified this limit can be taken in that says that the. So, this becomes P 0 into e raise to power r t.

So, it says if the interest rate is given continuously then the formula should be if this is to be done you have to compute what is this limit. So, limit of this exist and it is called the exponential that is what we have said earlier also and here is a precise that e is equal to 2.178, dot, dot, dot. So, that is a value so, but up short is; what is the value of P t on the right hand side. So, if i all to take limit inside this becomes P t becomes is equal to P t plus P t into one plus e raise to power r t.

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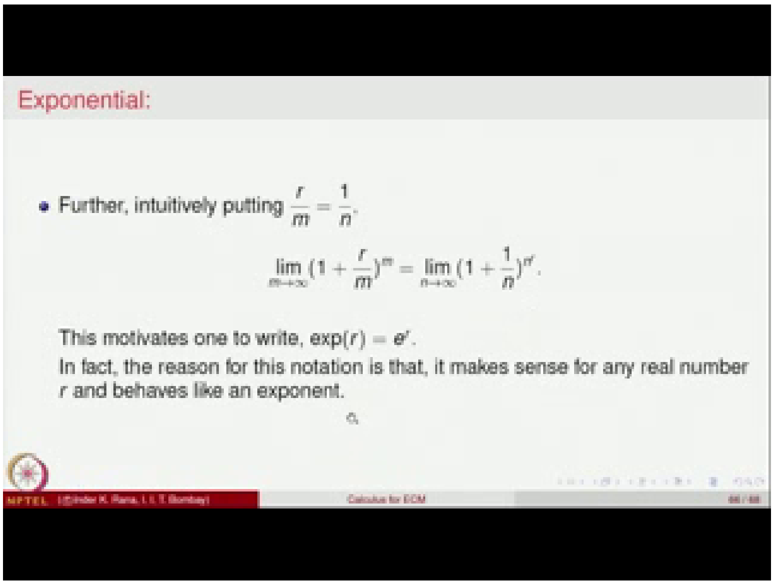

So, that is the up short. So, so that limit if it is taken as this; that means, this limit one can define e raise to power r that way and it says. So, it behave like an exponent.

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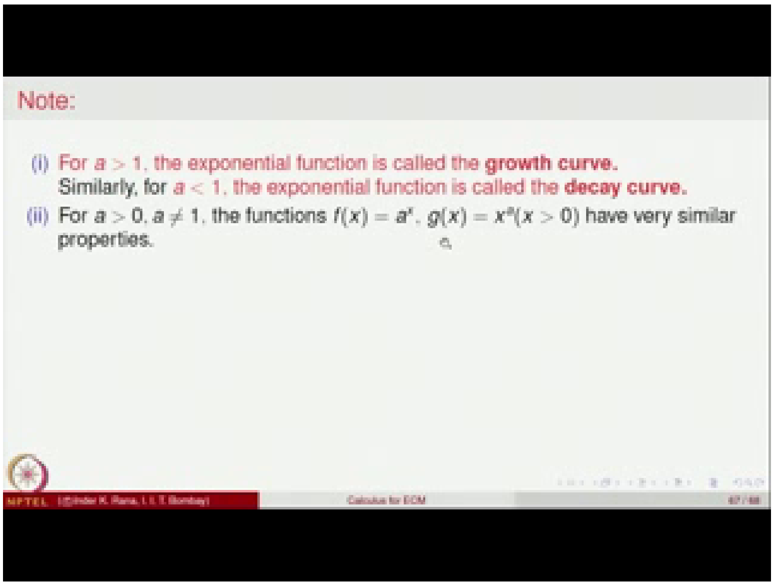

So, the growth the exponential function a raise to power x that we have said is a what is called a growth function and this is called when it a is less than one this is called the decay function. So, I think; what we will do is we stop here and we will look at the exponential and the growth function in detail in the next lecture.

Thank you.