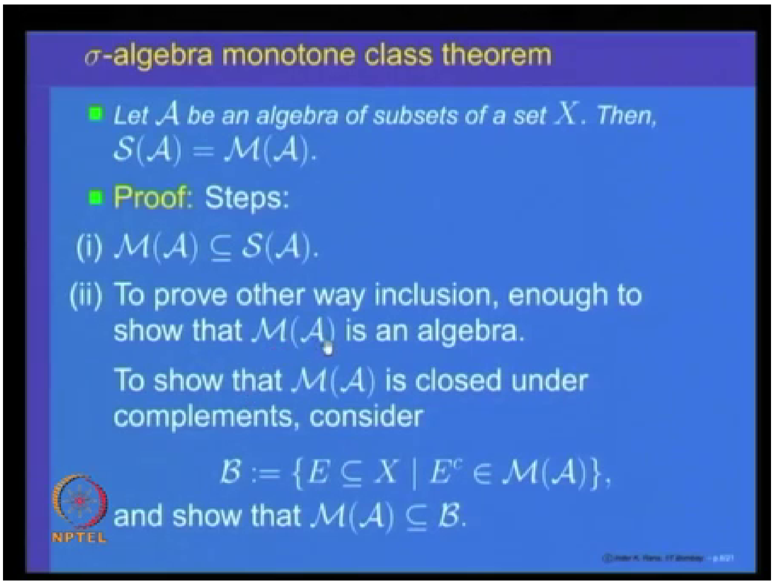
Measure & Integration
Prof. Inder K. Rana
Department of Mathematics
Indian Institute of Technology, Bombay

Lecture - 04 B
Monotone Class

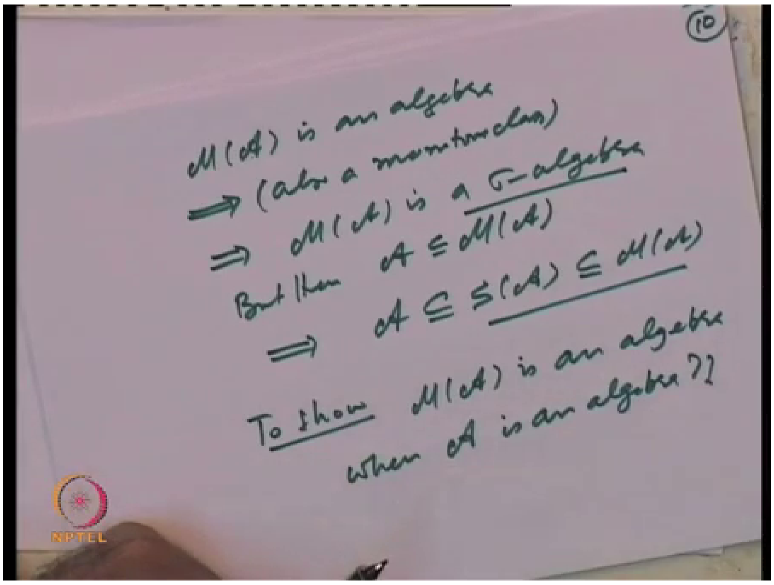
(Refer Slide Time: 00:15)



So, this is an important theorem called the sigma algebra monotone class theorem. It says if A is an algebra of subset of set x, then the sigma algebra generated by it is same as the monotone class generated by it. Just now we all observed that M of. So, the first part we have already observed that M of A is a subset of S of A. For that one need not have even a as algebra for any collection M of C is contained in S of C. So, in particular if A is an algebra then M of A is an subset of S of A.

So, we have to prove the second the other around inclusion namely M of A includes S of A, when A is an algebra and to prove the other way around inclusion let us observe it is enough to prove that M of A is an algebra, because why it is enough to prove this because. So, let us observe enough to show M of A is an algebra will imply, because M of A is an algebra also a monotone class.

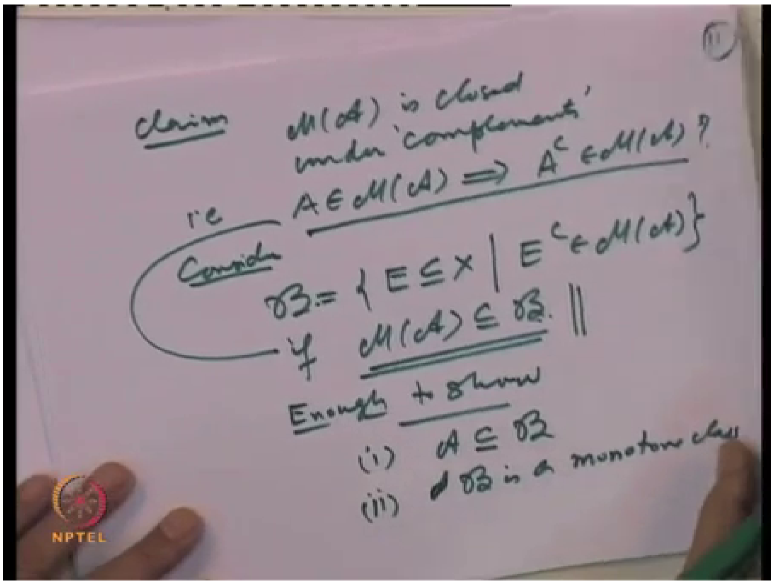
(Refer Slide Time: 01:21)



And just now we proved every algebra which is a monotone class will imply M of A is a sigma algebra. Because M of A is a monotone class and if you are able to show it is an algebra then it will be also be a sigma algebra. But then we have A is inside M of A and now if M of A is a sigma algebra that will imply the smallest one come inside it. So, S of A will be inside M of A. So, that will prove S of A is a subset of M of A and will be through. So, we have to only prove to show M of A is an algebra when A is an algebra. So, this is what we have to prove.

So, let us start looking at a proof of this. So, first of all to prove that M of A is an algebra, we should show that it is closed under compliments right. So, let us try to prove it is closed under compliments. So that means what? To show it is closed under compliments I have to show that for every subset in M of A it is compliment is also in M of A. So, this is a technique we are going to use very often. So, let us collect together all the sets B which have the property that whenever So, B is a collection of all those subsets say that E compliment belongs to M of A. So, to prove M of A is closed under compliments what we have to show is that M of A is a subset of B. So, M of A is a subset of B we have to show that. So, let us try to prove that.

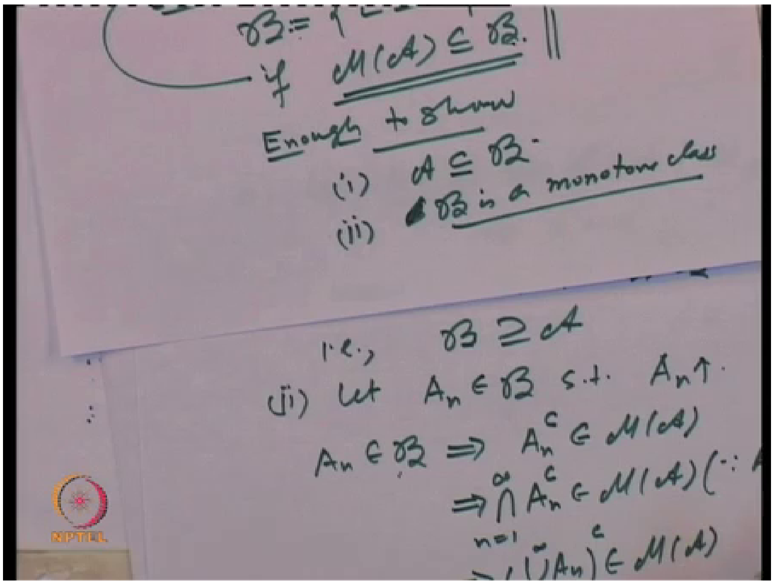
(Refer Slide Time: 03:40)



So, claim let me repeat that to we want to show that M of A is closed under the operation of compliments that is a belonging to M of A should imply a compliment belong to M of A. So, to show that, consider all those subsets. So, let us consider the collection B of all those subsets E contained in x such that E compliment belongs to M of A. So, to prove this, this will be true. So, the required claim will be true. So, let us so, this will be true if we can show M of A is contained in B. So, that is what we want to prove, we want to show because then for every set a belonging to M of A it will belong to B; that means, compliment belong to M of A. So, this is what we have to show.

Now let us observe we have trying to show that M of A is inside a collection B, and what is M of A? M of A is the smallest monotone class including A. So, suppose we are able to show that A is inside B and B is a monotone class then this claim will be true. So, for this enough to show. So, to prove this claim it is enough to show one, that A is inside B and second we should show that M of A. And Secondly, B is a monotone class because once B is a monotone class including a the smallest one will come inside. So, let us try to prove this 2 facts that A is so, proof of one that A is a subset of B.

(Refer Slide Time: 06:01)

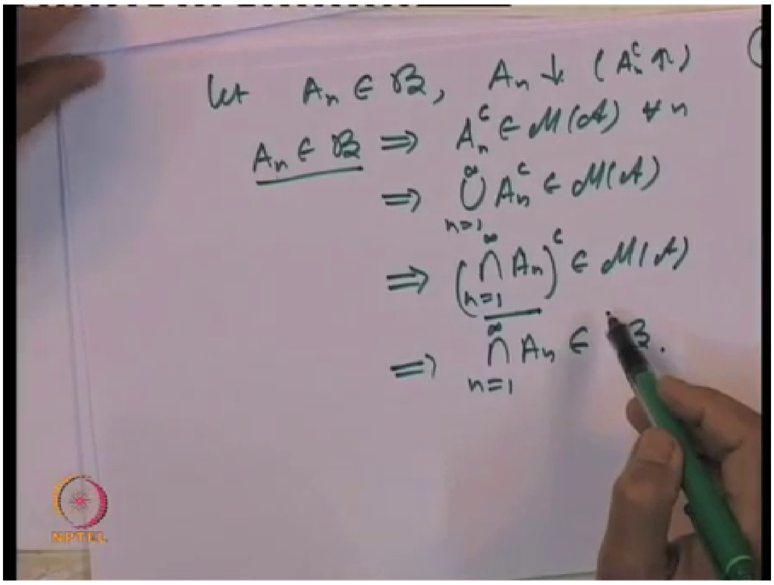


So, let Set a belong to a that implies A is a algebra, so that a compliments belong to A because A is algebra, because A is algebra. And note implies a compliment belongs to A which is inside M of A, because A is always inside M of A. So, what we have shown that if a belongs to A then it is compliment belongs to M of A. Hence that is same as say that a belongs to now the collection B. So that means so, that is so, we have proved that B includes A. So, first property is true.

Let us look at the second property. So, what is the second property we want to prove the second property we want to prove is the B is a monotone class? So, let us take let us take a collection A n a sequence belonging to B, such that it is decreasing or increasing. So, let us say A n is increasing, but A n belonging to B means, what that means? What A n compliments belong to M of A right. That is the definition of the class that is the definition of the class B. So, saying that we got a set A n in this; that means, A n compliments belong to a now M of A is a monotone class. So, that implies that A n compliments intersection will belong to M of A, provided you can say n compliments are decreasing and that is true because ans are increasing, because A n compliments are decreasing and M of A is a monotone class. So that means, this intersection belongs to it So that means, union of A n’s n equal to 1 to infinity compliment of this belongs to M of A. So, we have got whenever a sequence an and B and A n's are increasing you got the compliment union the compliment of the union belongs to it So that means, union of A n's n equal to 1 to infinity belongs to B. So, the collection B is closed under increasing unions.

And let us finally, prove that it is also closed under decreasing sequences. So, let us take a sequence of sets which is decreasing.

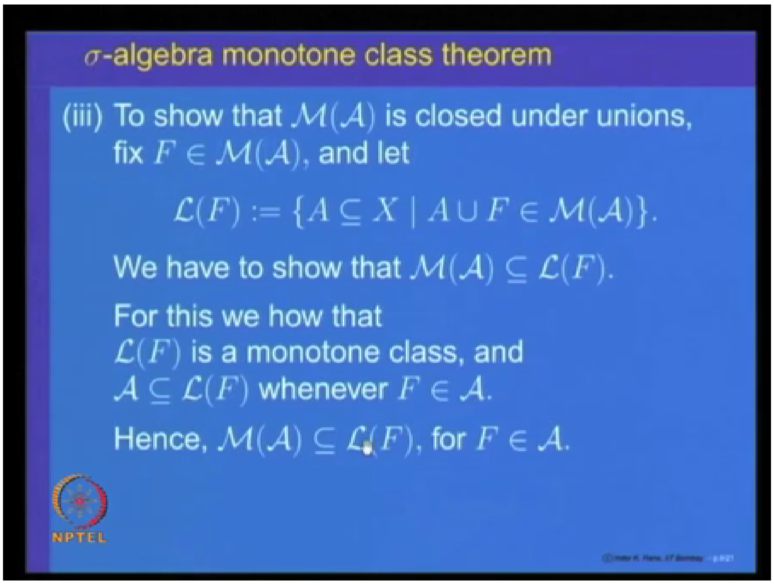
(Refer Slide Time: 08:49)



So, let A n So, let A n belong to B and A n's decrease. So, you want to show that intersection of A n's belong to it, but A n's belong to B implies that just now we observed A n by definition A n compliments belong to M of A. By definition A n belong to B means A n compliments belong to a for every n. And that implies now A n compliments is a sequence because A n's are decreasing that is same as A n compliments are increasing. So, union of A n compliments belong to M of A because M of A is a monotone class, and that implies that if the intersection of A n's n equals to 1 to infinity compliments belongs to M of A.

So, if whenever A n's belongs to it and take the intersection of A n's compliment belongs to it; that means, intersection of A n, n equal to 1 to infinity belongs to B. So, what we have shown is the collection B is closed under increasing union is closed under decreasing intersections; that means, B is a monotone class. So, A is inside B, B is a monotone class and that will prove that M of A is a subset of B. Because so this is a monotone class including A. So, it must include the smallest one. So, that proves the first step of our first step of our claim namely that the collection M of A is closed under compliments we wanted to show it is a algebra. So, what is the next step next step should be to show that M of A is closed under unions right.

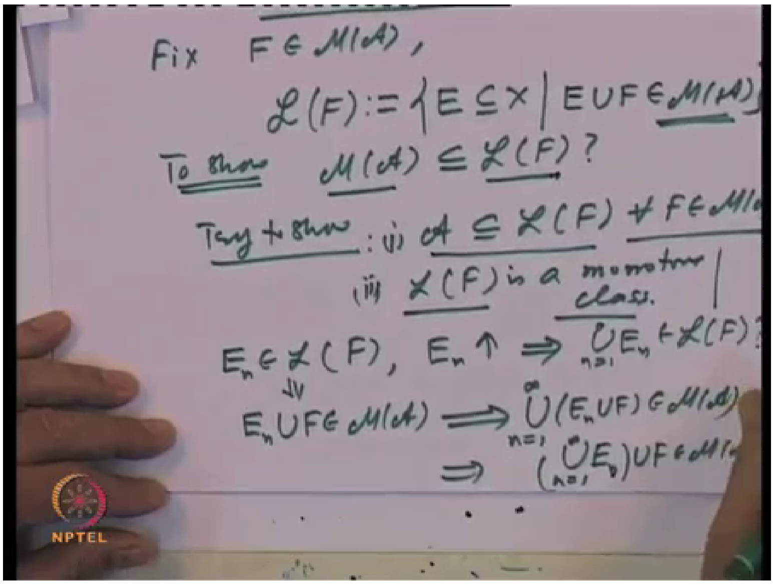
(Refer Slide Time: 10:46)



So that means, when ever 2 sets E and F belongs to M of A their union must belong to M of A.

So, let us fix one of them let us fix the set F in M of A, and let us look at the collection L of F such that it is the collection of all those sets say that a union F belongs to M of A. So, what we have to prove. So, in this we have to prove that M of A is closed under unions we have to prove that M of A is a subset of L of A. So, once again the required property that M of A is closed under unions we are translating into a property of a collection of subsets. So, let us try to show that M of A is contained in L of F. So, that is first we should try to show; so to show that the collection M of A is closed under unions.

(Refer Slide Time: 11:42)



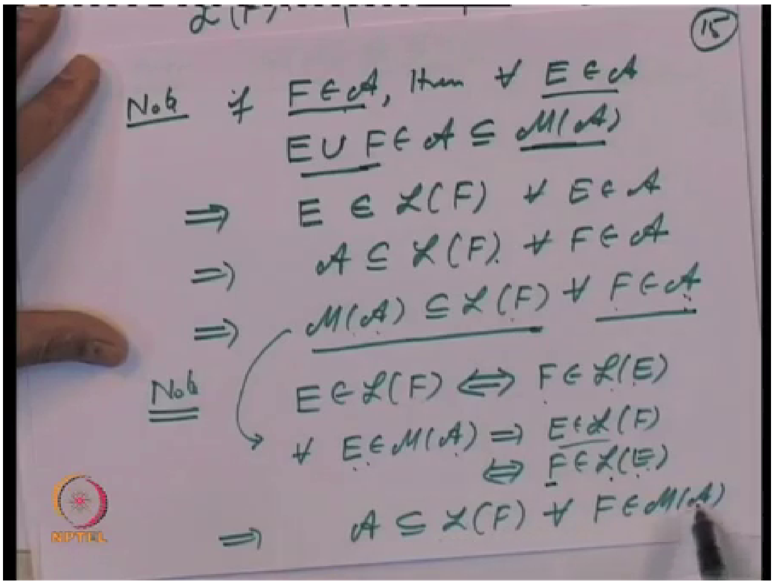
So, this is what we want to show. So, let us fix a set F belonging to M of A, and consider the collection which is let us call it L of F what is this collection? It is the collection of all those subsets in x such that E union F belongs to M of A. So, saying that M of A is closed under unions. So, to show so we should show that M of A is a subset of L of F; so that is what we should show.

So, once again we want to show M of A is a subset of L of F. M of A is a monotone class generated by A. And we want to show it comes under L of in L of A (Refer Time: 12:49) in the in some other collection L of F. So that means, we should try to show that A is inside this collection an this collection L of F is a monotone class. So, we should try to show that A is inside L of F for every F belonging to M of A. And second L of F is a monotone class right. Let us just observe second one which is quite obvious. So, let us observe L of F is a monotone class.

So, for that what we have to show let us take a sequence E n belonging to L of F E n's increasing, but that will mean if E n's are in L of F that will imply that E n union F belongs to M of A. That is by definition of L on F and M of A is a monotone class E n's are increasing. So, E n union F is also increasing. So, implies that union of E n union F belongs to M of A. Because E n's are increasing E n unions F are increasing and belong, and that means, it is same as saying that union of E n's union F belongs to M of A. And what does it mean? That means, union of E n's belong to the class L of F. So, whenever E n's belongs to L of F, E n is increasing this implies that union E n's belong to L of F. So, this is what we have just now proved a similar proof will work for decreasing also. So, saying that L of F is the monotone class is a straightforward argument, because M of A is a monotone class. Let us try to check that A is inside L of F for every F belonging to L M of A.

So, we want to check the first property namely. So, this is the property we want to check that this collection algebra is inside L of F for every F belonging to M of A.

(Refer Slide Time: 15:28)



Let us note For the time being we want to check this property for every F in M of A. Note if F belongs to A then for every E belonging to A E union F belongs to A. Because A is algebra, if E and F are 2 sets in A is algebra that it mean that the union belongs to Algebra and that is included in M of A. So, what does it imply this means for every F in A E union F belongs to M of A; that means, that the set E belongs to the collection L of F. Once again we are starting with very simple observation if a set F belongs to A and E belongs to A then their union belongs to A right. And A is always inside M of A so that means, E union F belongs to M of A E union F belongs to M of A for every E belonging to A; that means, for every E belonging to A. So that means, we have shown that A is inside L of F right.

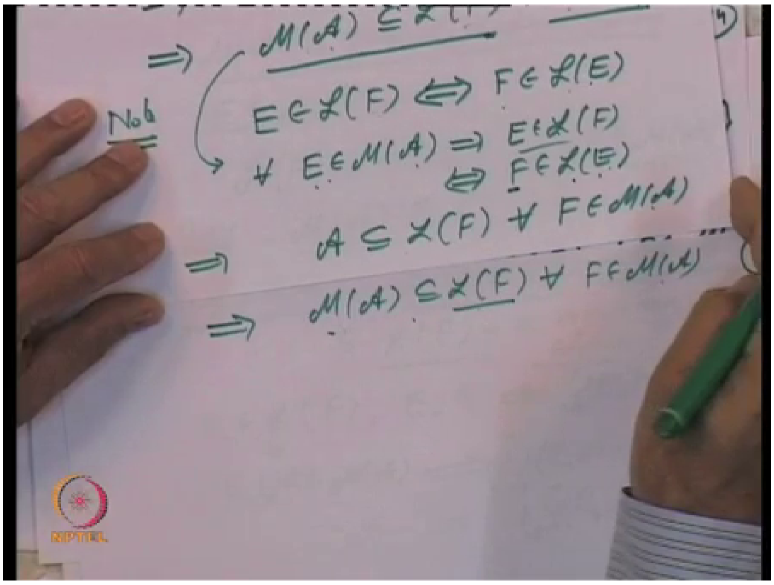
So now, A is inside L of F for every F in A, implies L of F is a monotone class just now proved implies M of A is inside L of F for every F belonging to A. So, what we have shown is M of A is a subset of L of F for every F belonging to A, but we wanted to check that M of A is instead as for every F belonging to M of A, we have got only for F belonging to A.

Now, here is a very simple observation which helps us note, a set E belongs to L of F if and only if F belongs to L of E. So, this is an observation which is going to be very important and very useful for us that a set E belongs to L of F means what? E union F belongs to M of A, but if E union F belongs to M of A that is same as F union E belongs to; that means, F belongs to L of E. So, saying that E belongs to L of F is same as F belongs to L of A.

So now, let us translate this property. So, here it says M of A is inside L of F for every F belonging to A. So that means, for every E belonging to M of A implies that E belongs to L of F, and that is if and only if F belongs to L of E and here F was in F was in the algebra F in the algebra. So, what we have got for every F in the algebra L it belongs to L of E whenever E belongs to L of A. So that means, what that means, A is inside L of F for every F belonging to M of A.

So, see how nicely we have turned the tables. Earlier we had M of A inside L of F for every F in A. So that means, every element a at E is element in L of F, but here E belongs to L of F means F belongs to L of E now F is in a; that means, A is in L of F for every F belonging to A, but that means, once that is true Now, A is inside A is inside L of F for every F in M of A and that implies that M of A is inside L of F for every F belonging to M of A.

(Refer Slide Time: 19:40)

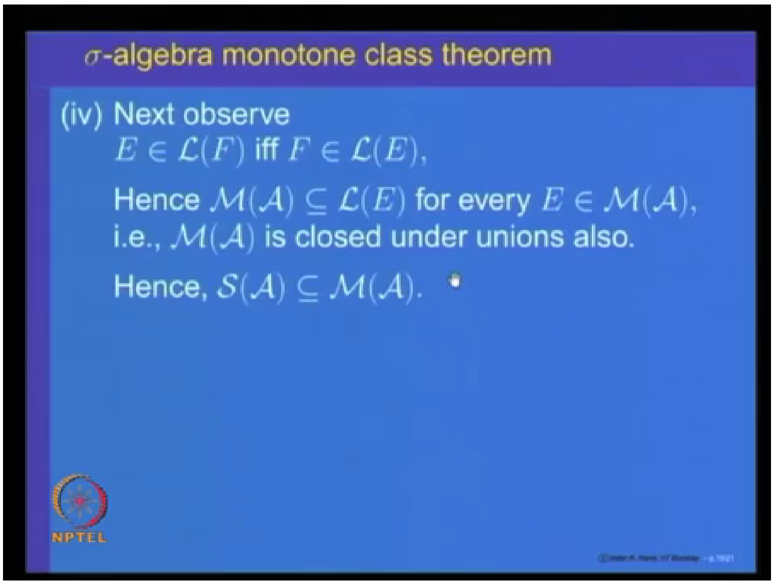


Because L of F is a monotone class it includes the algebra A. So, it must include the smallest one. So, M of A is inside L of F for every F belonging to M of A. So, we have proved the required thing namely M of A is inside L of F for every F belonging to M of A and hence; that means, M of A is also closed under. So, here is what we wanted to prove. So, L M of A is closed under unions that we translated into the property that M of A is inside L of F for every F in M of A, and that we just now proved finally, and see again and again.

We are whenever we want to show something is true we can converted into a property of a collection of objects and show generators come inside and everything comes inside. So, that proves that M of A is an algebra M of M of A is a algebra it is already a monotone class. So, it must be sigma algebra. And S of A is a sigma algebra. So, that will prove that the required theorem. So, that proves this step proves the required theorem namely.

So, for this Let us go through this proof again to show that M of A is closed under unions, we fix F in M of A and look at this collection. And we want to show M of A is inside L of F for this. We have to show that L of F is a monotone class right and A is inside L of F whenever F belongs to A. So, that says M of A will come inside for F belonging to A and now reverse the rolls of these 2 that M of A for F in A; that means, E belongs to L of F. So, F belongs to L of E and reverse the rolls and that comes.

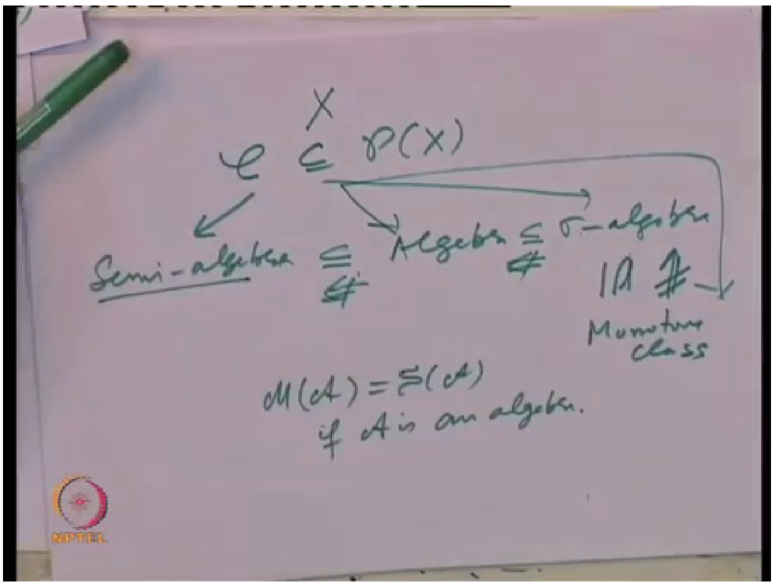
(Refer Slide Time: 21:48)



So, that gives you the property that M of A is inside L of E. So, it is closed under unions and hence it is a sigma algebra. So, it must include the smallest one and that. So, that proves the fact that the sigma algebra generated by a algebra is same as the monotone class generated by the algebra.

So, let us just recall what I have done till now we started with a set x looked at a collection C of subset c contained in p x.

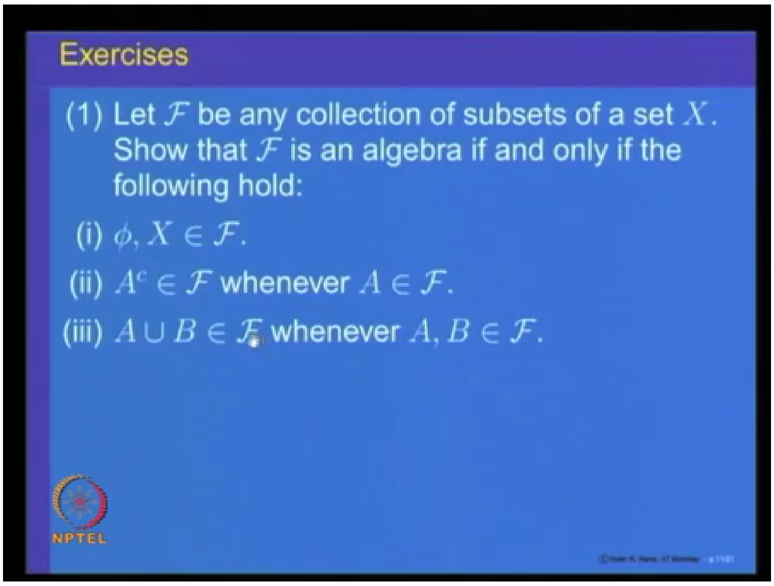
(Refer Slide Time: 22:17)



The first thing we looked at what is called a semi algebra. Then we looked at this collection c to be an algebra then we looked at this collection to be a sigma algebra, and then it to be a monotone class. So, a semi algebra every algebra is a semi algebra every sigma algebra is a also an algebra every sigma algebra is also a monotone class. So, this is something here this way around implication may not be true this way around implication may not be true and this way around implication may not be true.

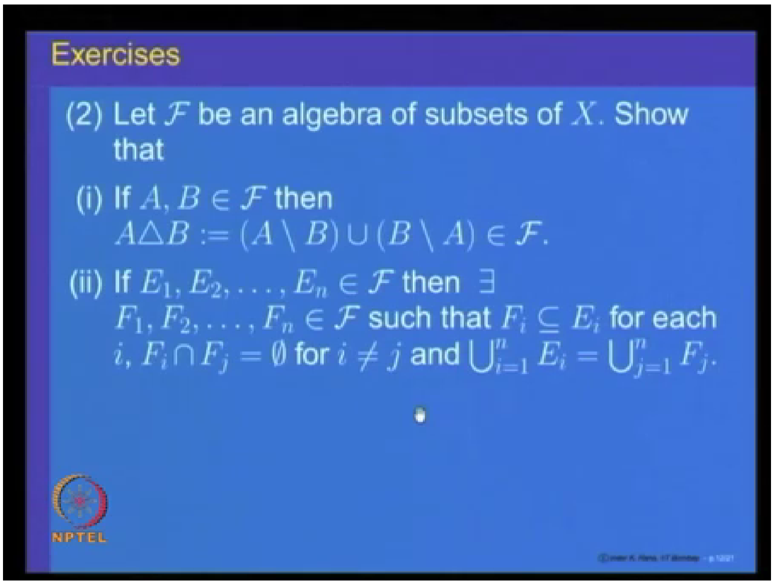
And finally, we have proved monotone class generated by an algebra is the sigma algebra generated by algebra if A is an algebra. So, that finishes our study of collection of subsets of x with special properties. I just want to leave you with some exercises which you should try which are important.

(Refer Slide Time: 23:42)



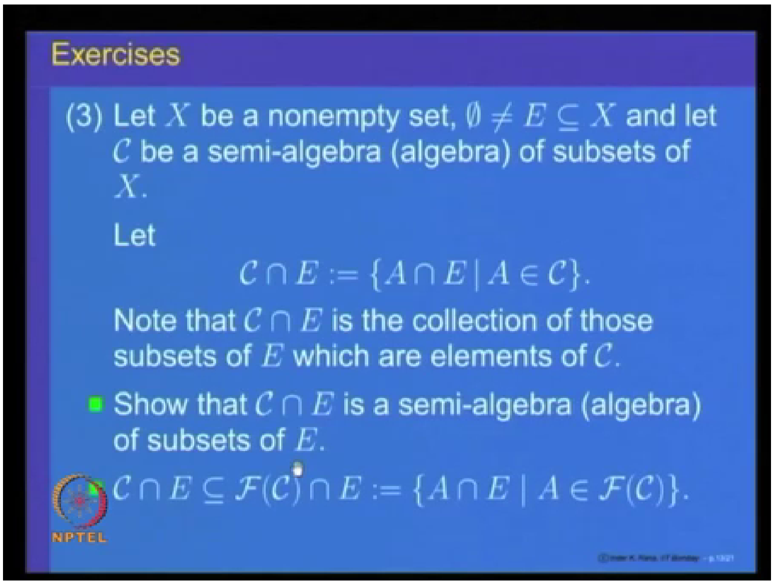
The first property, the first exercise would like to that whenever a collection is an algebra it is equivalent to saying that empty set in the whole space it collects is closed under complements it is closed under unions and that is equivalent to saying whether it is closed under complements because of this de Morgan's laws.

(Refer Slide Time: 24:07)



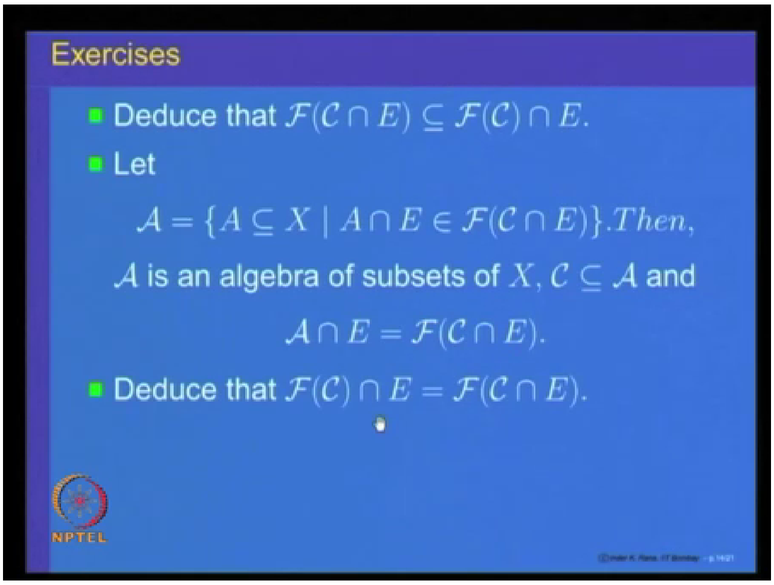
Another property that whenever something is an algebra a collection is algebra it is also closed under symmetric differences. And any finite union in algebra can be represented as a finite disjoint union whenever you are inside a algebra. That property you have seen, but you should try to prove this exercise yourself.

(Refer Slide Time: 24:27)



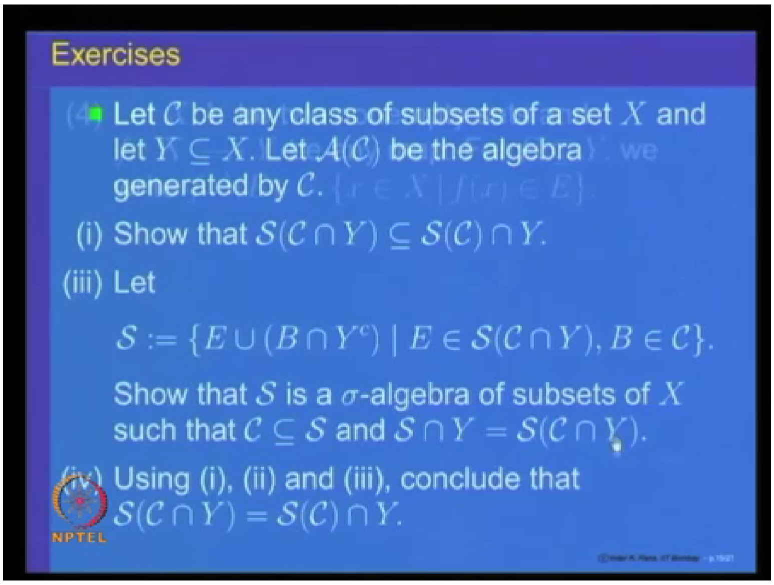
Another property about semi algebras is you can or sigma algebras is that you can restrict. So, take a collection c of subset service at x and restrict it to a set E; that means, take intersection of all sets in c with E, then this is c restricted to E and the property we want to prove is that if c is a semi algebra then c intersection E is a semi algebra of subsets of E.

(Refer Slide Time: 24:59)



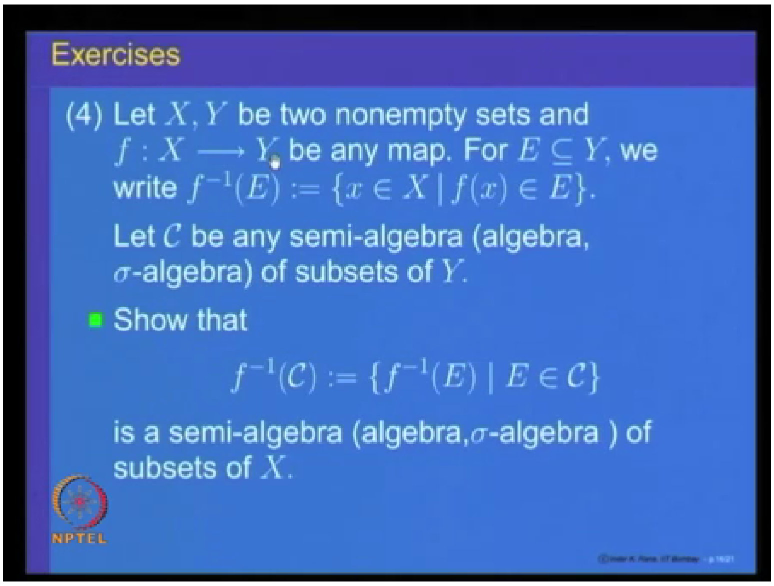
And similarly you proved the Property that the algebra generated by the restricted sets is same as the algebra generated by generate the algebra and restrict. So, F of c intersection E is equivalent to F of c intersection E. So, restrict and generate it is same as generate and restrict and the same property is true for algebras is true for semi sigma algebra.

(Refer Slide Time: 25:26)



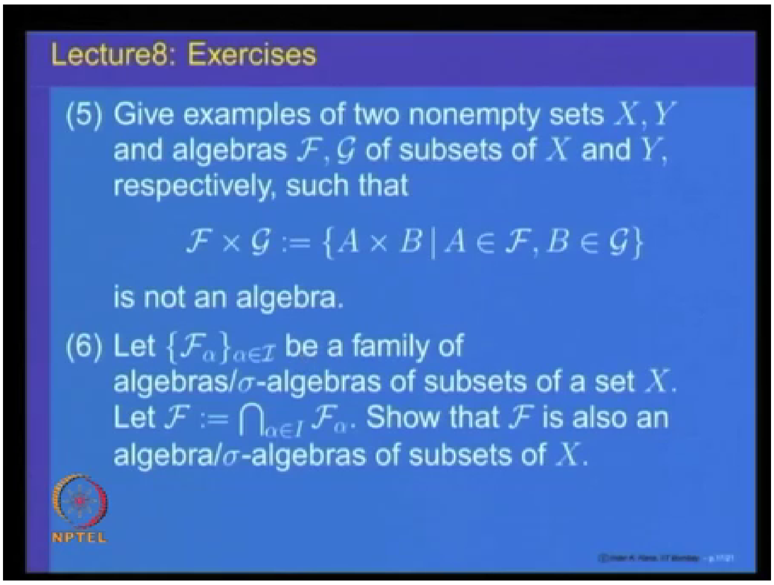
So, that is the property of sigma algebras and these exercises you should try to prove yourself the steps are outlined here for you to prove. We have already proved these things in our lectures, but I will strongly advise that you prove these things yourself.

(Refer Slide Time: 25:44)



And here is another example of Generating new sigma algebras semi algebras F is a function from x to y it would take sets in y and take inverse imagine that give you collection of subsets of y. So, try to show that whenever if c is a collection of subsets of y which is a semi algebra, algebra or a sigma algebra then the pull backsets also form a semi algebra, algebra or sigma algebra.

(Refer Slide Time: 26:07)



Here is another example of take 2 collection of subsets F and g of 2 sets x and y look at the cartition products of this collection. So, that in general it is not a algebra. Of course, if you take a collection of subsets F alpha, which are all algebras or sigma algebras the corresponding intersection also is a algebra or a semi algebra. So, this property is unions may not be true.

So, show that for union this property need not be true. So, that sigma algebra technique that same thing is inside then the generated sigma general comes inside that we use. So, it fuse that to prove that if you take the collection of all intervals in I and who left open and right closed intervals, then the sigma algebra generated by all intervals is same as the sigma algebra generated by all left open right closed intervals and same as the Borel sigma algebra.

So, I would strongly advise you to try these properties to get use to this concept of algebra semi, semi algebra, algebra sigma algebra and monotone class. So, let us stop here today.

Thank you very much.