Measure & Integration

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Lecture – 05 A
Set Functions

Welcome to lecture 5 on measure and integration. If you recall in the previous lectures, we have been looking at the various classes of subsets of a set X with various properties. We will look at what is an algebra, what is a sigma algebra, and a monotonic class. Today we will start looking at functions defined on classes of subsets of a set X. We will look at first what is called set functions, and then we will look at a very important example of a set function; namely the length function. So, let us start defining with what are called set functions.

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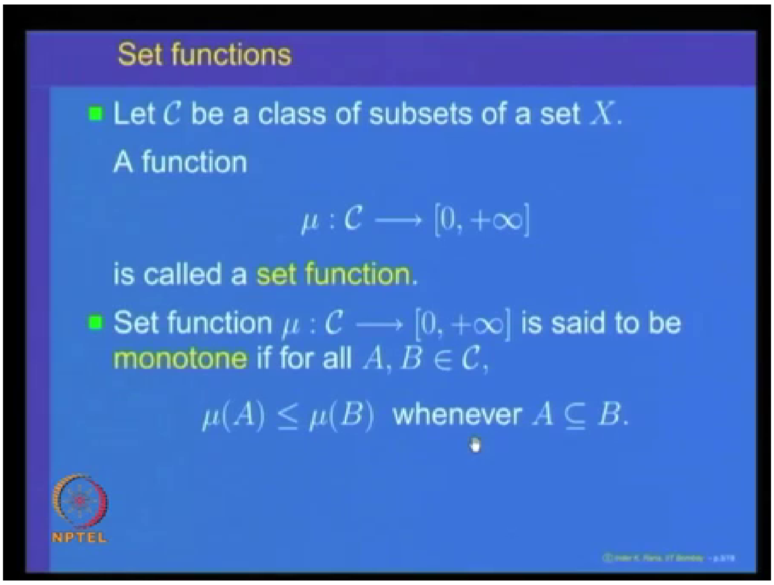

Let us start with C, a class of subsets of a set X any function mu. So, this is a Greek symbol called mu. So, a function mu defines on the class of subsets C of a set X, and taking non negative extended real valued functions.

So, this interval 0 plus infinity both included denotes the set of all non negative extended, really the extended real numbers. So, A function mu defined on this collection C of subsets of a set X taking values in non negative extended real numbers is going to be called a set function. So, it is a function, whose domain is a collection of sets; that is why it is called a set function. Next we will be looking at some special properties, which will be analyzing such functions. So, let us define a set function mu; of course, where C is a collection of subsets of a set X and 0 to plus infinity is a nonnegative extended real numbers. So, a set function mu is said to be monotone, if it has the following property; namely for any 2 sets A and B in C mu of A is less than or equal to mu of B, whenever A is a subset of B.

So, a some kind of a and monotone property, let whenever A is a subset of B and both are in the collection C. We want that mu of A should be less than or equal to mu of B. So, this is called the monotone property.

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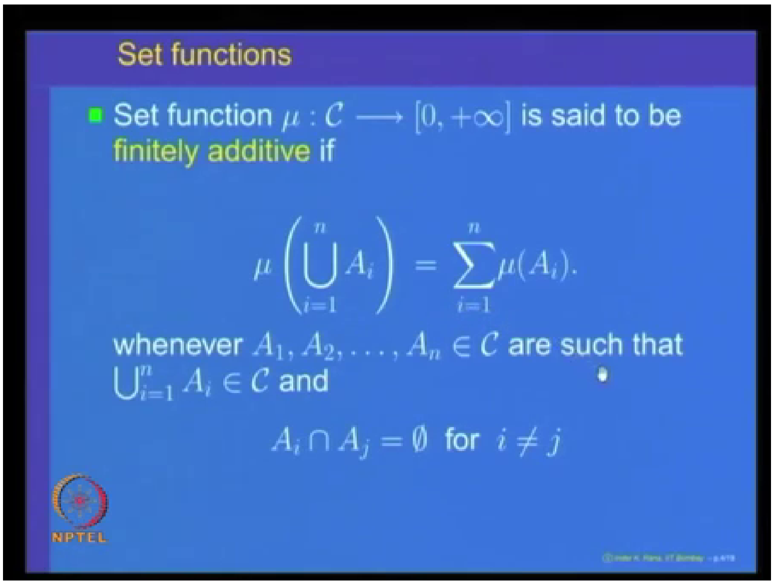

Next, we look at, what is called finite additivity property of a set function mu. So, a set function mu is said to be finitely additive. So, I am a emphasizing the point finitely and additive, if it has the following properties that mu of union of sets A i, i equal to 1 to n. So, given any finite collection of sets A 1 A 2 A n in C, we want mu of the union of these sets is equal to mu of A I's. Of course, this will be whenever A 1 A 2 A n, this is a finite collection of sets in C; such that there union also belongs to C for; otherwise this number on the left hand side of this equation will not be defined, and we want further this sets are pair wise with disjoint.

So, A i intersection A J is empty for I not equal to J. So, once again, let us see what is finite additivity. Finite additivity means for any finite collection of sets in C A 1 A 2 A n in C; such that their union is also an element in C, and these sets are pair wise disjoint for any such finite collection of sets, we want that mu of the union is equal to summation of mu of the individual A i's intuitively. Keep in mind mu in some sense is denoting the size of a set A. So, we are saying mu of the union is equal to sum of the individual sizes, whenever the sets A i's are disjoint, and this is, we are requiring it for any finite collection i 1 to n. So, A 1 A 2 A n is any infinite collection of sets in C, which are pair wise disjoint, and such that there union is element in C. We want mu of the union is equal to summation of mu of the individual A i's.

So, such a property is called finite additivity property of mu, or 1 says mu is finitely additive.

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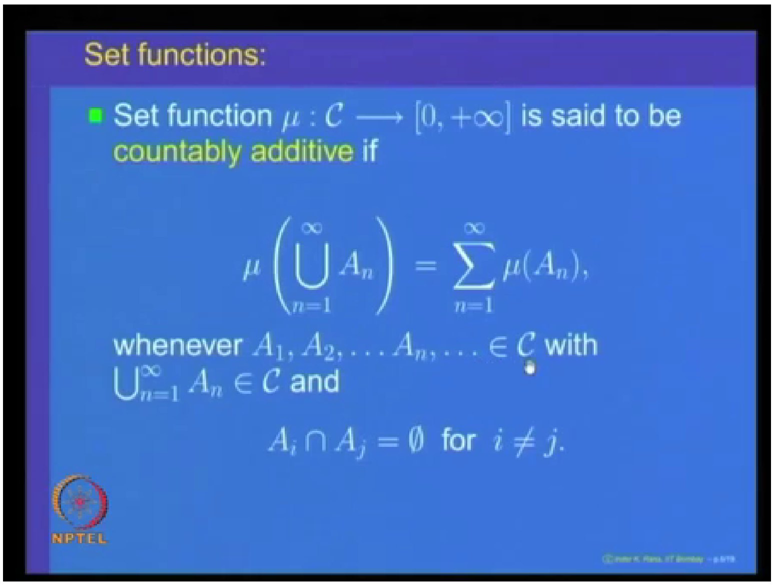

We can extend a generalization of this definition, we will say mu is countably additive. So, from finite we were going to countably additive. If mu of union A n's 1 to infinity is equal to summation of mu n A n's. Of course, whenever A 1 A 2 A n is a sequence of sets in C; such that the union is also an element of C, and there pair wise disjoint. So, countable additivity is a property about a sequence of sets A 1 A 2 A n. So, on in C which are pair wise disjoint, and there union is an element in C. Then we want for any such sequence of pair wise disjoint sets mu of the union, must be equal to summation of mu of A n's n equal to 1 to infinity. There is another notion of called countable is sub additive, if mu of A is less than or equal to summation 1 to infinity mu of A n's, whenever A is a set in C, and A is contained in union of A n's, where A n's also in C for every n.

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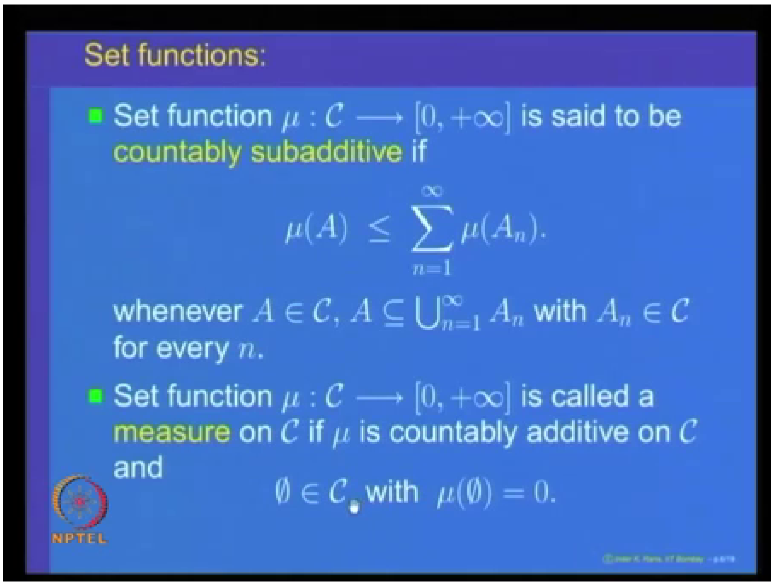

So, in some sense, if A is covered by A union of sets A n's, then we want the size. So, that is mu of A to be less than or equal to summation mu of A n's n equal to 1 to infinity. So, this is called countable sub additivity, because here we are just saying that mu of A is less than or equal to, and we are not requiring that A n's are pair wise A disjoint. So, this is called countable sub additivity property of, with the set function mu, a set function mu is called A measure on C A. C is a collection of subsets on C, if mu has a property that it is countably additive. So, it should be countably additive, and we want that empty set belongs to C, and with the property that mu of empty set is equal to 0. So, mu is defined on a collection C of subjects, and we want the properties at empty set should belong to C, mu of empty set should be 0, and mu on this collection should be countably additive. Such a set function is going to be called a measure on C.

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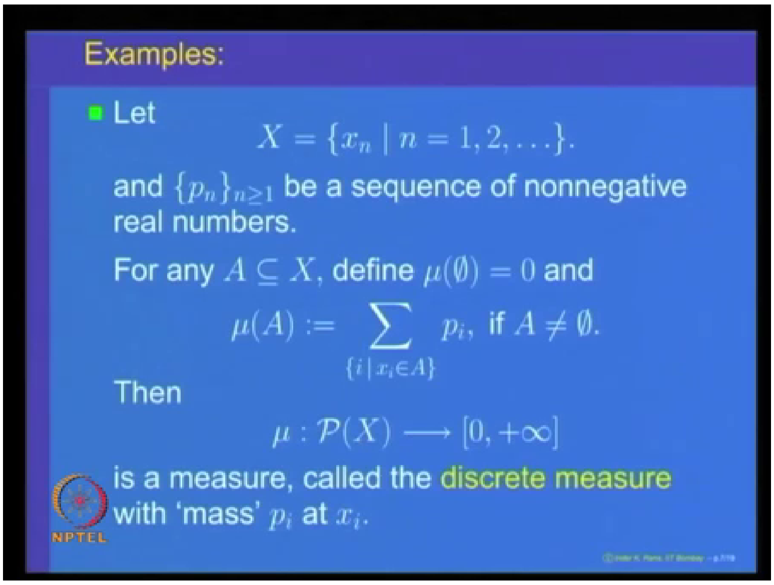

Let us look at some examples of set functions. So, let us start with a very simple 1, let us look a set X which is a countable set. So, its elements are X 1 X 2 X 3 and so on. So, X is equal to X n n 1 2 3 so on, and let us fix P n a sequence of non negative real numbers. So, X is a set which is a countable set with the elements X 1 X 2 X 3 and so on, and we are fixing arbitrarily some sequence of nonnegative real numbers. So, let us define for any set A contained in X for any subset A contained in A X. Let us define mu of the empty set to be equal to 0, and for the set a, if it is non empty, let us define mu of A to be equal to summation over those P i's; such that X i belongs to A.

So, A is a subset of X So, some of the X I's will belong to. So, look at those indices I; such that X i belongs to a. Pick up those p I's from those given sequence p n and add them up, and that is called mu of a. So, mu of A is defined as summation over those p I's; such that xi belongs to a. We want to check that this is a measure on the collection of all subsets of the set x. Well that is quite obvious, because mu of empty set is defined to be equal to 0, and A let us observed that mu of the singleton. If A is a singleton set, then mu of the singleton set is going to be the number P i. So, if a set A is a countable disjoint union of sets. So, let us check that this mu is a measure.

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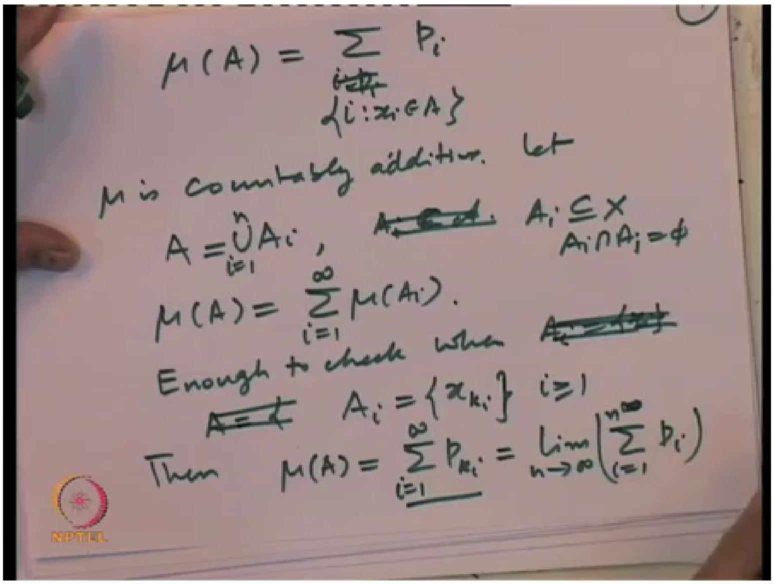

So, we are defining mu of A to be equal to submission P i where I is such that p i. Sorry I is such that X i belongs to A . So, mu is countably additive.

So, let us take a set A. So, let us take a set A, which is union of A i's, I equal to 1 to n a, I's belonging A i a subset of a subset of, where A i is any subset of X To check that mu of A is equal to A as belong to, and we want that A i intersection A J to be empty. So, beyond this to be equal to mu of A I's, I equal to 1 to infinity. Let us observe enough to check when each A i is a singleton X i. So, let us check that case first. So, what is A, A is let us when A i's are some not X i. Let us, because X itself is X 1 X 2 X n. So, this will be the whole space.

So, let us look at the special case, when say A i is equal to sum X k I i bigger then is k I I bigger than or equal to 1, then the set mu of A is going to be equal to summation p of k I i equal to 1 to infinity, which can be written as limit n going to infinity of which can be written as n, going to infinity I equal to 1 to infinity P i up to n. So, sum of a series these are non negative numbers. So, this sum is nothing, but the limit of the partial sums there, the non negative. So, there is no problem in writing that way.

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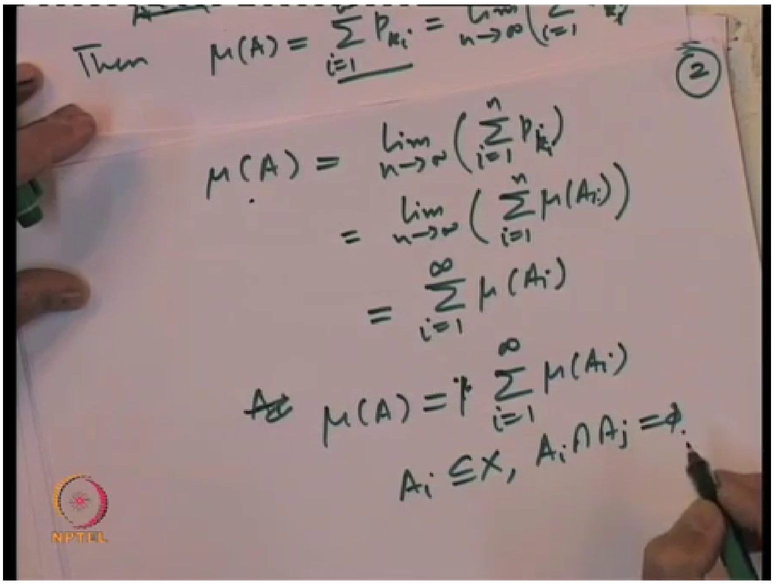

So; that means, mu of A is equal to limit n going to infinity of sigma I equal to 1 to n of p i, but; that means, what we want to check that, it is summation of mu of each A i. So, this is limit n going to infinity of summation I equal to 1 to n mu of A I, because each 1 is P i p of is p of k i. Sorry this is summation p of k i. So, this is k I and this is mu of A i.

So, that is equal to I equal to 1 to infinity mu of A i. So, mu of A is equal to summation mu of A I's, whenever they are, whenever A i is a singleton set, if not it is a finite set, then each finite is A union of finite sets, and then, because for a nonnegative series you can add it in any way you like, it is easy to check that mu of A is l. So, equal to. So, it is easy to check that mu of A is equal to summation mu of A i i equal to 1 to infinity, whenever A i's are contained in X and A i intersection A J is empty. So, that says that the set function mu that we have defined is countably additive. So, this is what is called A discrete measure, because it is given by a sequence, and P i is called the mass at the point X i.

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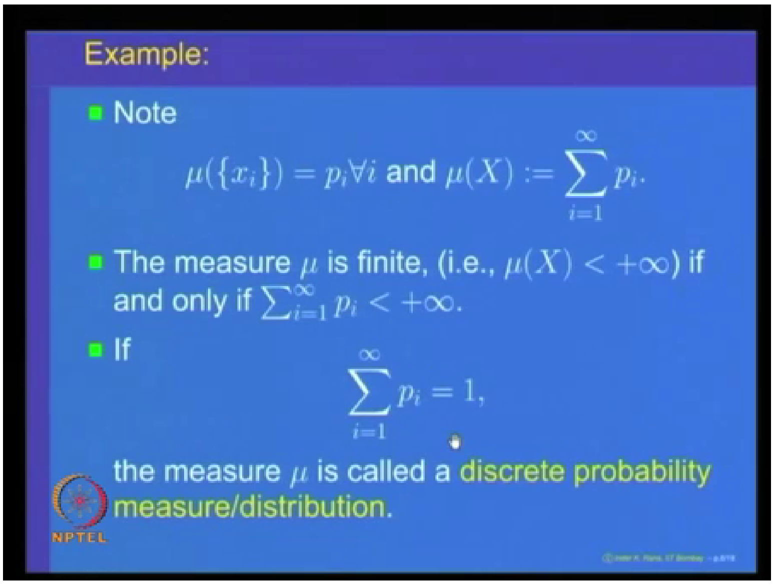

So as we observed mu of the singleton X i is equal to P i for every I and mu of the whole space, which is equal to summation mu of the singletons mu, all the singletons; that is P i.

So, mu of X is equal to summation of P i's. So, obvious consequence of this is that mu of X is finite, whenever this series is convergent. So, the mu is, we have, 1 says this discrete measured mu is finite; that is mu of X is less than infinity mu of the whole space is finite, if and only if summation mu of P i's is less than infinity. If this summation of P i's, if the series P i is convergent and sum is equal to 1, then this measure mu is called A discrete probability distribution on the set X which is X 1 X 2 X n.

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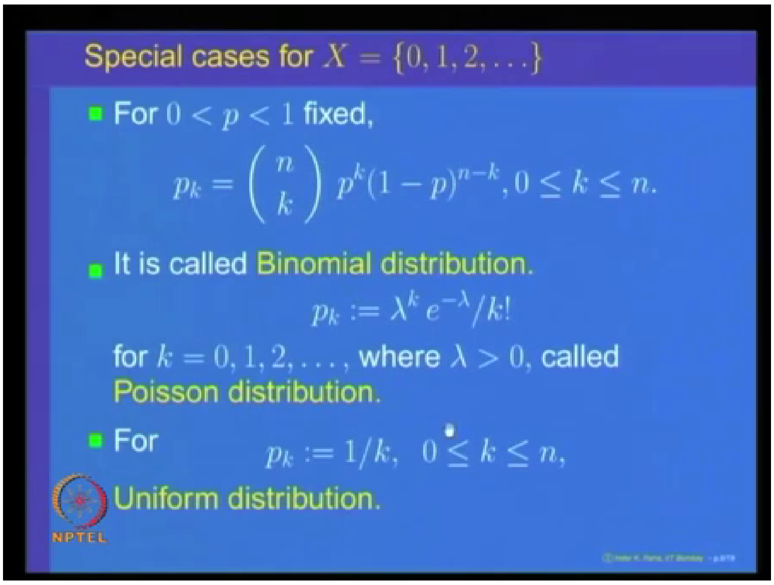

A very special case which plays important role in the theory of probability and so on. X is the set of the numbers 0 1 2 and so on.

Let us fix any number p, which is between 0 and 1, and define p k to be equal to n choose k. So, this is the binomial coefficient n choose k p to the power k 1 minus p raise to the power n minus k, k between 0 and n, this is called the binomial distribution, because of this binomial coefficient appearing in the definition of p k and is quite easy to check the summation of this P k's is equal to 1; that is by summation of this p k is summation k equal to 0 to n, and this side is nothing, but the sum is equal to of p plus 1 minus p raise to power n, and that is equal to 1. So, this is a distribution which plays a very important role in probability, this is a probability distributions. Supposing you have got a coin, and your tossing a coin with probability p for head, appearing then in n toss is, lets p k represents the probability that in n tosses you will get k heads, and another special case of this discrete distribution is called the binomial distribution is called the poisson distribution, which is characterized by the definition that pk is equal to lambda to the power k e raise to power minus lambda divided by k factorial.

This is called poisson distribution, this is another important distribution in the theory of probability. And finally, when we take only the finite number of points 0 1 up to n and p k is 1 over k that each point is given the same mass 1 over k, then this is called the uniform distribution. So, the special cases of discrete probability distributions, when next video and important example of a measure, which is defined on the collection of all intervals in the real line. So, do that, let us fix our notations.

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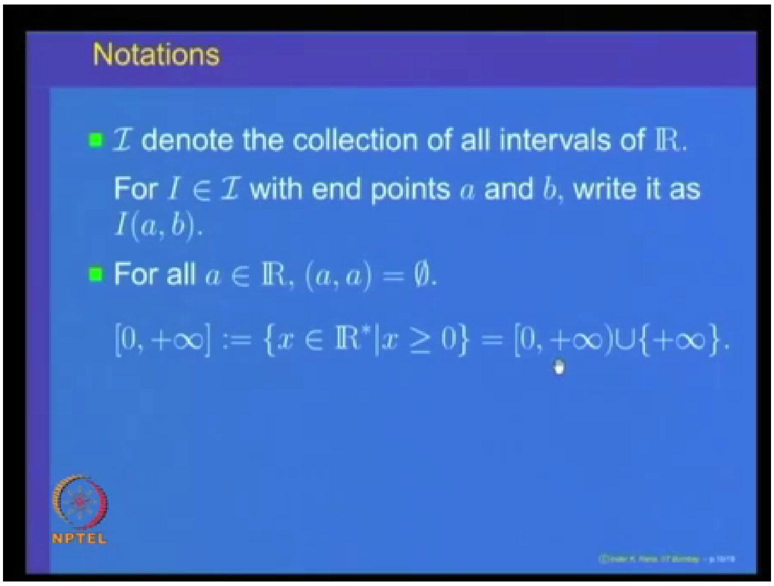

So, we will denote by I the collection of all intervals on the real line for A interval I with end points A and B the left hand point being A the right end point being B, where we will write it as I A comma B. So, A will denote the left end point and B will denote the right end point.

We are not saying that this is a open interval A comma B, we are just saying that it is a interval with left end point A, right end point B, where the left or the right may or may not be, or both may or may not be included in that interval. So, it is just an interval with end points A comma A and B left end point is a, the right end point is B. So, on this collection of all intervals we are going to define a function. So, for example, recall that the open interval A comma A is the empty set and the interval 0 to plus infinity. So, this square brackets indicate that we are including 0, and we are including plus infinity. So, this is the closed interval in R star X belonging to R star, the extended real numbers X bigger than or equal to 0, which is same as the open interval closed on the left, 0 open on the right, infinity in the real line union. The special symbol plus infinity that we have added in the extended real numbers. So, with these notations we define what is called the length function on the class of intervals.

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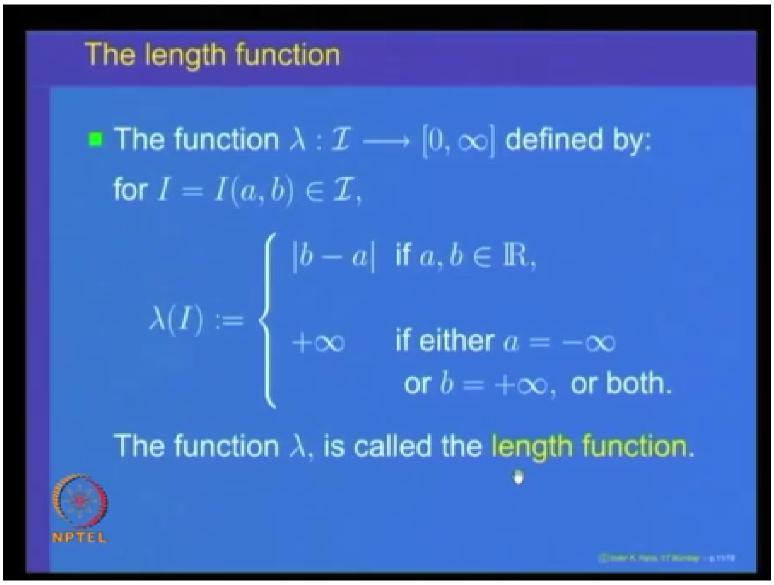

So, it is a set function lambda defined on I taking values in 0 to infinity and is defined by take any interval I with left end point a,, right end point B. So, we defined it as the absolute value of B minus a. If A and B are both real numbers; that means, if a, if the interval I is a finite interval with the end points A and B, then it is length is defined as B minus a, and we defined it will, plus infinity in case either the left end point A is minus infinity or the right end point B is equal to plus infinity or both. So, length of I for A unbounded interval is defined as plus infinity. So, this function is called a length function on the class of all intervals and this length function going to play an important role for in our subject. So, let us study its properties.

So, next we will be studying properties of this length function. The first properties, the length function has a property that lambda of the empty set is 0.

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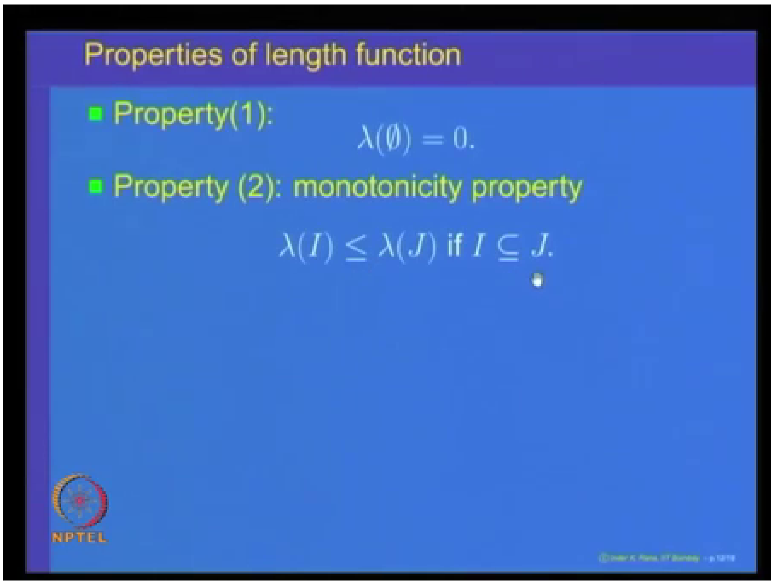

Because empty set is A interval is an open interval with left end point, say A right end point a. So, both is open interval A comma A which is empty set. So, by very definition that is equal to A minus A, which is equal to 0, let. Next let us check that this is a monotone set function. So, namely length of I is less than or equal to length of J. If I is A subset of J. So, to prove that, we want to check, whenever you have got intervals I comma J and I is A subset of J, this should imply that length of I is less than or equal to length of j.

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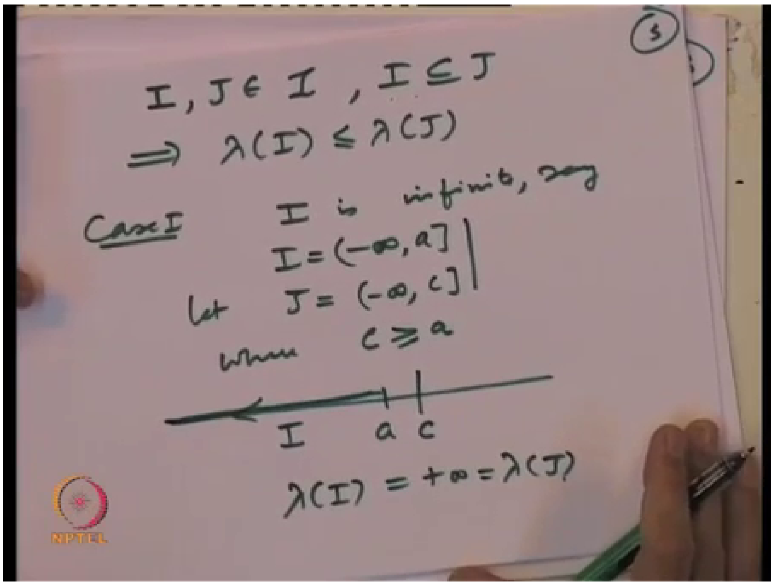

Since the intervals are defined characterized by the end points. So, let in case 1, let us say I is say infinite, say I is equal to minus infinity to say a.

Let J B equal to, now since I is subset of j; obviously, J has to start with minus infinity, and can go up to some point say C, where C is bigger than or equal to a. So, essentially what we are saying is, if this is a, and this side is the interval on this side is the interval, all of it is the interval I, and if J is to contain I, then J must be somewhere ending somewhere here. So, that is C. So, clearly both are infinite. So, length of I is equal to plus infinity is length of J. So, that case is obvious, let us look at the next case, I is A subset of j.

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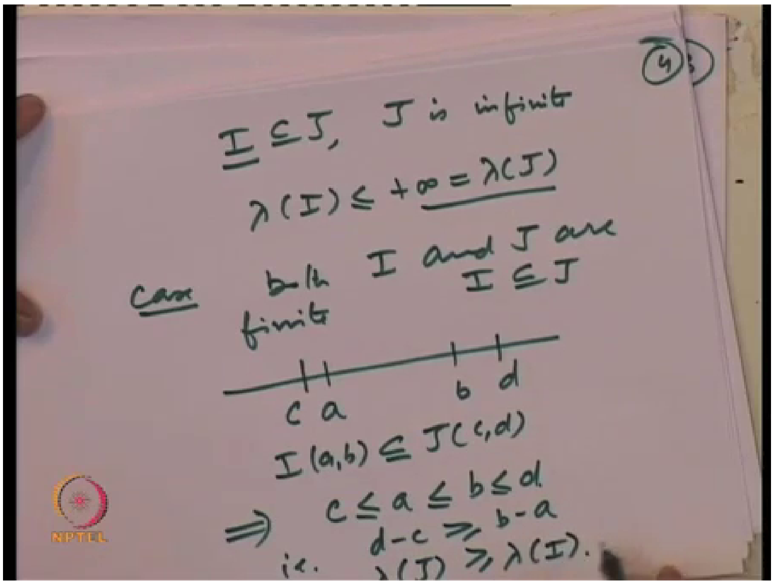

J is infinite, whether I is infinite or not does not matter, because the length of I is always less than or equal to plus infinity, which is equal to length of J, J being infinite its length is always going to be plus infinity.

So, this is obvious, if this is the case. So, finally, it has took at the case when both I and J are finite. So, here is. So, let us say I has got the end point A and B. Now I is subset of j; that means, the endpoints of J has to be somewhere here, and has to be somewhere here. So, we if I is with the end points A B, and J is with the endpoint d, then we should have C is less than or equal to a, less than or equal to B, is less than or equal to d. So, this implies this, and that is same as saying that d minus C is bigger than or equal to B minus a, and that is saying that the length of J is bigger than or equal to length of i. So, the monotone property is check.

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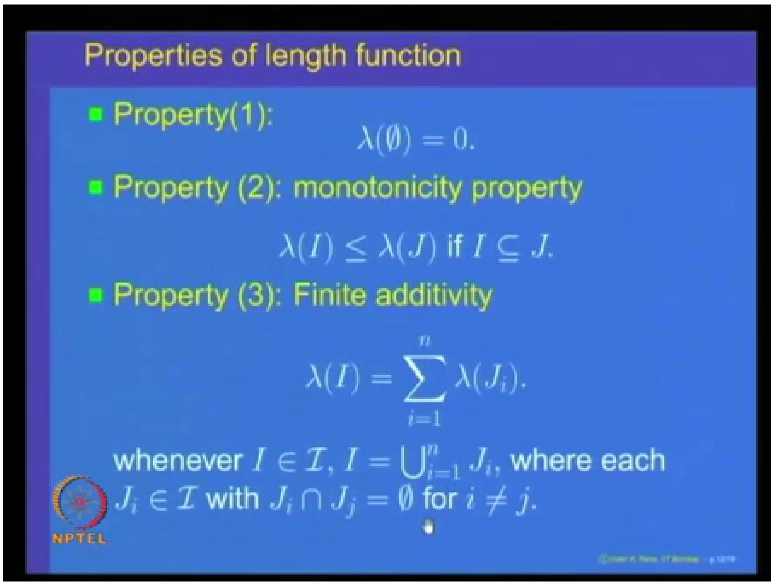

So, the length function lambda is a monotone property; namely if I is a subset of J, whenever A interval I is contained a another interval J, length of I is less than or equal to length of J.

Next let us look at another properly is called additivity property. So, what should be finite additivity property we want, whenever A interval I is A disjoint union of some other intervals, then the length of I should be summation of length of J's. So, what we are saying, if A interval I is written as A finite union of intervals J I J I 1 to n, where this J I's are pair wise disjoint, then we want length of I to be equal to length of summation length of J I's. So, that is going to be called finite additivity property. So, let us check the finite additivity property we want to check if I is equal to union of J I's all are intervals.

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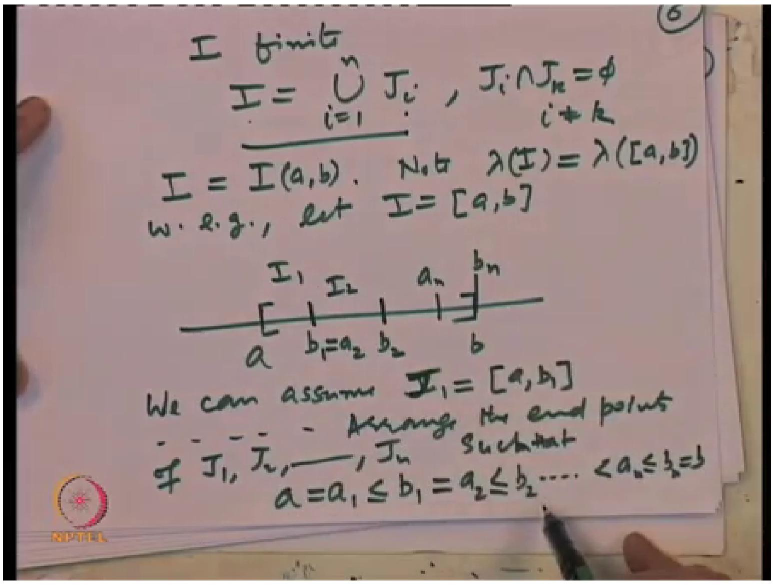

Where J I intersection J k is empty, then that should imply length of I is equal to summation length of J I's, I equal to 1 to n. So, let us assume if I is infinite and I is equal to union of J I's, J I's 1 to n, then that implies at least 1 of J I's is infinite, because if all of them are finite intervals, there union will be again A finite interval.

So, I infinite implies I equal to union of J I's implies at least 1 of these J I's, just to infinite. So, that implies lambda of I is equal to plus infinity is equal to summation lambda of J I's, I equal to 1 to n, because 1 of them is plus infinity. So, the case when I is infinite is, let is look at the case when I is finite. So, I finite I equal to union of J I's and J I's are pair wise disjoint. Now let us say the interval I has got endpoints A and B, and let us say it will the left end point is A the right end point is B. So, here is a, and here is B, we want to compute the length of i. So, note length of I is same as the length of the closed interval A comma B. I can include the end points in the interval I, because the length depends only on the values of the endpoints, it does not matter whether the endpoints are inside or not.

So, what we are saying is without loss of generality, let I B equal to A comma B. So, this is the interval A comma B right now, I is equal to union of J I's. So, the point A belongs to to this union. So, it should belong to 1 of the intervals J I's. So, it belongs tp 1 of the intervals J I's, and actually it has to be end point of 1 of the intervals of J I's, because interval cannot sort with somewhere else. So, let us say. So, without we can say, we can assume I 1 sorry the interval J 1 is starting at A and ending somewhere. Let us call, let us B 1 with the point endpoint may or may not be included.

So, the first interval I 1, we can assume it starts here, and n somewhere here; that is B 1, right now the point B 1 is again in that union. So, either it is already included in the interval I 1, or it should be an endpoint of another interval in the union J 1 J 2 J n's. So, the second 1 must start with here, and end somewhere here; that is B 2. So, what we are saying is I 2 some other interval, we can rename it as I 2. So, it should start somewhere again at B 1, and n somewhere here and so on. So, here will be the last 1 A n and that should B eb n. So, what we are saying is we can assume this is. So, going this way we can arrange the endpoints of J 1 J 2 and J n; such that A is same as A 1 less than or equal to B 1 is equal to A 2, less than or equal to B 2 and so on.

So, A n less than or equal to B n, which is equal to B. So, we can rearrange this endpoints of this intervals, because this is A union and that is a disjoint union. So, this is what its possible for us to arrange.

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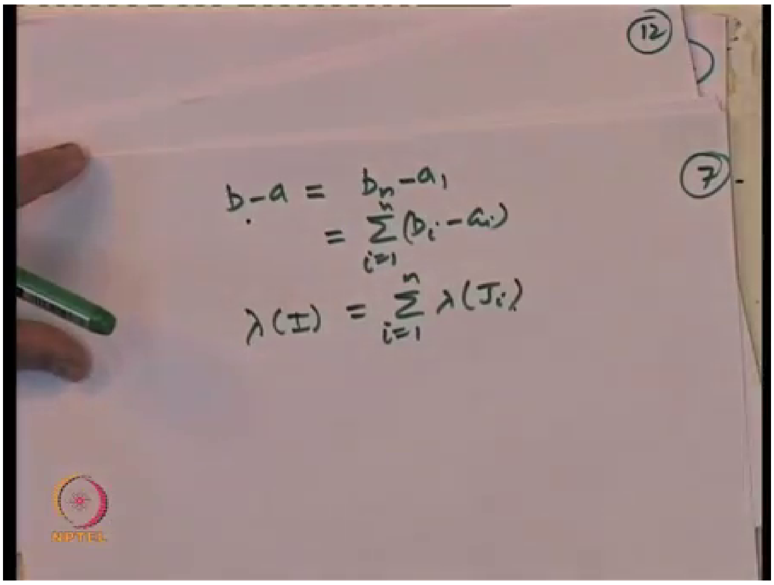

So; that means, and that clearly says that B minus A is equal to B n minus A 1; that is equal to summation B I minus A i I equal to 1 to n, adding and subtracting these terms in between; that is same as I equal to 1 to n lambda of J i. So, whenever I is a finite interval I is equal to union of J I's, there pair wise disjoint we have gotten that the length of this B minus A is the length of the interval I is equal to summation length of J I's. So; that means, that the length function lambda is finitely additive. So, this is the property of lambda the length function being finitely additive; namely if an interval I is A finite union of pair wise disjoint intervals, then length of the interval I is equal to summation length of J I's.